Lecture 5.
Nonholonomic constraint.

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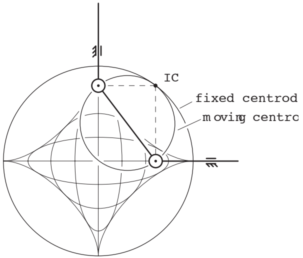
Mechanics of Manipulation

Outline

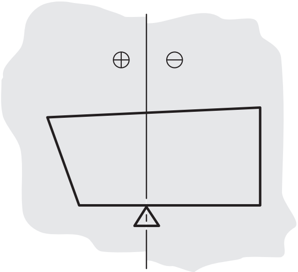
Review planar bilateral constraint

- Procedure for bilateral planar constraints:
  - Identify point A and feasible velocity dA, for each constraint.
  - Construct $\perp$ to dA at A, for each constraint.
  - Intersection of all $\perp$'s is candidate IC.
- More abstractly:
  - Express each constraint as a set of feasible IC's.
  - Intersect all sets, to get candidate IC's.



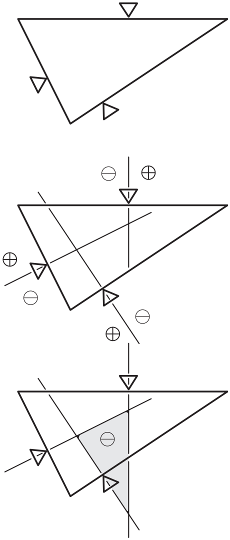
Generalizing to unilateral constraints

- The abstract procedure:
  - Express each constraint as a set of feasible IC's.
  - Intersect all sets, to get candidate IC's.
- It won't work: *all* IC's are feasible.
- The solution: *signed* IC's!
  - Construct inward pointing normal, at contact.
  - Label the normal both $+$ and $-$.
  - Label the right half plane $-$.
  - Label the left half plane $+$.



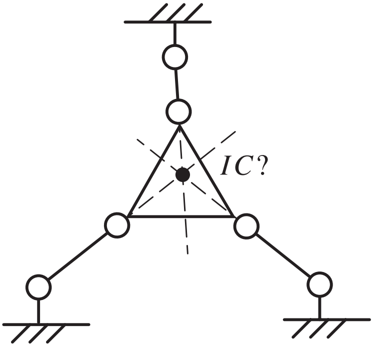
Multiple unilateral constraints (Reuleaux's method)

- Can this triangle move?
- Construct positive and negative half-planes for each contact.
- Keep consistently labelled points.
- Triangle can rotate CW about any − point.

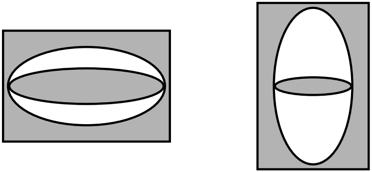


But watch for false positives

- The process identifies *candidate* IC's.



False positive for bilateral constraints.



False positive for unilateral constraints.

Nonholonomic constraint

Definition (Holonomic constraint)

A kinematic constraint is a holonomic constraint if it can be expressed in the form

$$f(q, t) = 0.$$

That is, expressed as a *bilateral* constraint on *configuration*.

- Not everybody recognizes that a unilateral constraint is nonholonomic.

Nonholonomic constraint

Why study nonholonomic constraint?

It is *fundamental* to all of robotics . . .

- The robot has only a few motors, say k.
- The task has many degrees of freedom, say n.
- How many independent motions can the robot produce? At most k.
- How many DoFs in the task does the robot wish to control? Perhaps all n.
- The difference implies nonholonomic constraint.
- (Often true in locomotion problems, almost always true in manipulation problems.)

Integrable constraints

Look again at the definition:

Definition (Holonomic constraint)

A kinematic constraint is a holonomic constraint if it can be expressed in the form

$$f(q, t) = 0.$$

Suppose you have a constraint of the form

$$f(q, \dot{q}, t) = 0$$

Is it nonholonomic? Perhaps it *can be* expressed in the form

$$f(q, t) = 0$$

in which case we say the constraint is integrable. It's a holonomic constraint, *disguised* as a nonholonomic constraint.

The unicycle: counting constraints

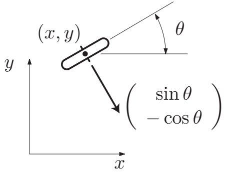
The unicycle cannot move sideways. Let

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$

and let

$$\mathbf{w}_1 = (\sin\theta, -\cos\theta, 0)$$

so there is one constraint, written $\mathbf{w}_1 \dot{\mathbf{q}} = 0$.



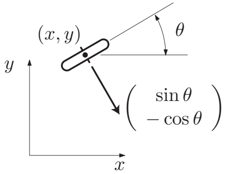
The unicycle: counting freedoms

The unicycle can move in two directions, expressed by defining

$$\mathbf{g}_1(\mathbf{q}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{g}_2(\mathbf{q}) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

and noting that the unicycle's motion is (missing from book)

$$\dot{\mathbf{q}} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2$$



where u_1 and u_2 are arbitrary reals. They are the *controls*.

The robot has two controls. How many freedoms?

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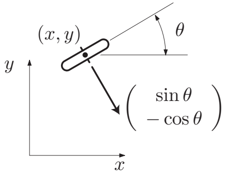


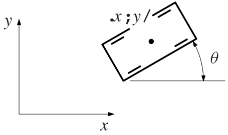
where u_1 and u_2 are arbitrary reals. They are the *controls*.

The robot has two controls. How many freedoms? Three.

Unsteered cart constraint and freedom

The unsteered cart cannot turn, and cannot move sideways. Let

$$\mathbf{w}_1 = (\sin\theta, -\cos\theta, 0)\,, \mathbf{w}_2 = (0, 0, 1)$$

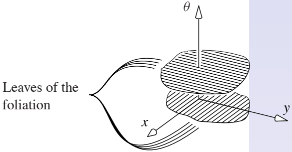


so the two constraints are written $\mathbf{w}_1\dot{\mathbf{q}} = 0$, $\mathbf{w}_2\dot{\mathbf{q}} = 0$. Expanding the products:

$$\begin{aligned} \dot{x}\sin\theta - \dot{y}\cos\theta &= 0 \\ \dot{\theta} &= 0 \end{aligned}$$

These can be integrated:

$$\begin{aligned} \theta &= \theta_0 \\ (x - x_0)\sin\theta_0 - (y - y_0)\cos\theta_0 &= 0 \end{aligned}$$

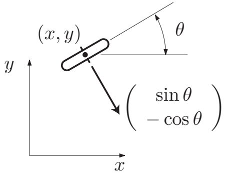


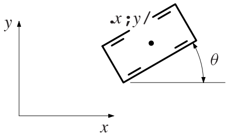
Planar contact constraints (Reuleaux)
Nonholonomic constraint
Example: the unicycle
Integebrable and nonintegrable constraints
Vector fields and distributions
Frobenius's theorem

Unicycle versus cart

- Unicycle.
  - One velocity constraint.
  - Three freedoms.
- Unsteered cart
  - Two velocity constraints.
  - *Integrable*. Equivalent to two configuration constraints.
  - One freedom.

System is nonholonomic if the constraint *cannot* be written in the form $f(q, t) = 0$.





Holonomic does not mean unconstrained!!!

If your universe consists of planar mobile robots with all three freedoms, then either the robot is unconstrained, or it is nonholonomic. You will think holonomic means unconstrained.

- Holonomic means the constraints can be written as equations independent of $\dot{q}$

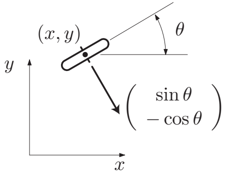
$$f(q, t) = 0$$

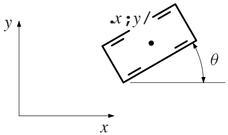
- A mobile robot with no constraints is holonomic.
- A mobile robot capable of only translations is also holonomic.

Integrable? Or not? How can you tell?

How can you tell whether a velocity constraint is integrable?

1. Try to integrate it. As we did for the cart.
2. Determine whether the DOFs were reduced. As we did for the unicycle.
3. Either technique might work for simple systems. But we need a systematic technique: Lie brackets!!! (Frobenius's theorem.)





Pfaffian constraints

Definition (Pfaffian constraints)

A set of k Pfaffian constraints are of the form

$$\mathbf{w}_i(\mathbf{q})\dot{\mathbf{q}} = 0, i = 1 \dots k$$

where the $\mathbf{w}_i$ are linearly independent row vectors, and $\dot{\mathbf{q}}$ is a column vector.

All the velocity constraints we have considered for the unicycle and the cart are Pfaffian.

Vector fields

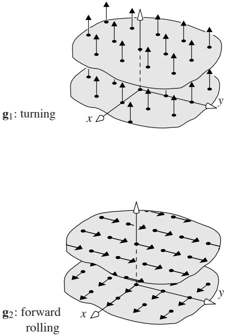
Definition (Vector field)

A vector field is a smooth map

$$f(\mathbf{q}) : \mathbf{C} \mapsto \mathbf{T_q C}$$

from configurations $\mathbf{q}$ to velocity vectors $\dot{\mathbf{q}}$.

Note: In differential geometry "vector" sometimes means specifically "velocity vector".

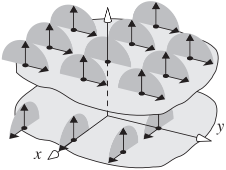


Distributions

Definition (Distribution)

A distribution is a smooth map assigning a linear subspace of $\mathbf{T_q C}$ to each configuration $\mathbf{q}$ of $\mathbf{C}$.

Example: The linear span of $\mathbf{g}_1$ and $\mathbf{g}_2$.



Recall that for the unicycle

$$\dot{\mathbf{q}} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2$$

for $u_1, u_2 \in \mathbf{R}$. So the figure shows the feasible velocities for every $\mathbf{q}$.

(Well, it only shows a circular patch where it should show a whole plane at every $\mathbf{q}$.)

Regular distributions and Lie brackets

Definition (Regular distribution)

A distribution is regular if its dimension is constant over the configuration space.

Definition (Lie bracket)

Let $\mathbf{f}$, $\mathbf{g}$ be two vector fields on $\mathbf{C}$. Define the Lie bracket $[\mathbf{f}, \mathbf{g}]$ to be the vector field

$$\frac{\partial \mathbf{g}}{\partial \mathbf{q}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\mathbf{g}$$

What is this thing written $\frac{\partial \mathbf{g}}{\partial \mathbf{q}}$ or $\frac{\partial \mathbf{f}}{\partial \mathbf{q}}$? Matrix. Each column is partial of velocity w.r.t. configuration variable.

Lie brackets, example.

Let's take the Lie bracket $[\mathbf{g}_1, \mathbf{g}_2]$.

$$\frac{\partial \mathbf{g}_1}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix}$$

For the new vector field defined by the Lie bracket we obtain

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2] = \frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}}\mathbf{g}_2$$
$$= \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

Lie brackets example continued

$$\mathbf{g}_3 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

Physically, $\mathbf{g}_3$ moves sideways. It is linearly independent of $\mathbf{g}_1$ and $\mathbf{g}_2$, and it violates the constraint $\mathbf{w}_1$.

What is its physical significance? Given two vector fields $\mathbf{f}$ and $\mathbf{g}$,

1. Follow $\mathbf{f}$ for some time ϵ;
2. Follow $\mathbf{g}$ for ϵ;
3. Follow $-\mathbf{f}$ for ϵ;
4. Follow $-\mathbf{g}$ for ϵ.

In the limit as ϵ approaches zero, the result of the above motion approaches the Lie bracket $[\mathbf{f}, \mathbf{g}]$. The Lie bracket could have been called “parallel parking product”.

Involutive distribution

Definition (Involutive)

A distribution is involutive if it is closed under Lie brackets.

Definition (Involutive closure)

The involutive closure of a distribution Δ is the closure $\overline{\Delta}$ of the distribution under Lie bracketing.

- Given a distribution take all Lie brackets.
- If you get new fields, add them to the distribution.
- Repeat until you get nothing new.

Frobenius's theorem

Theorem (2.8, Frobenius's theorem)

A regular distribution is integrable if and only if it is involutive.

Proof.

- (integrable $\to$ involutive.) Take the Taylor series of parallel parking as a function of ϵ. Second order terms are Lie brackets! If the distribution is involutive, the Lie brackets must also be contained in the distribution.
- (involutive $\to$ integrable). Too involved for us. By induction over dimension.

□

Restating Frobenius's theorem: A set of constraints is nonholonomic $\leftrightarrow$ parallel parking is useful.

Lessons

- Robots usually have less motors than task freedoms. There will be constraints.
- A holonomic constraint is a constraint on configuration: it says there are places you cannot go. That is a reduction in freedoms. That's (usually) *bad*.
- A nonholonomic constraint is a constraint on velocity: there are directions you cannot go. But you can still get wherever you want. That's (usually) *good*!
- Parallel parking is *general*. If you want to move in the constrained direction, pick a pair of controls and interleave oscillations. Or do it mathematically with Lie brackets.
- If parallel parking doesn't help, you are truly stuck on the leaf of a foliation. Rearrange your motors, or buy more.