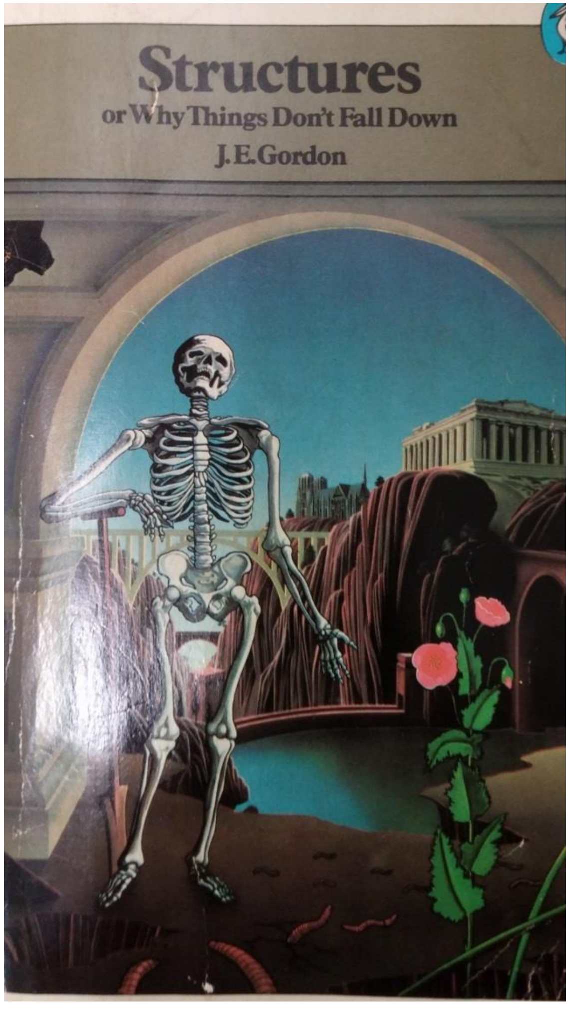
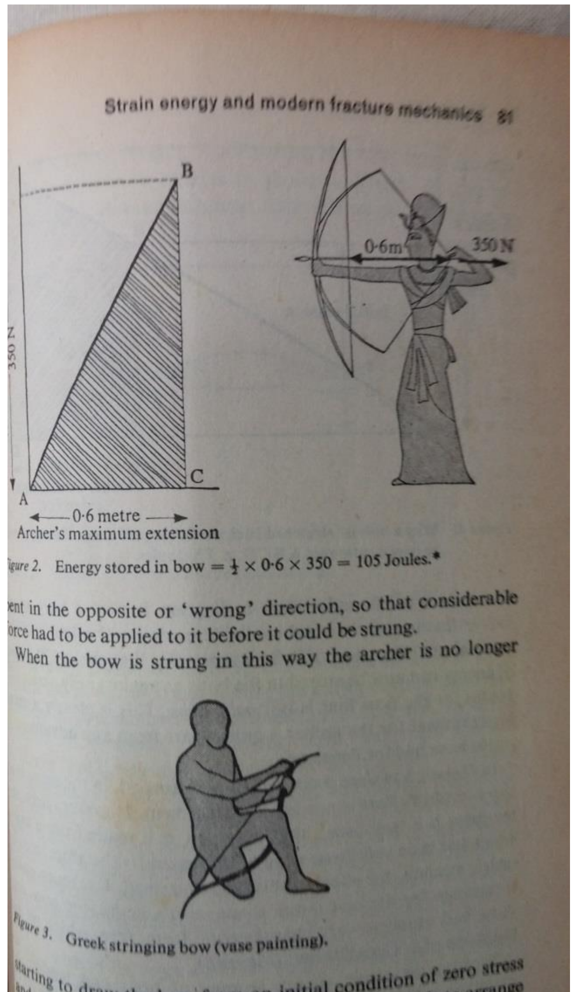
Teoremas de Energia

Energia de deformação



Energia de deformação



igure 2. Energy stored in bow = $\frac{1}{2} \times 0{\cdot}6 \times 350 = 105$ Joules.*

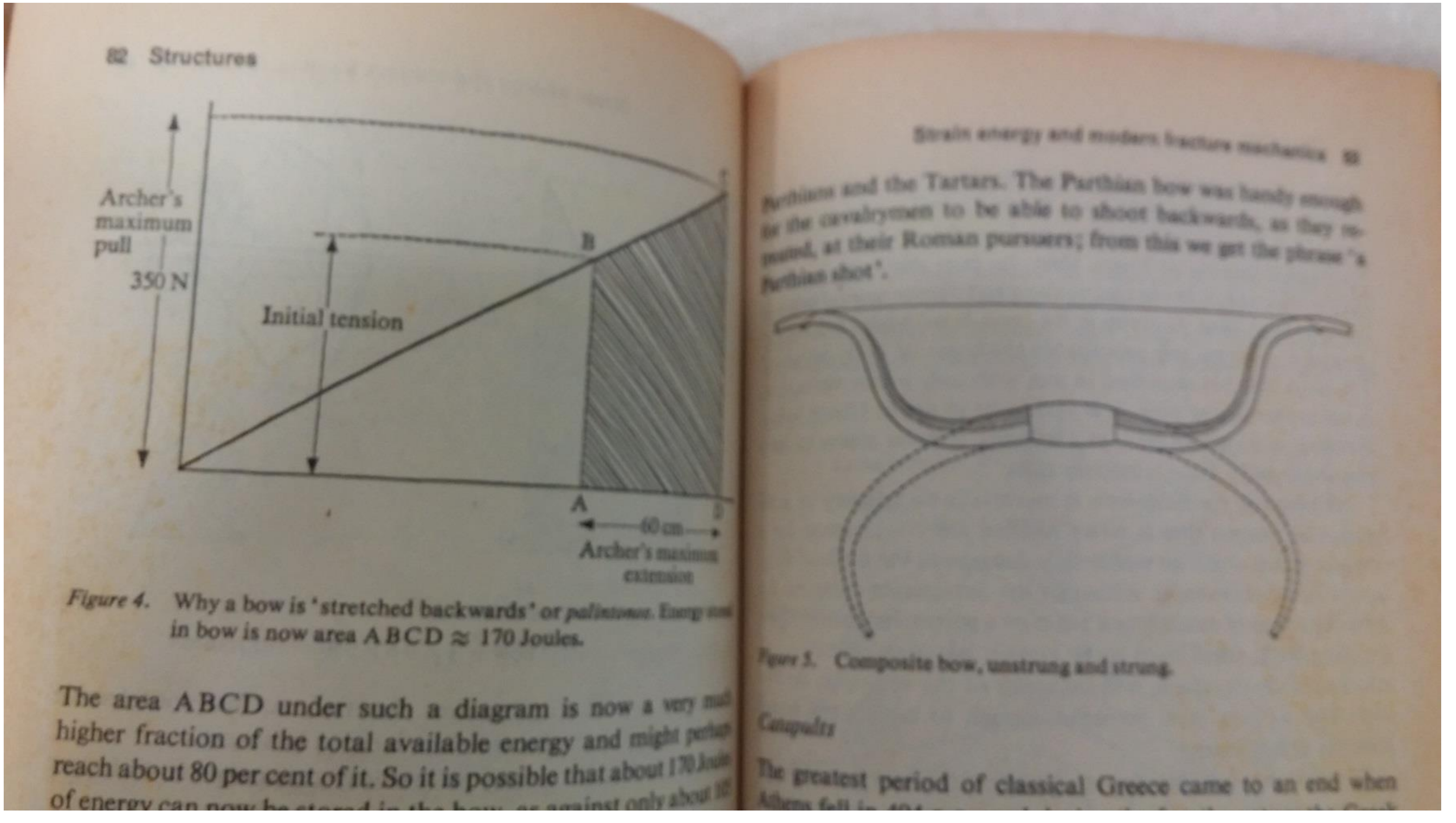
ent in the opposite or 'wrong' direction, so that considerable orce had to be applied to it before it could be strung.

When the bow is strung in this way the archer is no longer

igure 3. Greek stringing bow (vase painting).

tarting to draw the bow from an initial condition of zero stress
nd st ... ssible to arrange

Energia de deformação



82 Structures

Archer's maximum pull

350 N

Initial tension

B

A

D

60 cm

Archer's maximum extension

Figure 4. Why a bow is 'stretched backwards' or *palintonos*. Energy sto[...] in bow is now area ABCD ≈ 170 Joules.

The area ABCD under such a diagram is now a very mu[...] higher fraction of the total available energy and might perha[...] reach about 80 per cent of it. So it is possible that about 170 Joul[...] of energy can now be stored in the bow, as against only about 10[...]

Strain energy and modern fracture mechanics 83

Parthians and the Tartars. The Parthian bow was handy enough for the cavalrymen to be able to shoot backwards, as they retreated, at their Roman pursuers; from this we get the phrase 'a Parthian shot'.

Figure 5. Composite bow, unstrung and strung.

Catapults

The greatest period of classical Greece came to an end when Athens fell in 404 [...]

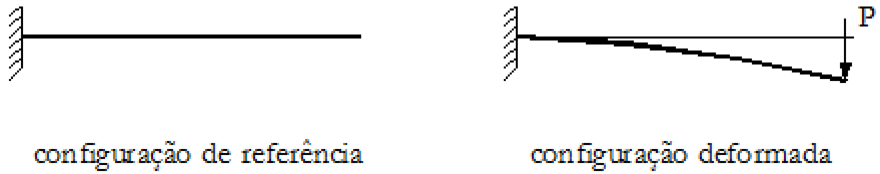
Energia de deformação

TABLE 3

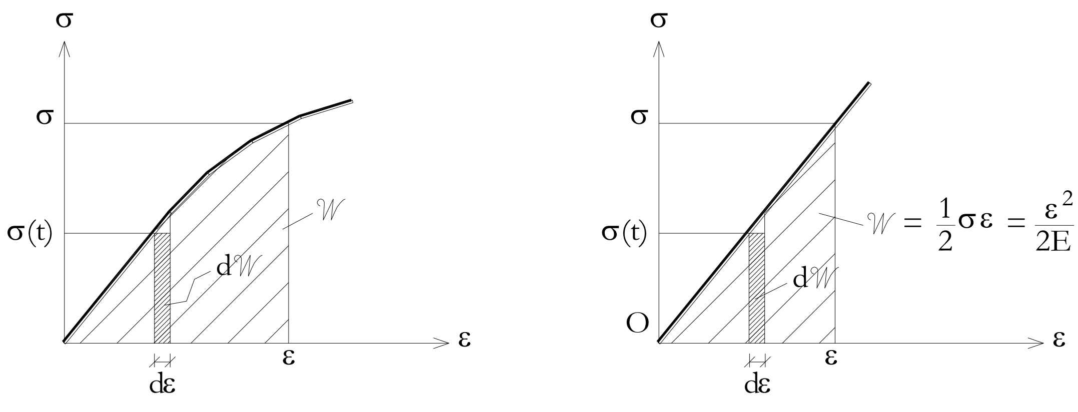
Approximate strain energy storage capacities of various solids

Material	Working strain %	Working stress p.s.i.	Working stress MN/m²	Strain energy stored Joules × 10⁶ per cubic metre	Density kilograms per cubic metre	Energy stored Joules per kilogram
Ancient iron	0·03	10,000	70	0·01	7,800	1·3
Modern spring steel	0·3	100,000	700	1·0	7,800	130
Bronze	0·3	60,000	400	0·6	8,700	70
Yew wood	0·9	18,000	120	0·5	600	900
Tendon	8·0	10,000	70	2·8	1,100	2,500
Horn	4·0	13,000	90	1·8	1,200	1,500
Rubber	300	1,000	7	10·0	1,200	8,000

Energia de deformação



$$\mathcal{U} = T_{\text{int}} \qquad \mathcal{U} = \int_V \mathcal{W}\, dV$$



Energia de deformação

Teoria elementar de barra plana

$$\mathcal{U} = \frac{1}{2}\int_V \sigma\varepsilon dV = \frac{1}{2}\int_V E\varepsilon^2 dV = \frac{1}{2}\int_V E\left(u' - zw''\right)^2 dV = \frac{1}{2}\int_{est}\int_A E\left(u' - zw''\right)^2 dA dx$$

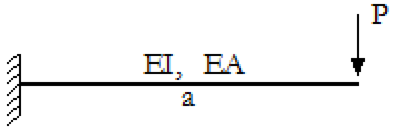
$$\mathcal{U} = \frac{1}{2}\int_{est} EI_y\left(w''\right)^2 ds + \frac{1}{2}\int_{est} EA\left(u'\right)^2 ds = \int_{est} \frac{M_y{}^2}{2EI_y} ds + \int_{est} \frac{N^2}{2EA} ds$$

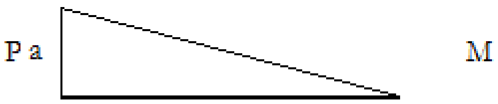
Teoria elementar de barra espacial

$$\mathcal{U} = \frac{1}{2}\int_{est} EI_y\left(w''\right)^2 ds + \frac{1}{2}\int_{est} EI_z\left(v''\right)^2 ds + \frac{1}{2}\int_{est} EA\left(u'\right)^2 ds + \frac{1}{2}\int_{est} GI_T\left(\theta'\right)^2 ds$$

$$\mathcal{U} = \int_{est} \frac{M_y{}^2}{2EI_y} ds + \int_{est} \frac{M_z{}^2}{2EI_z} ds + \int_{est} \frac{N^2}{2EA} ds + \int_{est} \frac{T^2}{2GI_T} ds$$

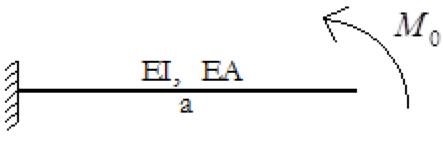
Exemplo 1 - Cálculo de energia de deformação

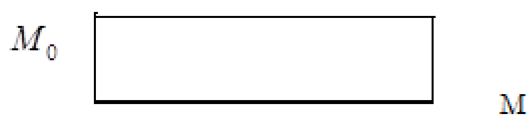




$$\mathcal{U} = \int_{est} \frac{{M_y}^2}{2EI_y} ds = \frac{P^2 a^3}{3EI}$$

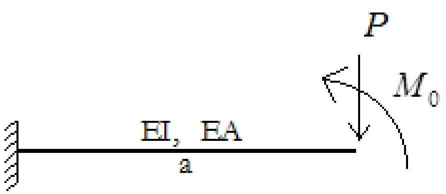
Exemplo 1 - Cálculo de energia de deformação

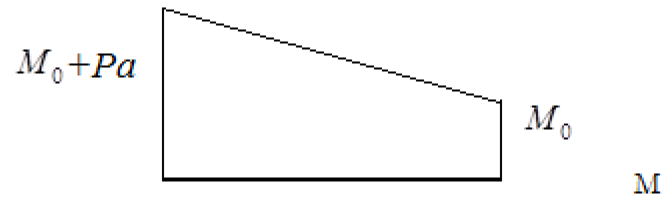




$$\mathcal{U} = \int_{est} \frac{M_y^{\,2}}{2EI_y} ds = \frac{M_0^{\,2} a}{2EI}$$

Exemplo 1 - Cálculo de energia de deformação

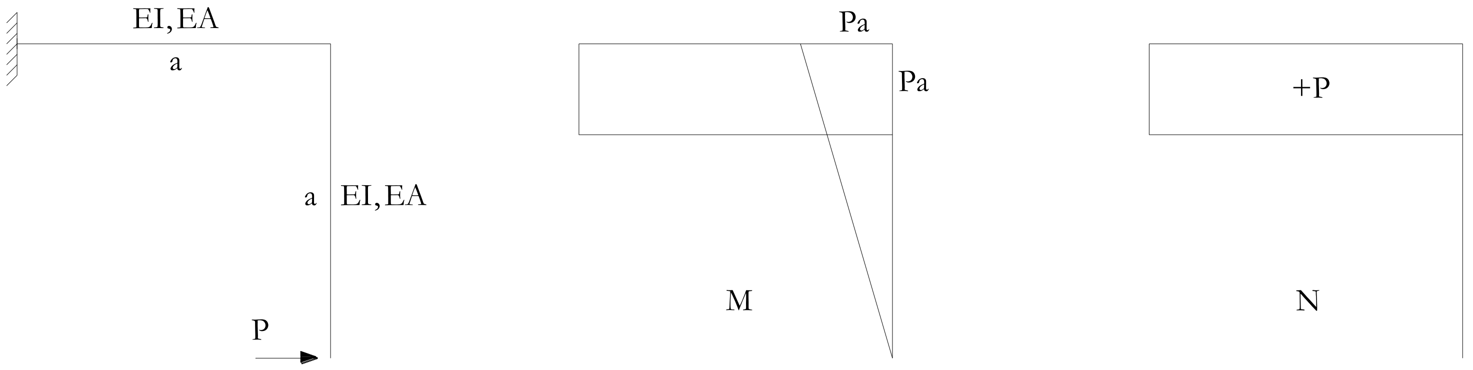




$$\mathcal{U} = \int_{est} \frac{M_y^{\,2}}{2EI_y} ds = \frac{P^2 a^3}{6EI} + \frac{M_0^{\,2} a}{2EI} + \underbrace{\frac{M_0 P a^2}{2EI}}_{\text{trabalho cruzado}}$$

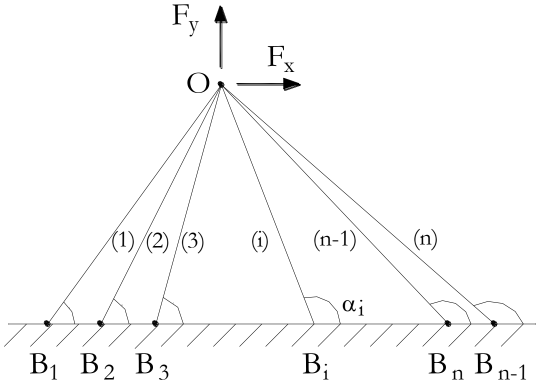
Cuidado! Em geral não vale superposição de efeitos!

Exemplo 2 - Cálculo de energia de deformação



$$\mathcal{U} = \int_{est} \frac{M_y^{\,2}}{2EI_y} ds + \int_{est} \frac{N^2}{2EA} ds = \frac{2P^2a^3}{3EI} + \frac{P^2 a}{2EA} = \frac{2P^2a^3}{3EI}\left(1 + \frac{h^2}{16a^2}\right)$$

Exemplo 3 - Cálculo de energia de deformação



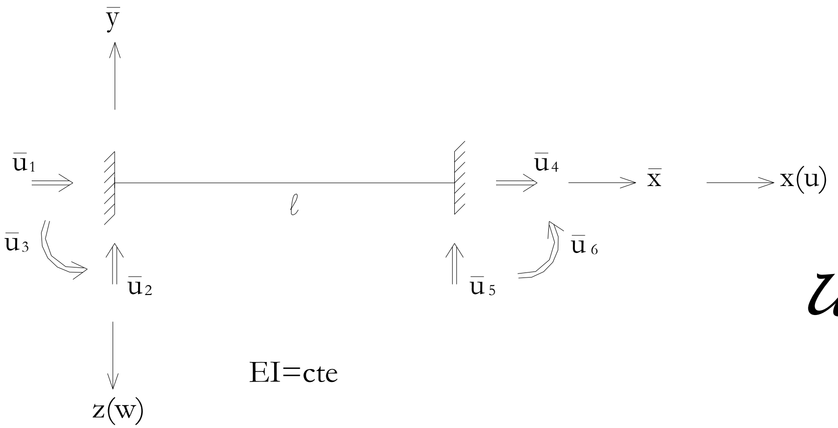
$$\mathcal{U} = \int_{est} \frac{N^2}{2EA} ds = \sum_{i=1}^{n} \int_0^{\ell_i} \frac{N^2}{2EA} ds = \sum_{i=1}^{n} \frac{N_i^{\,2} \ell_i}{2EA_i}$$

$$N_i = \frac{EA_i}{\ell_i} \left(U_1 \cos\alpha_\mathrm{i} + U_2 \mathrm{sen}\,\alpha_\mathrm{i} \right)$$

$$\mathcal{U} = \frac{1}{2} \left(K_{11} U_1^2 + K_{12} U_1 U_2 + K_{21} U_2 U_1 + K_{22} U_2^2 \right) = \frac{1}{2} \{U\}^T [K] \{U\}$$

$$[K] = \begin{bmatrix} \displaystyle\sum_{i=1}^{n} \frac{EA_i}{\ell_i} \cos^2\alpha_i & \displaystyle\sum_{i=1}^{n} \frac{EA_i}{\ell_i} \cos\alpha_i \sin\alpha_i \\ \displaystyle\sum_{i=1}^{n} \frac{EA_i}{\ell_i} \cos\alpha_i \sin\alpha_i & \displaystyle\sum_{i=1}^{n} \frac{EA_i}{\ell_i} \mathrm{sen}^2\alpha_i \end{bmatrix}$$

Exemplo 4 - Cálculo de energia de deformação



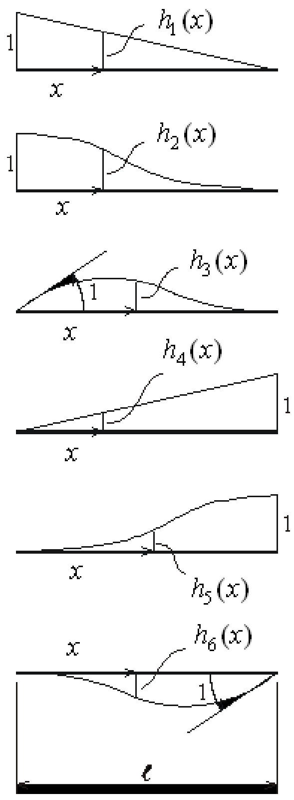
$$\mathcal{U} = \frac{1}{2}\int_{est} EI_y \left(w''\right)^2 ds + \frac{1}{2}\int_{est} EA\left(u'\right)^2 ds$$

$$w = h_2(x)\bar{u}_2 + h_3(x)\bar{u}_3 + h_5(x)\bar{u}_5 + h_6(x)\bar{u}_6$$
$$u = h_1(x)\bar{u}_1 + h_4(x)\bar{u}_4$$

$$\mathcal{U}_M = \frac{1}{2}\int_0^\ell EI_y \left(w''\right)^2 ds = \frac{EI_y}{2}\left(\sum_{i=2,3,5,6}\sum_{j=2,3,5,6}\left(\int_0^\ell h_i'' h_j'' dx\right)\bar{u}_i \bar{u}_j\right)$$

$$\mathcal{U}_N = \frac{1}{2}\int_0^\ell EA\left(u'\right)^2 ds = \frac{EA}{2}\left(\sum_{i=1,4}\sum_{j=1,4}\left(\int_0^\ell h_i' h_j' dx\right)\bar{u}_i \bar{u}_j\right)$$

Exemplo 4 - Cálculo de energia de deformação



$h_1(x) = 1 - \frac{x}{\ell}$ $\qquad \bar{u}(x) = h_1(x)$ *para* $\bar{U}_1 = 1$

$h_2(x) = 1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}$ $\qquad \bar{w}(x) = h_2(x)$ *para* $\bar{U}_2 = 1$

$h_3(x) = x - 2\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$ $\qquad \bar{w}(x) = h_3(x)$ *para* $\bar{U}_3 = 1$

$h_4(x) = \frac{x}{\ell}$ $\qquad \bar{u}(x) = h_4(x)$ *para* $\bar{U}_4 = 1$

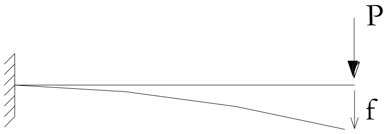
$h_5(x) = 3\frac{x^2}{\ell^2} - 2\frac{x^3}{\ell^3}$ $\qquad \bar{w}(x) = h_5(x)$ *para* $\bar{U}_5 = 1$

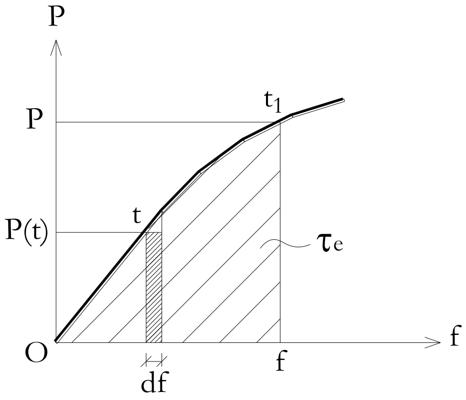
$h_3(x) = -\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$ $\qquad \bar{w}(x) = h_6(x)$ *para* $\bar{U}_6 = 1$

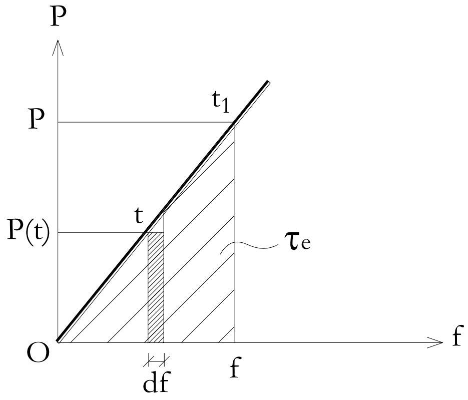
Exemplo 4 - Cálculo de energia de deformação

$$\mathcal{U} = \frac{1}{2}\{\bar{u}\}^T\left[\bar{k}\right]\{\bar{u}\} \text{ com } \left[\bar{k}\right] = \begin{bmatrix} \frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\ 0 & \frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} & 0 & -\frac{12EI}{\ell^3} & \frac{6EI}{\ell^2} \\ 0 & \frac{6EI}{\ell^2} & \frac{4EI}{\ell} & 0 & -\frac{6EI}{\ell^2} & \frac{2EI}{\ell} \\ -\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 \\ 0 & -\frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} & 0 & \frac{12EI}{\ell^3} & -\frac{6EI}{\ell^2} \\ 0 & -\frac{6EI}{\ell^2} & \frac{2EI}{\ell} & 0 & -\frac{6EI}{\ell^2} & \frac{4EI}{\ell} \end{bmatrix}$$

Teorema de Clapeyron







Para comportamento linear: $$T_e = \frac{1}{2} P f$$

Generalizando: $$T_e = \frac{1}{2} \sum_{i=1}^{n} R_i U_i = \frac{1}{2} \{U\}^T \{R\}$$

Teorema do Trabalho

Particularizando para teoria de barras:

$$T_{int} = T_{ext}$$

$$T_{int} = \mathcal{U} = \frac{1}{2}\{U\}^T [K]\{U\}$$

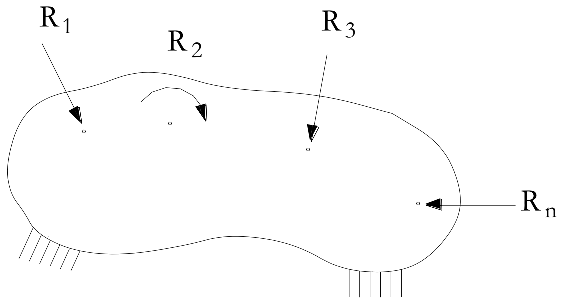
$$T_e = \frac{1}{2}\sum_{i=1}^{n} R_i U_i = \frac{1}{2}\{U\}^T \{R\}$$

$$\{U\}^T \left([K]\{U\} - \{R\}\right) = \{0\} \xrightarrow[\{U\}\neq\{0\}]{} [K]\{U\} = \{R\}$$

Energia potencial total

$$\Pi = \mathcal{U}(U_i) + \mathcal{P}(U_i) = \mathcal{U}(U_i) - \sum_{i=1}^{n} R_i U_i$$

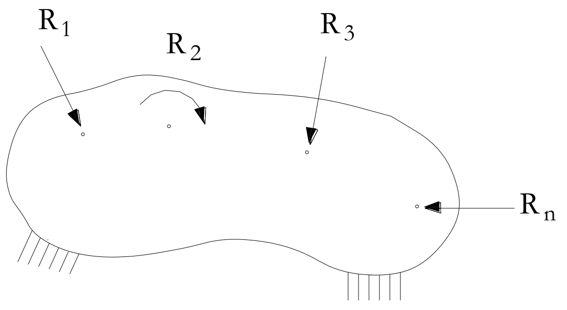
Teorema: “entre todas as soluções compatíveis, aquela que também é equilibrada torna estacionária a energia potencial total”



$$\frac{\partial \Pi}{\partial U_k} = \frac{\partial \mathcal{U}}{\partial U_k} - R_k = 0$$

Primeiro teorema de Castigliano

(por Alberto Castigliano em 1873)



$$\frac{\partial \mathcal{U}}{\partial U_k} = R_k$$

Teorema: "Se a energia de deformação for expressa em função de n deslocamentos independentes entre si, então a sua derivada parcial em relação a um desses deslocamentos é igual ao esforço conjugado"

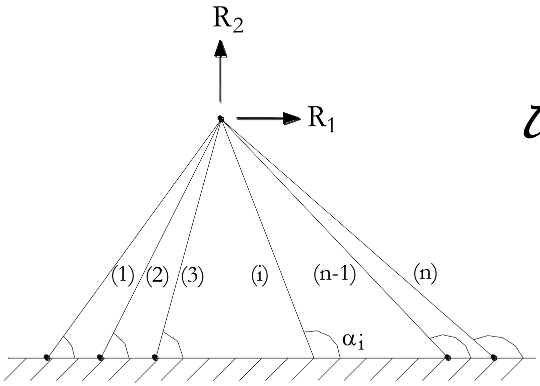
Aplicações do primeiro teorema de Castigliano

- Estabelecer equações de equilíbrio

- análise de estruturas pelo método dos deslocamentos que permite escolher entre todas as soluções compatíveis aquela também equilibrada

- Nas duas aplicações é importante observar que os deslocamentos devem se independentes

Exemplo: Aplicação do primeiro teorema de Castigliano



$$\mathcal{U} = \frac{1}{2}\left(K_{11}U_1^2 + K_{12}U_1U_2 + K_{21}U_2U_1 + K_{22}U_2^2\right) = \frac{1}{2}\{U\}^T[K]\{U\}$$

$$\mathcal{P} = -R_1U_1 - R_2U_2$$

$$\mathbf{\Pi} = \frac{1}{2}\left(K_{11}U_1^2 + K_{12}U_1U_2 + K_{21}U_2U_1 + K_{22}U_2^2\right) - R_1U_1 - R_2U_2$$

$$\frac{\partial \mathcal{U}}{\partial U_1} = R_1 \quad ou \quad \frac{\partial \Pi}{\partial U_1} = 0 \quad \Rightarrow \quad K_{11}U_1 + K_{12}U_2 = R_1$$

$$\frac{\partial \mathcal{U}}{\partial U_2} = R_2 \quad ou \quad \frac{\partial \Pi}{\partial U_2} = 0 \quad \Rightarrow \quad K_{21}U_1 + K_{22}U_2 = R_2$$

$$\Rightarrow [K]\{U\} = \{R\}$$

Método dos deslocamentos

- Formulação direta das equações de equilíbrio, que envolve grandezas vetoriais - forças e deslocamentos
- Formulação baseada no teorema dos deslocamentos virtuais que envolve grandeza escalar - trabalho
- Formulação variacional baseada no teorema da energia potencial total ou no primeiro teorema de Castigliano que envolve grandeza escalar - energia