Numerical methods

Roots finding

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Incremental search method

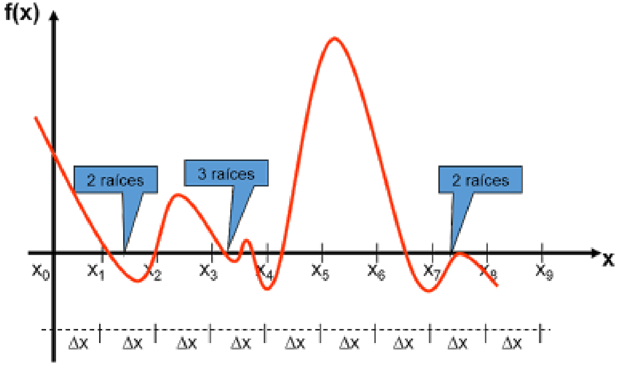


Figure: Incremental search method.

The choice of the increment size can influence the results.

Bisection method

Bisection method is based on the **Intermediate Value Theorem**.

Intermediate Value Theorem

If f is continuous of a closed interval $[a, b]$, and u is any number between $f(a)$ and $f(b)$ inclusive, there is at least one number $c \in [a, b]$ so that $f(c) = u$.

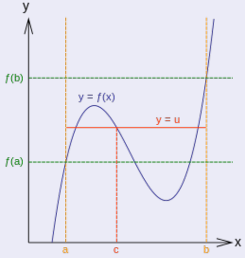


Figure: Intermediate value theorem representation.

See: Intermediate Value Theorem - Khan Academy

Bisection method

To begin, let p be the middle point of the interval $[a, b]$:

$$p_1 = \frac{a+b}{2} \tag{1}$$

If $f(p_1) = 0$, $p = p_0$, and **"That's all folks!"**
If $f(p_1) \neq 0$:
  If $sign(f(p_1)) = sign(f(a))$, $a = p_1$
  If $sign(f(p_1)) = sign(f(b))$, $b = p_1$
Then reapply the process to the new interval $[a, b]$.

Bracketing methods

Bracketing methods are based on two initial guesses that "bracket" the root. If $f(x)$ is a real and continuous in the interval $[x_l, x_u]$ and

$$f(x_l)f(x_u) < 0 \tag{2}$$

then there is at least one real root between x_l and x_u

Bisection method

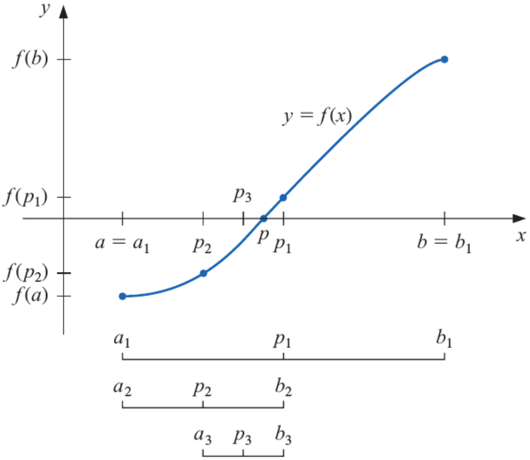


Figure: Representation of bisection method.

Stopping inequalities

Typical stopping inequalities:

$$|p_n - p_{n-1}| < \varepsilon \tag{3}$$

$$\frac{|p_n - p_{n-1}|}{|p_n|} < \varepsilon, \quad p_n \neq 0 \tag{4}$$

$$|f(p_n)| < \varepsilon \tag{5}$$

Fixed-point method

The number p is a **fixed point** for a given function g if $g(p) = p$.

Theorem

If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has at least one fixed point on $[a,b]$.
If, in addition, $g'(x)$ exists on $[a,b]$ and a positive constant $k > 1$ exists with $|g'(x)| < k$, then there is exactly one fixed point in $[a,b]$.

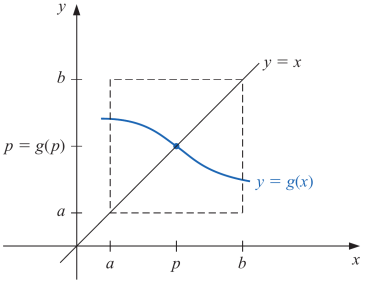


Figure: Fixed point for a function.

Fixed-point method

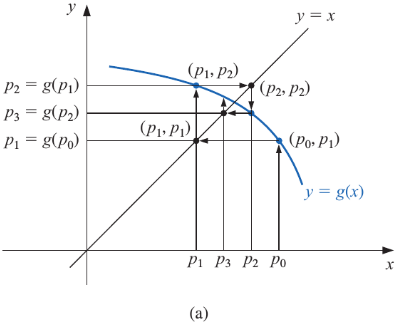


Figure: Fixed point method representation.

Fixed-point method - Algorithm

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

> *Step 3* Set $p = g(p_0)$. (*Compute* p_i.)
>
> *Step 4* If $|p - p_0| < TOL$ then
> OUTPUT (p); (*The procedure was successful.*)
> STOP.
>
> *Step 5* Set $i = i + 1$.
>
> *Step 6* Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Figure: Fixed point algorithm.

Newton Raphson method

Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi) \tag{6}$$

where $\xi \in [p_0, p]$.
Taking a first order approximation:

$$0 \approx f(p_0) + (p - p_0)f'(p_0) \tag{7}$$

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \tag{8}$$

For an iterative process, we have:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad n \geq 1 \tag{9}$$

Newton Raphson

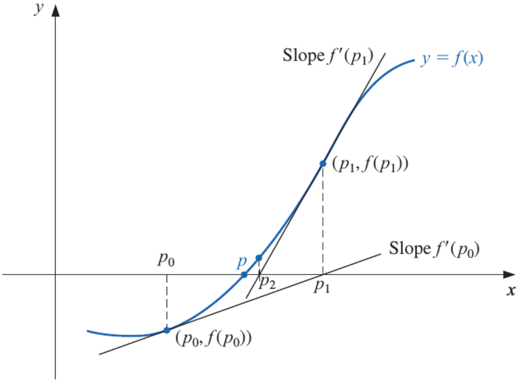


Figure: Representation of Newton Raphson method.

Newton Raphson - Algorithm

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute* p_i.)

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Figure: Newton Raphson algorithm.

Secant method

To circumvent the problem of derivative evaluation at each approximation in Newton's method, the Secant method gives an alternative. By definition:

$$f'(p_{n-1}) = \lim \frac{f(x) - f(p_{n-1})}{x - p_{n-1}} \tag{10}$$

Considering p_{n-2} is close to p_{n-1}, so:

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}} \tag{11}$$

Using the Newton formula with this new derivative approximation, we obtain:

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})} \tag{12}$$

Secant method

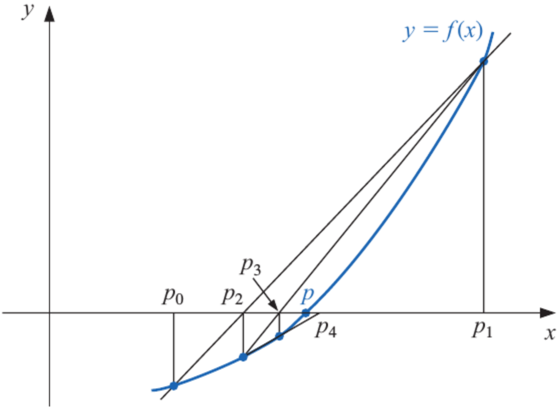


Figure: Secant method representation.

Secant method - Algorithm

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; (*Update* p_0, q_0, p_1, q_1.)
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Figure: Secant method algorithm.

Order of convergence

Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exists with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_n|}{|p_n - p|^{\alpha}} = \lambda \tag{13}$$

then $\{p_n\}_{n=0}^{\infty}$ converges to p of order α, with asymptotic error constant λ.

Newton Raphson (single variable)

Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi) \tag{14}$$

where $\xi \in [p_0, p]$.
Taking a first order approximation:

$$0 \approx f(p_0) + (p - p_0)f'(p_0) \tag{15}$$

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \tag{16}$$

For an iterative process, we have:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad n \geq 1 \tag{17}$$

Newton Raphson (single variable)

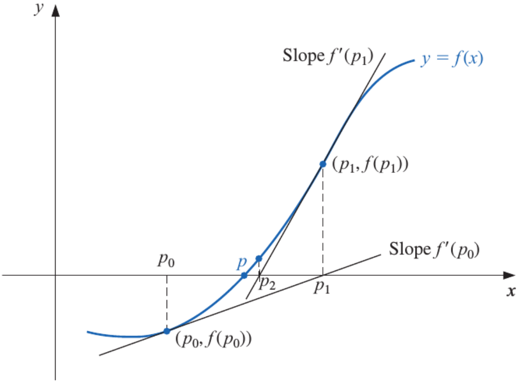


Figure: Representation of Newton Raphson method.

Newton-Raphson method (multivariate)

The method described for 1D functions can be generalized for a system of non-linear equations:

$$
\begin{aligned}
f_1(\mathbf{x}) &= 0 \\
f_2(\mathbf{x}) &= 0 \\
&\dots \\
f_N(\mathbf{x}) &= 0
\end{aligned}
$$

where

$$\mathbf{x} = [x_1 \quad x_2 \quad \cdots x_N]^T \tag{18}$$

Defining a function vector:

$$\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}) \quad f_2(\mathbf{x}) \quad \cdots \quad f_N(\mathbf{x})] \tag{19}$$

The system can be rewritten as:

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{20}$$

Newton-Raphson method (multivariate)

Considering N = 2 (2D problem), the multidimensional equation can be geometrically interpreted as:

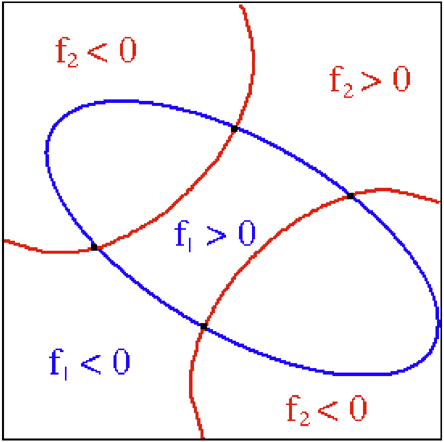


Figure: Visualization of the root finding problem in 2D.

Newton-Raphson method (multivariate)

The Taylor expansion for each function f_i can be written as:

$$f_i(\mathbf{x}+\delta\mathbf{x}) = f_i(\mathbf{x})+\sum_{j=1}^{N}\frac{\partial f_i(x_j)}{\partial x_j}+O(\delta\mathbf{x}^2) \approx f_i(\mathbf{x})+\sum_{j=1}^{N}\frac{\partial f_i(x_j)}{\partial x_j}\,\delta\mathbf{x} \quad (21)$$

In the vector form, the above equation can be written as:

$$\mathbf{f}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{f}(\mathbf{x})+\mathbf{J}(\mathbf{x})\,\delta\mathbf{x} \quad (22)$$

where $\mathbf{J}(\mathbf{x})$ is the *Jacobian matrix*, which is defined as:

$$\mathbf{J}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix} \quad (23)$$

Newton-Raphson method (multivariate)

Assuming $\mathbf{f}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{0}$, the roots are $\mathbf{x}+\delta\mathbf{x}$, where $\delta\mathbf{x}$ can be obtained from:

$$\mathbf{f}(\mathbf{x}+\delta\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\,\delta\mathbf{x} \implies \tag{24}$$

$$\delta\mathbf{x} = \mathbf{J}(\mathbf{x})^{-1}[\mathbf{f}(\mathbf{x}+\delta\mathbf{x}) - \mathbf{f}(\mathbf{x})] = -\mathbf{J}(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x}) \tag{25}$$

And, from an starting point $\mathbf{x}$:

$$\mathbf{x}+\delta\mathbf{x} = \mathbf{x} - \mathbf{J}(\mathbf{x})^{-1}\mathbf{f}(\mathbf{x}) \tag{26}$$

For nonlinear equations, the result above is only an approximation:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \delta\mathbf{x}_k = \mathbf{x}_k - \mathbf{J}(\mathbf{x}_k)^{-1}\mathbf{f}(\mathbf{x}_k) \tag{27}$$

Newton-Raphson method (multivariate)

Example 1

Computes the roots of:

$$\begin{cases} x_1^2 - 2x_1 + x_2 + 7 = 0 \\ 3x_1 - x_2 + 1 = 0 \end{cases} \tag{28}$$

Starting point: $\mathbf{x} = [1.00 \quad 1.00]^T$

Newton-Raphson method (multivariate)

Example 2

Determine the points of intersection between the circle $x^2 + y^2 = 3$ and the hyperbola $xy = 1$
Starting point: $x = 0.5; y = 1.5$

Solution: $\pm(0.618, 1.618)$ and $\pm(1.618, 0.618)$

Newton-Raphson method (multivariate)

Example 3

Computes the roots of:

$$\begin{cases} 3x_1 - \cos(x_2 x_3) - 3/2 = 0 \\ 4x_1^2 - 625x_2^2 + 2x_3 - 1 = 0 \\ 20x_3 + \exp(-x_1\, x_2) + 9 \end{cases} \tag{29}$$

Starting point: $\mathbf{x} = [1.00 \quad 1.00 \quad 1.00]^T$

Solution: $x = [0.833282 \quad 0.035335 \quad -0.498549]^T$

Exercises

Exercise 1

■ The natural frequencies of a uniform cantilever beam are related to the roots β_i of the frequency equation $f(\beta) = \cosh\beta\cos\beta + 1 = 0$, where

$$\beta_i^4 = (2\pi f_i)^2 \frac{mL^3}{EI}$$

$$f_i = i\text{th natural frequency (cps)}$$

$$m = \text{mass of the beam}$$

$$L = \text{length of the beam}$$

$$E = \text{modulus of elasticity}$$

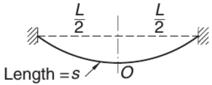
$$I = \text{moment of inertia of the cross section}$$

Determine the lowest two frequencies of a steel beam 0.9 m. long, with a rectangular cross section 25 mm wide and 2.5 mm in. high. The mass density of steel is 7850 kg/m^3 and $E = 200$ GPa.

Exercises

Exercise 2

■



A steel cable of length s is suspended as shown in the figure. The maximum tensile stress in the cable, which occurs at the supports, is

$$\sigma_{max} = \sigma_0 \cosh \beta$$

where

$$\beta = \frac{\gamma L}{2\sigma_0}$$

σ_0 = tensile stress in the cable at O

γ = weight of the cable per unit volume

L = horizontal span of the cable

The length to span ratio of the cable is related to β by

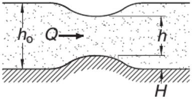
$$\frac{s}{L} = \frac{1}{\beta} \sinh \beta$$

Find σ_{max} if $\gamma = 77 \times 10^3$ N/m^3 (steel), $L = 1000$ m and $s = 1100$ m.

Exercises

Exercise 3

■



Bernoulli's equation for fluid flow in an open channel with a small bump is

$$\frac{Q^2}{2gb^2h_0^2} + h_0 = \frac{Q^2}{2gb^2h^2} + h + H$$

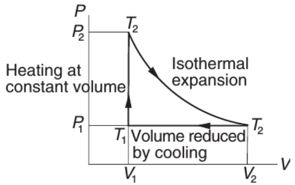
where

$$\begin{aligned}
Q &= 1.2\ \mathrm{m^3/s} = \text{volume rate of flow} \\
g &= 9.81\ \mathrm{m/s^2} = \text{gravitational acceleration} \\
b &= 1.8\ \mathrm{m} = \text{width of channel} \\
h_0 &= 0.6\ \mathrm{m} = \text{upstream water level} \\
H &= 0.075\ \mathrm{m} = \text{height of bump} \\
h &= \text{ water level above the bump}
\end{aligned}$$

Determine h.

Exercises

Exercise 4

- 

The figure shows the thermodynamic cycle of an engine. The efficiency of this engine for monoatomic gas is

$$\eta = \frac{\ln(T_2/T_1) - (1 - T_1/T_2)}{\ln(T_2/T_1) + (1 - T_1/T_2)/(\gamma - 1)}$$

where T is the absolute temperature and $\gamma = 5/3$. Find T_2/T_1 that results in 30% efficiency ($\eta = 0.3$).

Exercises

Exercise 5

- The equations

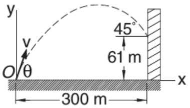
$$\sin x + 3\cos x - 2 = 0$$

$$\cos x - \sin y + 0.2 = 0$$

have a solution in the vicinity of the point $(1, 1)$. Use the Newton–Raphson method to refine the solution.

Exercises

Exercise 6

- 

A projectile is launched at O with the velocity v at the angle θ to the horizontal. The parametric equations of the trajectory are

$$x = (v\cos\theta)t$$

$$y = -\frac{1}{2}gt^2 + (v\sin\theta)t$$

where t is the time measured from the instant of launch, and $g = 9.81\ \mathrm{m/s^2}$ represents the gravitational acceleration. If the projectile is to hit the target at the 45° angle shown in the figure, determine v, θ and the time of flight.

Exercises

Exercise 7

- The equation of a circle is

$$(x-a)^2 + (y-b)^2 = R^2$$

where R is the radius and (a, b) are the coordinates of the center. If the coordinates of three points on the circle are

x	8.21	0.34	5.96
y	0.00	6.62	−1.12

determine R, a and b.