14.5 SOLUÇÕES

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1. $z = x^2 + y^2, x = t^3, y = 1 + t^2 \quad \Rightarrow$

$$\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = \left(2t^3\right)\left(3t^2\right) + 2\left(1+t^2\right)(2t) = 6t^5 + 4t^3 + 4t$$

2. $z = x^2y^3, x = 1 + \sqrt{t}, y = 1 - \sqrt{t} \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= 2xy^3\frac{dx}{dt} + 3x^2y^2\frac{dy}{dt} \\ &= 2xy^3\frac{1}{2\sqrt{t}} + 3x^2y^2\left(-\frac{1}{2\sqrt{t}}\right) \\ &= 2\left(1+\sqrt{t}\right)\left(1-\sqrt{t}\right)^3\frac{1}{2\sqrt{t}} \\ &\quad + 3\left(1+\sqrt{t}\right)^2\left(1-\sqrt{t}\right)^2\left(-\frac{1}{2\sqrt{t}}\right) \\ &= \frac{(1-t)\left(1-\sqrt{t}\right)^2}{\sqrt{t}} - \frac{3(1-t)^2}{2\sqrt{t}}\end{aligned}$$

3. $z = \ln\left(x + y^2\right), x = \sqrt{1+t}, y = 1 + \sqrt{t} \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{\left(x+y^2\right)}\frac{1}{2\sqrt{1+t}} + \frac{1}{\left(x+y^2\right)}2y\frac{1}{2\sqrt{t}} \\ &= \frac{1}{\sqrt{1+t}+1+\sqrt{t}}\left(\frac{1}{2\sqrt{1+t}} + \frac{1+\sqrt{t}}{\sqrt{t}}\right)\end{aligned}$$

4. $z = xe^{x/y}, x = \cos t, y = e^{2t} \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= e^{x/y}\left(1 + \frac{x}{y}\right)(-\operatorname{sen} t) - x^2y^{-2}e^{x/y}\left(2e^{2t}\right) \\ &= -e^{\cos t/e^{2t}}\left[\left(1 + \frac{\cos t}{e^{2t}}\right)\operatorname{sen} t - \frac{2e^{2t}\cos^2 t}{e^{4t}}\right]\end{aligned}$$

5. $z = 6x^3 - 3xy + 2y^2, x = e^t, y = \cos t \quad \Rightarrow$

$$\begin{aligned}\frac{dz}{dt} &= \left(18x^2 - 3y\right)e^t + (-3x + 4y)(-\operatorname{sen} t) \\ &= \left(18e^{2t} - 3\cos t\right)e^t + \left(3e^t - 4\cos t\right)\operatorname{sen} t\end{aligned}$$

6. $z = x\sqrt{1-y^2}, x = te^{2t}, y = e^{-t} \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial t} &= \sqrt{1+y^2}\left(e^{2t} + 2te^{2t}\right) \\ &\quad + \tfrac{1}{2}x\left(1+y^2\right)^{-1/2}(2y)\left(-e^{-t}\right) \\ &= e^{2t}\sqrt{1+e^{-2t}}\,(1+2t) - t\sqrt{1+e^{-2t}}\end{aligned}$$

7. $w = xy^2z^3, x = \operatorname{sen} t, y = \cos t, z = 1 + e^{2t} \quad \Rightarrow$

$$\frac{dw}{dt} = y^2z^3(\cos t) + 2xyz^3(-\operatorname{sen} t) + 3xy^2z^2\left(2e^{2t}\right)$$

8. $w = \dfrac{x}{y} + \dfrac{y}{z}, x = \sqrt{t}, y = \cos 2t, z = e^{-3t} \quad \Rightarrow$

$$\begin{aligned}\frac{dw}{dt} &= \frac{1}{y}\frac{1}{2\sqrt{t}} + \left(\frac{-x}{y^2} + \frac{1}{z}\right)(-2\operatorname{sen} 2t) + \frac{-y}{z^2}\left(-3e^{-3t}\right) \\ &= \frac{1}{2y\sqrt{t}} + 2(\operatorname{sen} 2t)\left(\frac{x}{y^2} - \frac{1}{z}\right) + \frac{3y}{z^2e^{3t}}\end{aligned}$$

9. $z = x^2\operatorname{sen} y, x = s^2 + t^2, y = 2st \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= (2x\operatorname{sen} y)(2s) + \left(x^2\cos y\right)(2t) \\ &= 4sx\operatorname{sen} y + 2tx^2\cos y \\ \frac{\partial z}{\partial t} &= (2x\operatorname{sen} y)(2t) + \left(x^2\cos y\right)(2s) \\ &= 4xt\operatorname{sen} y + 2sx^2\cos y\end{aligned}$$

10. $z = \operatorname{sen} x\cos y, x = (s-t)^2, y = s^2 - t^2 \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= (\cos x\cos y)\,2(s-t) - (\operatorname{sen} x\operatorname{sen} y)(2s) \\ &= 2(s-t)\cos x\cos y - (2s)\operatorname{sen} x\operatorname{sen} y \\ \frac{\partial z}{\partial t} &= (\cos x\cos y)(-2)(s-t) - (\operatorname{sen} x\operatorname{sen} y)(-2t) \\ &= 2(t-s)\cos x\cos y + 2t\operatorname{sen} x\operatorname{sen} y\end{aligned}$$

11. $z = x^2 - 3x^2y^3, x = se^t, y = se^{-t} \quad \Rightarrow$

$$\begin{aligned}\partial z/\partial s &= \left(2x - 6xy^3\right)\left(e^t\right) + \left(-9x^2y^2\right)\left(e^{-t}\right) \\ &= \left(2x - 6xy^3\right)e^t - 9x^2y^2e^{-t} \\ \partial z/\partial t &= \left(2x - 6xy^3\right)\left(se^t\right) + \left(-9x^2y^2\right)\left(-se^{-t}\right) \\ &= \left(2x - 6xy^3\right)se^t + 9x^2y^2se^{-t}\end{aligned}$$

12. $z = x\operatorname{tg}^{-1}(xy), x = t^2, y = se^t \quad \Rightarrow$

$$\begin{aligned}\frac{\partial z}{\partial s} &= \left[\operatorname{tg}^{-1}(xy) + \frac{x}{1+x^2y^2}y\right](0) + \frac{x^2}{1+x^2y^2}e^t \\ &= \frac{x^2e^t}{1+x^2y^2} \\ \frac{\partial z}{\partial t} &= \left[\operatorname{tg}^{-1}(xy) + \frac{xy}{1+x^2y^2}\right](2t) + \frac{x^2}{1+x^2y^2}se^t\end{aligned}$$

13. $z = 2^{x-3y}, x = s^2t, y = st^2 \quad \Rightarrow$

$$\begin{aligned}\partial z/\partial s &= (z\ln 2)(2st) + z(-3\ln 2)\left(t^2\right) \\ &= \left(2^{x-3y}\ln 2\right)\left(2st - 3t^2\right) \\ \partial z/\partial t &= (z\ln 2)\left(s^2\right) + z(-3\ln 2)(2st) \\ &= \left(2^{x-3y}\ln 2\right)\left(s^2 - 6st\right)\end{aligned}$$

14. $z = xe^y + ye^{-x}, x = e^t, y = st^2 \quad \Rightarrow$

$$\begin{aligned}\partial z/\partial s &= \left(e^y - ye^{-x}\right)(0) + \left(xe^y + e^{-x}\right)\left(t^2\right) \\ &= \left(xe^y + e^{-x}\right)t^2 \\ \partial z/\partial t &= \left(e^y - ye^{-x}\right)\left(e^t\right) + \left(xe^y + e^{-x}\right)(2st)\end{aligned}$$

15. $w = x^2 + y^2 + z^2, x = st, y = s\cos t, z = s\operatorname{sen} t \quad \Rightarrow$

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= 2xt + 2y\cos t + 2z\operatorname{sen} t\end{aligned}$$

Quando $s = 1, t = 0$, temos $x = 0$, $y = 1$ e $z = 0$, então $\partial w/\partial s = 2\cos 0 = 2$. Analogamente

$$\begin{aligned}\partial w/\partial t &= 2xs + 2y(-s\operatorname{sen} t) + 2z(s\cos t) \\ &= 0 + (-2)\operatorname{sen} 0 + 0 = 0\end{aligned}$$

quando $s = 1$ e $t = 0$.

16. $u = xy + yz + zx$, $x = st$, $y = e^{st}$, $z = t^2 \Rightarrow$
$\partial u/\partial s = (y+z)\,t + (x+z)\,te^{st} + (x+y)(0)$ e
$\partial u/\partial t = (y+z)\,s + (x+z)\,se^{st} + (x+y)(2t)$. Quando $s = 0, t = 1$, temos $x = 0, y = 1, z = 1$, então
$\partial u/\partial s = 2 + 1 + 0 = 3$ e $\partial u/\partial t = 0 + 0 + (1)(2) = 2$.

17. $z = y^2 \operatorname{tg} x$, $x = t^2 uv$, $y = u + tv^2 \Rightarrow$
$\partial z/\partial t = \left(y^2 \sec^2 x\right) 2tuv + (2y \operatorname{tg} x)\, v^2$,
$\partial z/\partial u = \left(y^2 \sec^2 x\right) t^2 v + 2y \operatorname{tg} x$,
$\partial z/\partial v = \left(y^2 \sec^2 x\right) t^2 u + (2y \operatorname{tg} x)\, 2tv$. Quando $t = 2$, $u = 1$ e $v = 0$, temos $x = 0, y = 1$, então $\partial z/\partial t = 0$, $\partial z/\partial u = 0$, $\partial z/\partial v = 4$.

18. $z = \dfrac{x}{y}$, $x = re^{st}$, $y = rse^t \Rightarrow$

$$\frac{\partial z}{\partial r} = \frac{1}{y}e^{st} + \frac{-x}{y^2}se^t,\quad \frac{\partial z}{\partial s} = \frac{1}{y}rte^{st} - \frac{x}{y^2}re^t,$$

$\dfrac{\partial z}{\partial t} = \dfrac{1}{y}rse^{st} - \dfrac{x}{y^2}rse^t$. Quando $r = 1, s = 2$ e $t = 0$, temos $x = 1, y = 2$, então $\partial z/\partial r = \frac{1}{2} + \frac{-1}{4} \cdot 2 = 0$,
$\partial z/\partial s = 0 - \frac{1}{4} = -\frac{1}{4}$ e $\partial z/\partial t = \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 2 = \frac{1}{2}$.

19. $u = \dfrac{x+y}{y+z}$, $x = p + r + t$, $y = p - r + t$, $z = p + r - t \Rightarrow$

$$\frac{\partial u}{\partial p} = \frac{1}{y+z} + \frac{(y+z)-(x+y)}{(y+z)^2} - \frac{x+y}{(y+z)^2} = \frac{(y+z)+(z-x)-(x+y)}{(y+z)^2} = 2\frac{z-x}{(y+z)^2} = 2\frac{-2t}{4p^2} = -\frac{t}{p^2}$$

$$\frac{\partial u}{\partial r} = \frac{1}{y+z} + \frac{z-x}{(y+z)^2}(-1) - \frac{x+y}{(y+z)^2} = 0, \text{ e}$$

$$\frac{\partial u}{\partial t} = \frac{1}{y+z} + \frac{z-x}{(y+z)^2} + \frac{x+y}{(y+z)^2} = 2\frac{y+z}{(y+z)^2} = \frac{2}{2p} = \frac{1}{p}$$

20. $t = z\sec(xy)$, $x = uv$, $y = vw$, $z = wu \Rightarrow$

$$\frac{\partial t}{\partial u} = [zy\sec(xy)\operatorname{tg}(xy)]\,v + [zx\sec(xy)\operatorname{tg}(xy)](0) + [\sec(xy)]\,w = \sec(xy)\,[w + vzy\operatorname{tg}(xy)]$$

$$\frac{\partial t}{\partial v} = [zy\sec(xy)\operatorname{tg}(xy)]\,u + [zx\sec(xy)\operatorname{tg}(xy)]\,w + [\sec(xy)](0) = z\sec(xy)\operatorname{tg}(xy)\,[yu + xw]$$

$$\frac{\partial t}{\partial w} = [zy\sec(xy)\operatorname{tg}(xy)](0) + [zx\sec(xy)\operatorname{tg}(xy)]\,v + [\sec(xy)]\,u = \sec(xy)\,[u + vzx\operatorname{tg}(xy)]$$

21.

$$\frac{\partial w}{\partial r} = [-\operatorname{sen}(x-y)]\,s^2t^3\operatorname{sen}\theta + [\operatorname{sen}(x-y)]\,2rst\cos\theta = st\operatorname{sen}(x-y)\left[2r\cos\theta - st^2\operatorname{sen}\theta\right]$$

$$\frac{\partial w}{\partial s} = [-\operatorname{sen}(x-y)]\,2rst^3\operatorname{sen}\theta + [\operatorname{sen}(x-y)]\,r^2t\cos\theta = [rt\operatorname{sen}(x-y)]\left(r\cos\theta - 2st^2\operatorname{sen}\theta\right)$$

$$\frac{\partial w}{\partial t} = [-\operatorname{sen}\,x-y)]\,3rs^2t^2\operatorname{sen}\theta + [\operatorname{sen}(x-y)]\,r^2s\cos\theta = [sr\operatorname{sen}(x-y)]\left(r\cos\theta - 3st^2\operatorname{sen}\theta\right)$$

$$\frac{\partial w}{\partial \theta} = [-\operatorname{sen}(x-y)]\,rs^2t^3\cos\theta + [\operatorname{sen}(x-y)]\left(-r^2st\operatorname{sen}\theta\right) = [-rst\operatorname{sen}(x-y)]\left(st^2\cos\theta + r\operatorname{sen}\theta\right)$$

22.

$$\frac{\partial u}{\partial x} = \left(q - 2pr^2s\right) + p + \left(-2p^2rs\right)\frac{1}{y^4} + \left(-p^2r^2\right)\left(2y^{3/2}\right) = p + q - 2pr^2s - \frac{2p^2rs}{y^4} - 2p^2r^2y^{3/2} = 2x - \frac{8x^2(x+2y)(x+y)}{y^{13/2}}$$

$$\frac{\partial u}{\partial y} = \left(q - 2pr^2s\right)(2) + p(-2) + \left(-2p^2rs\right)\left(-\frac{4x}{y^5}\right) + \left(-p^2r^2\right)3xy^{1/2} = 2(q-p) - 4pr^2s + \frac{8p^2rsx}{y^5} - 3p^2r^2xy^{1/2} = -8y + \frac{(x+2y)\,x^3y^{1/2}\,(5x+2y)}{y^8}$$

23. $x^2 - xy + y^3 = 8$, então considere $F(x, y) = x^2 - xy + y^3 - 8 = 0$.

Logo, $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = \dfrac{-(2x-y)}{-x+3y^2} = \dfrac{y-2x}{3y^2-x}$.

24. $y^5 + 3x^2y^2 + 5x^4 = 12$, então considere $F(x, y) = y^5 + 3x^2y^2 + 5x^4 - 12 = 0$. Logo $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{6xy^2 + 20x^3}{5y^4 + 6x^2y}$.

25. $x\cos y + y\cos x = 1$, então considere $F(x, y) = x\cos y + y\cos x - 1 = 0$. Logo, $\dfrac{dy}{dx} = -\dfrac{\cos y - y\operatorname{sen} x}{-x\operatorname{sen} y + \cos x} = \dfrac{y\operatorname{sen} x - \cos y}{\cos x - x\operatorname{sen} y}$.

26. $2y^2 + \sqrt[3]{xy} = 3x^2 + 17$, então considere $F(x, y) = 2y^2 + \sqrt[3]{xy} - 3x^2 - 17 = 0$. Então

$$\frac{dy}{dx} = -\frac{y\Big/\left[3(xy)^{2/3}\right] - 6x}{4y + x\Big/\left[3(xy)^{2/3}\right]} = \frac{18x - x^{-2/3}y^{1/3}}{12y + x^{1/3}y^{-2/3}}.$$

27. Seja $F(x, y, z) = xy + yz - xz = 0$. Logo

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y-z}{y-x} = \frac{z-y}{y-x},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{y-x} = \frac{x+z}{x-y}.$$

28. $x^2+y^2-z^2=2x(y+z)$. Seja
$F(x,y,z)=x^2+y^2-z^2-2x(y+z)=0$. Então
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{2x-2y-2z}{-2z-2x}=\frac{x-y-z}{z+x},$$
$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2y-2x}{-2z-2x}=\frac{y-x}{z+x}$$

29. $xy^2z^3+x^3y^2z=x+y+z$. Seja
$F(x,y,z)=xy^2z^3+x^3y^2z-(x+y+z)$.
Então $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{y^2z^3+3x^2y^2z-1}{3xy^2z^2+x^3y^2-1}$,
$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2xyz^3+2x^3yz-1}{3xy^2z^2+x^3y^2-1}.$$

30. Seja $F(x,y,z)=y^2ze^{x+y}-\operatorname{sen}(xyz)=0$. Então,
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{y^2ze^{x+y}-yz\cos(xyz)}{y^2e^{x+y}-xy\cos(xyz)}$$
$$=\frac{z\cos(xyz)-yze^{x+y}}{ye^{x+y}-x\cos(xyz)}$$
$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{xz\cos(xyz)-e^{x+y}\left(2yz+y^2z\right)}{y^2e^{x+y}-xy\cos(xyz)}$$

31. $xy^2+yz^2+zx^2=3$, então considere
$F(x,y)=xy^2+yz^2+zx^2-3=0$.
Logo, $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{y^2+2zx}{2yz+x^2}$ e
$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{2xy+z^2}{2yz+x^2}.$$

32. Seja $F(x,y,z)=xe^y+yz+ze^x=0$. Logo,
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{e^y+ze^x}{y+e^x},\ \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{xe^y+z}{y+e^x}.$$

33. $\ln(x+yz)=1+xy^2z^3$, então considere
$F(x,y)=\ln(x+yz)-1-xy^2z^3=0$. Logo,
$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{1/(x+yz)-y^2z^3}{y/(x+yz)-3xy^2z^2}$$
$$=\frac{y^2z^3(x+yz)-1}{y-3xy^2z^2(x+yz)}$$
$$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{z/(x+yz)-2xyz^3}{y/(x+yz)-3xy^2z^2}$$
$$=\frac{2xyz^3(x+yz)-z}{y-3xy^2z^2(x+yz)}$$

34. $dr/dt=-1{,}2$, $dh/dt=3$, $V=\pi r^2h$ e
$dV/dt=2\pi rh\,(dr/dt)+\pi r^2\,(dh/dt)$.
Assim, quando $r=80$ e $h=150$,
$dV/dt=(-28\,800)\,\pi+(19\,200)\,\pi=-9600\,\pi\ \text{cm}^3/\text{s}$.