INF3580/4580 – Semantic Technologies – Spring 2018

Lecture 5: Mathematical Foundations

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13th February 2018

Department of Informatics

University of Oslo

Mandatory exercises

- Remember: Hand-in Oblig 3 by tomorrow.
- Oblig 4 published after next lecture.


MSc project in Brazil?

Today's Plan

1. Basic Set Algebra
2. Pairs and Relations
3. Propositional Logic

Outline

1 Basic Set Algebra

2 Pairs and Relations

3 Propositional Logic

Motivation

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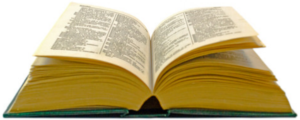

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- RDF has a mathematically defined semantics

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- There are some problems with this, but it's good enough for us!

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$$\{1, \triangle, \triangle\} = \{1, \triangle\}$$

- Sets with different elements are different:

$$\{1, 2\} \neq \{2, 3\}$$

Element of-relation

- We use $\in$ to say that something is element of a set:

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- $x \notin \emptyset$, whatever x is!

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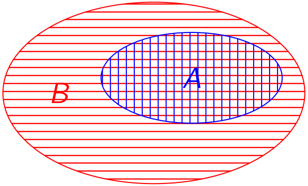

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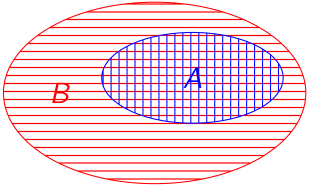

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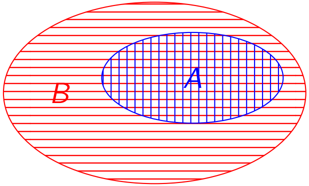

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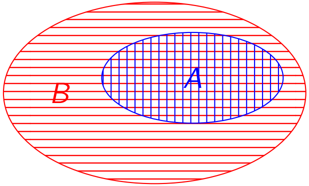

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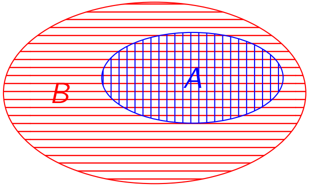

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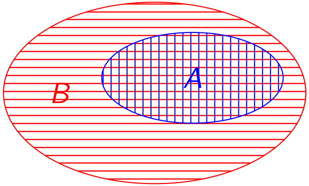

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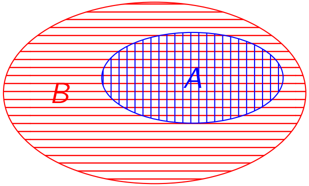

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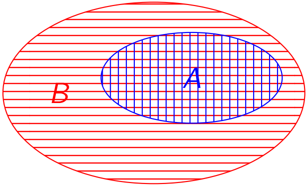

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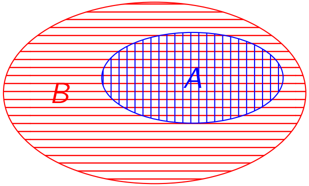

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- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

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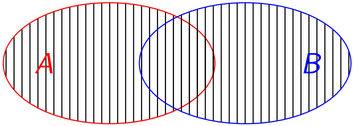

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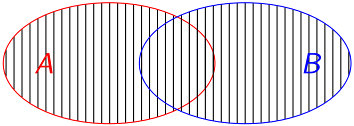

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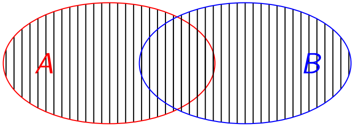

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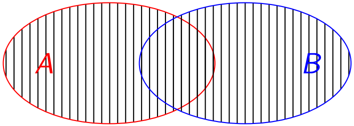

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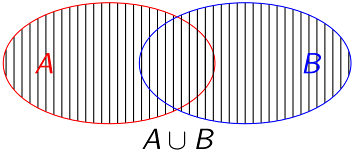

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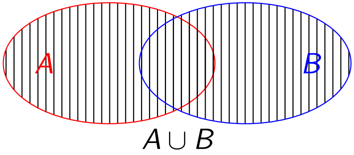

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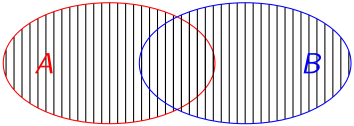

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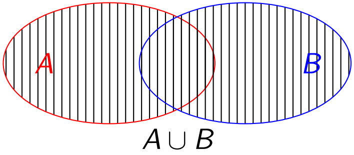

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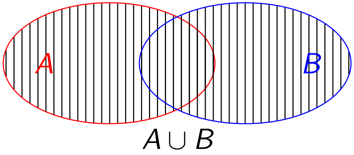

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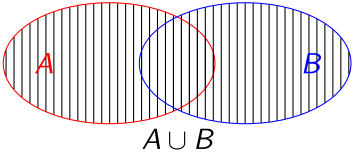

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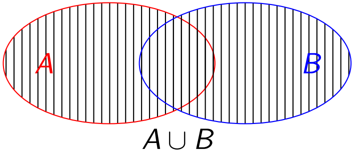

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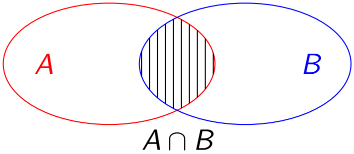

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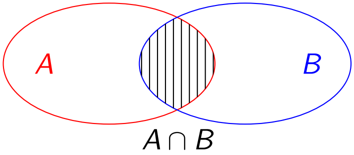

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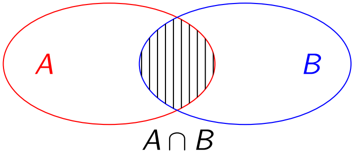

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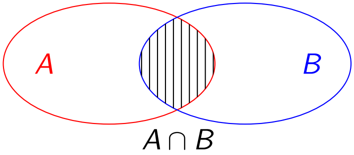

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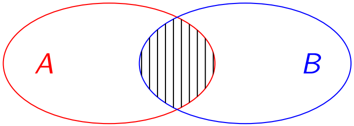

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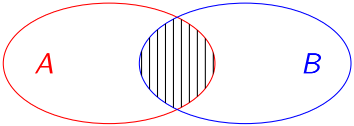

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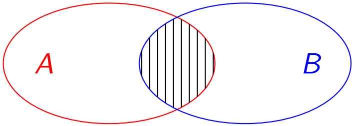

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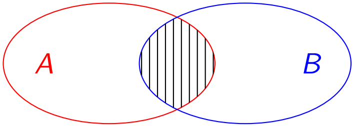

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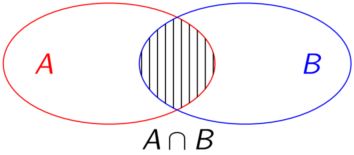

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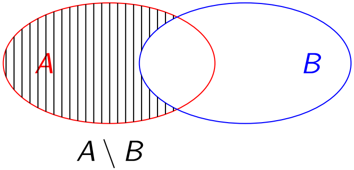

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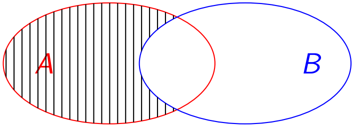

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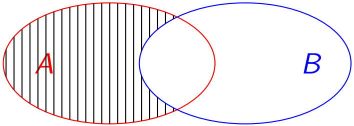

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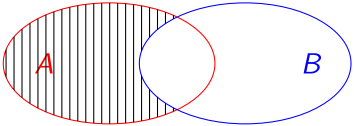

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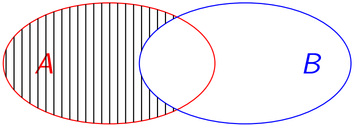

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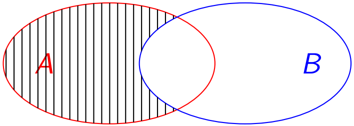

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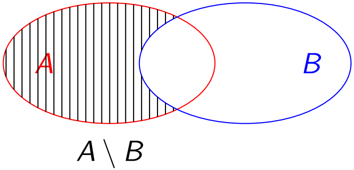

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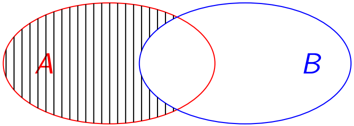

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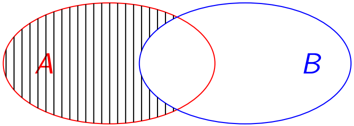

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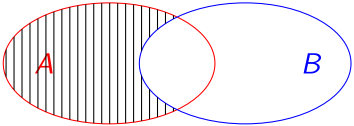

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Or:

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Outline

1. Basic Set Algebra
2. Pairs and Relations
3. Propositional Logic

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- Sets are good to group objects with some properties!
- How do we talk about relations between objects?

Pairs

- A pair is an *ordered* collection of two objects

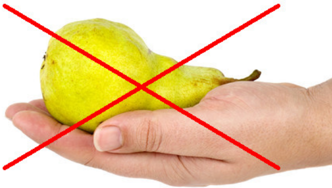

Image ©Colourbox.no

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- $\langle x, y \rangle$ is a pair, no matter if $x = y$ or not.

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- Instead of “$\langle a, b \rangle$ is a pair of a natural number and a person in this room”...

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  - $A \times B = \{ \langle 1, \text{'a'} \rangle, \langle 2, \text{'a'} \rangle, \langle 3, \text{'a'} \rangle, \langle 1, \text{'b'} \rangle, \langle 2, \text{'b'} \rangle, \langle 3, \text{'b'} \rangle \}$
- Why bother?
- Instead of "$\langle a, b \rangle$ is a pair of a natural number and a person in this room"...
- ...$\langle a, b \rangle \in \mathbb{N} \times P_{3580}$

The Cross Product

- Let A and B be sets.
- Construct the set of all pairs $\langle a, b \rangle$ with $a \in A$ and $b \in B$.
- This is called the *cross product* of A and B, written

$$A \times B$$

$\times$

- Example:
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  - $< = \{\langle x, y\rangle \in \mathbb{N}^2 \mid x \text{ is less than } y\}$

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16 relations on A. Generally: $2^{(|A|^2)}$

Outline

1. Basic Set Algebra
2. Pairs and Relations
3. Propositional Logic

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  - talks about sets and relations
- Basic concepts can be explained using predicate logic

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  - but every formula can be "parsed" uniquely.

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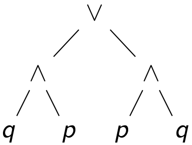

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- Let's formalize this context, a.k.a. interpretation, a.k.a. model

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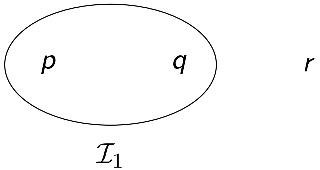

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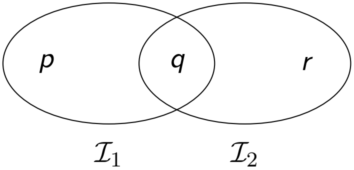

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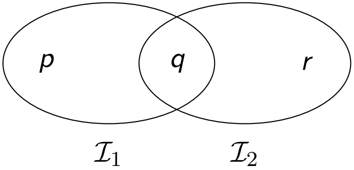

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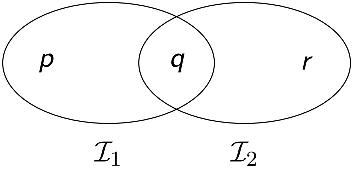

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- In other words, for all letters p:

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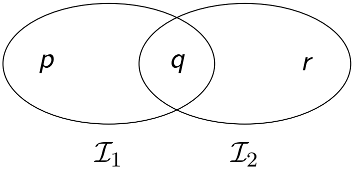

$\mathcal{I}_1 \models p \land q$ $\qquad$ $\mathcal{I}_2 \not\models p \land q$

Validity of Compound Formulas, cont.

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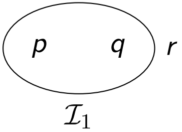
- For instance, if $\mathcal{I}_1 = \{p, q\}$:



$$\mathcal{I}_1 \not\models ((q \wedge r) \wedge (p \wedge q))$$

$$\mathcal{I}_1 \not\models (q \wedge r) \qquad \mathcal{I}_1 \models (p \wedge q)$$

$$\mathcal{I}_1 \models q \qquad \mathcal{I}_1 \not\models r \qquad \mathcal{I}_1 \models p \qquad \mathcal{I}_1 \models q$$

Semantics for $\neg$, $\rightarrow$ and $\vee$

- The complete definition of $\models$ is as follows:
- For any interpretation $\mathcal{I}$, letter p, formulas A, B:
  - $\mathcal{I} \models p$ iff $p \in \mathcal{I}$
  - $\mathcal{I} \models \neg A$ iff $\mathcal{I} \not\models A$
  - $\mathcal{I} \models (A \wedge B)$ iff $\mathcal{I} \models A$ and $\mathcal{I} \models B$
  - $\mathcal{I} \models (A \vee B)$ iff $\mathcal{I} \models A$ or $\mathcal{I} \models B$ (or both)
  - $\mathcal{I} \models (A \rightarrow B)$ iff $\mathcal{I} \models A$ implies $\mathcal{I} \models B$
- Semantics of $\neg$, $\wedge$, $\vee$, $\rightarrow$ often given as *truth table*:

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$
f	f	t	f	f	t
f	t	t	f	t	t
t	f	f	f	t	f
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$$\mathcal{I}_2 \models (p \vee \neg p)$$

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$$\mathcal{I}_2 \models p$$

Some Formulas Are Truer Than Others

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- $(p \lor \neg p)$ is true in *all* interpretations!

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- Therefore: $(\neg p \vee (\neg q \vee (p \wedge q)))$ is a tautology!

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p	q	$\neg p$	$\neg q$	$(p \land \neg q)$	$(\neg p \lor q)$	$\neg(\neg p \lor q)$
f	f	t	t	f	t	f
f	t	t	f	f	t	f
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- So $(p \land \neg q) \models \neg(\neg p \lor q)$

Checking Entailment

- SAT solvers can be used to check entailment:

$$A \models B \qquad \text{if and only if} \qquad \models (A \rightarrow B)$$

- We can check simple cases with a truth table:

$$(p \wedge \neg q) \models \neg(\neg p \vee q) \quad ?$$

p	q	$\neg p$	$\neg q$	$(p \wedge \neg q)$	$(\neg p \vee q)$	$\neg(\neg p \vee q)$
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- So $(p \wedge \neg q) \models \neg(\neg p \vee q)$
- And $\neg(\neg p \vee q) \models (p \wedge \neg q)$

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- Any formula is equivalent to a formula containing only the connectives $\neg$ and $\wedge$.

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