University of São Paulo
São Carlos School of Engineering
Department of Aeronautical Engineering

# Integration and differentiation

Ricardo Afonso Angélico
raa@sc.usp.br

# Finite differences

- Forward

$$\Delta_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + (n-i)h),$$

- Backward

$$\nabla_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x - ih),$$

- Central

$$\delta_h^n[f](x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\left(x + \left(\frac{n}{2} - i\right) h\right).$$

$$\frac{d^n f}{dx^n}(x) = \frac{\Delta_h^n[f](x)}{h^n} + O(h) = \frac{\nabla_h^n[f](x)}{h^n} + O(h) = \frac{\delta_h^n[f](x)}{h^n} + O(h^2).$$

# Central finite differences

| Derivative | Accuracy | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | | | | −1/2 | 0 | 1/2 | | | |
| | 4 | | | 1/12 | −2/3 | 0 | 2/3 | −1/12 | | |
| | 6 | | −1/60 | 3/20 | −3/4 | 0 | 3/4 | −3/20 | 1/60 | |
| | 8 | 1/280 | −4/105 | 1/5 | −4/5 | 0 | 4/5 | −1/5 | 4/105 | −1/280 |
| 2 | 2 | | | | 1 | −2 | 1 | | | |
| | 4 | | | −1/12 | 4/3 | −5/2 | 4/3 | −1/12 | | |
| | 6 | | 1/90 | −3/20 | 3/2 | −49/18 | 3/2 | −3/20 | 1/90 | |
| | 8 | −1/560 | 8/315 | −1/5 | 8/5 | −205/72 | 8/5 | −1/5 | 8/315 | −1/560 |
| 3 | 2 | | | −1/2 | 1 | 0 | −1 | 1/2 | | |
| | 4 | | 1/8 | −1 | 13/8 | 0 | −13/8 | 1 | −1/8 | |
| | 6 | −7/240 | 3/10 | −169/120 | 61/30 | 0 | −61/30 | 169/120 | −3/10 | 7/240 |
| 4 | 2 | | | 1 | −4 | 6 | −4 | 1 | | |
| | 4 | | −1/6 | 2 | −13/2 | 28/3 | −13/2 | 2 | −1/6 | |
| | 6 | 7/240 | −2/5 | 169/60 | −122/15 | 91/8 | −122/15 | 169/60 | −2/5 | 7/240 |
| 5 | 2 | | −1/2 | 2 | −5/2 | 0 | 5/2 | −2 | 1/2 | |
| 6 | 2 | | 1 | −6 | 15 | −20 | 15 | −6 | 1 | |

# Forward and backward finite differences

| Derivative | Accuracy | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | −1 | 1 | | | | | | | |
| | 2 | −3/2 | 2 | −1/2 | | | | | | |
| | 3 | −11/6 | 3 | −3/2 | 1/3 | | | | | |
| | 4 | −25/12 | 4 | −3 | 4/3 | −1/4 | | | | |
| | 5 | −137/60 | 5 | −5 | 10/3 | −5/4 | 1/5 | | | |
| | 6 | −49/20 | 6 | −15/2 | 20/3 | −15/4 | 6/5 | −1/6 | | |
| 2 | 1 | 1 | −2 | 1 | | | | | | |
| | 2 | 2 | −5 | 4 | −1 | | | | | |
| | 3 | 35/12 | −26/3 | 19/2 | −14/3 | 11/12 | | | | |
| | 4 | 15/4 | −77/6 | 107/6 | −13 | 61/12 | −5/6 | | | |
| | 5 | 203/45 | −87/5 | 117/4 | −254/9 | 33/2 | −27/5 | 137/180 | | |
| | 6 | 469/90 | −223/10 | 879/20 | −949/18 | 41 | −201/10 | 1019/180 | −7/10 | |
| 3 | 1 | −1 | 3 | −3 | 1 | | | | | |
| | 2 | −5/2 | 9 | −12 | 7 | −3/2 | | | | |
| | 3 | −17/4 | 71/4 | −59/2 | 49/2 | −41/4 | 7/4 | | | |
| | 4 | −49/8 | 29 | −461/8 | 62 | −307/8 | 13 | −15/8 | | |
| | 5 | −967/120 | 638/15 | −3929/40 | 389/3 | −2545/24 | 268/5 | −1849/120 | 29/15 | |
| | 6 | −801/80 | 349/6 | −18353/120 | 2391/10 | −1457/6 | 4891/30 | −561/8 | 527/30 | −469/240 |

# Integration

- Quadrature rules

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

$$\omega(x) = \frac{1}{\sqrt{1-x^2}}$$ Gauss-Chebyshev

$$\omega(x) = e^{-x^2}$$ Gauss-Hermite

Gauss-Legendre

| n | $P_n(x)$ |
|---|---|
| 1 | $x$ |
| 2 | $\frac{1}{2}[3x^2-1]$ |
| 3 | $\frac{1}{2}[5x^3-3x]$ |
| 4 | $\frac{1}{8}[35x^4-30x^2+3]$ |
| 5 | $\frac{1}{8}[63x^5-70x^3+15x]$ |

$$w_i = \frac{2}{(1-x_i^2)\,[P_n'(x_i)]^2}$$

| $n$ | $w_i$ | $x_i$ |
|---|---|---|
| 1 | 2.0 | 0.0 |
| 2 | 1.0 | ±0.5773502692 |
| 3 | 0.5555555556 | ±0.7745966692 |
| | 0.8888888889 | 0.0 |
| 4 | 0.3478548451 | ±0.8611363116 |
| | 0.6521451549 | ±0.3399810436 |
| 5 | 0.2369268851 | ±0.9061798459 |
| | 0.4786286705 | ±0.5384693101 |
| | 0.5688888889 | 0.0 |
| 6 | 0.1713244924 | ±0.9324695142 |
| | 0.3607615730 | ±0.6612093865 |
| | 0.4679139346 | ±0.2386191861 |
| 7 | 0.1294849662 | ±0.9491079123 |
| | 0.2797053915 | ±0.7415311856 |
| | 0.3818300505 | ±0.4058451514 |
| | 0.4179591837 | 0.0 |

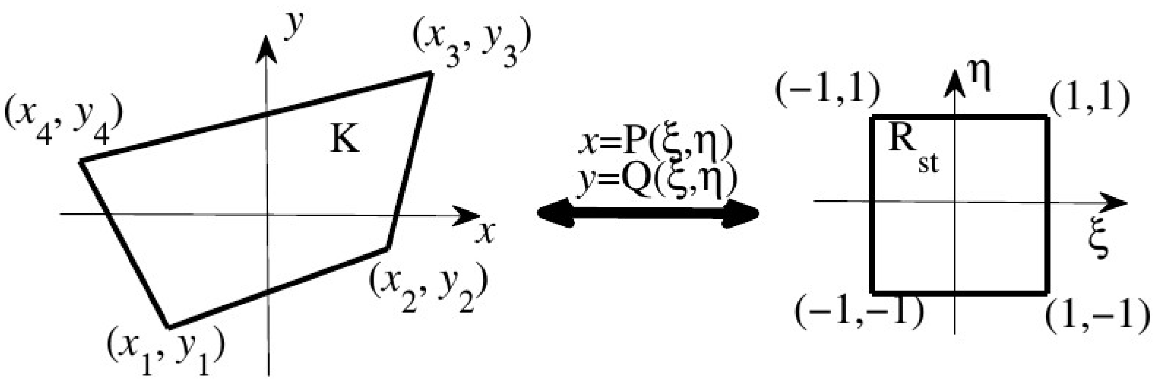
| $n$ | $w_i$ | $x_i$ |
|---|---|---|
| 8 | 0.1012285363 | ±0.9602898565 |
| | 0.2223810345 | ±0.7966664774 |
| | 0.3137066459 | ±0.5255324099 |
| | 0.3626837834 | ±0.1834346425 |
| 9 | 0.0812743883 | ±0.9681602395 |
| | 0.1806481607 | ±0.8360311073 |
| | 0.2606106964 | ±0.6133714327 |
| | 0.3123470770 | ±0.3242534234 |
| | 0.3302393550 | 0.0 |
| 10 | 0.0666713443 | ±0.9739065285 |
| | 0.1494513492 | ±0.8650633667 |
| | 0.2190863625 | ±0.6794095683 |
| | 0.2692667193 | ±0.4333953941 |
| | 0.2955242247 | ±0.1488743390 |

Change of interval

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)dx$$

# Integration in 2D

- Quadrature rules $$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j g(\xi_i,\xi_j)$$




$$
\begin{aligned}
N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta),\\
N_2(\xi,\eta) &= \frac{1}{4}(1+\xi)(1-\eta),\\
N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta),\\
N_4(\xi,\eta) &= \frac{1}{4}(1-\xi)(1+\eta).
\end{aligned}
$$

$$x = P(\xi,\eta) = \sum_{i=1}^{4} x_i N_i(\xi,\eta) = x_1N_1(\xi,\eta) + x_2N_2(\xi,\eta) + x_3N_3(\xi,\eta) + x_4N_4(\xi,\eta),$$

$$y = Q(\xi,\eta) = \sum_{i=1}^{4} y_i N_i(\xi,\eta) = y_1N_1(\xi,\eta) + y_2N_2(\xi,\eta) + y_3N_3(\xi,\eta) + y_4N_4(\xi,\eta).$$

$$\boxed{\iint_K F(x,y)\ \mathrm{d}x\mathrm{d}y = \iint_{R_{\mathrm{st}}} F\left(P(\xi,\eta),Q(\xi,\eta)\right)|J(\xi,\eta)|\ \mathrm{d}\xi\mathrm{d}\eta}$$

$$J(\xi,\eta) = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}$$