Lista de Integrais Impróprias

Calcule as seguintes integrais, classificando-as em convergente ou divergente:

a) $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x}}$

$$\lim_{t\to+\infty}\int_1^t \frac{dx}{x^{1/2}} = \lim_{t\to+\infty}\int_1^t x^{-1/2}\,dx$$

Usando o 1º Teorema Fundamental do Cálculo, temos que:

$$\lim_{t\to+\infty}\int_1^t x^{-1/2}\,dx = \lim_{t\to+\infty}\left[2x^{1/2}\right]_1^t = \lim_{t\to+\infty}\left(2t^{1/2} - 2\cdot 1^{1/2}\right) = \overbrace{\lim_{t\to+\infty} 2t^{1/2}}^{+\infty} - \lim_{t\to+\infty} 2\cdot 1^{1/2} =$$

$= +\infty$ $\therefore$ a integral é divergente

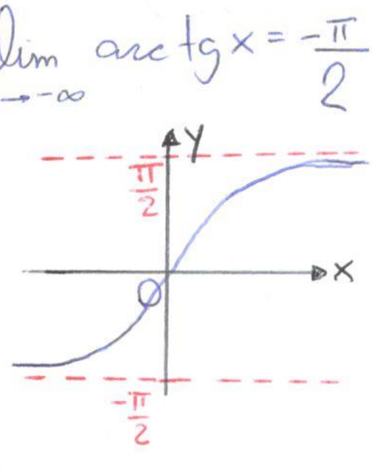
b) $\displaystyle\int_0^{+\infty} \frac{\text{arc tg}\,x}{x^2+1}\,dx$

Utilizando o método de integração por substituição de variáveis, vem:

Fazendo $u = \text{arc tg}\,x$; $du = \dfrac{1}{x^2+1}\,dx \Rightarrow dx = (x^2+1)\,du$

OBS: $\displaystyle\lim_{x\to+\infty} \text{arc tg}\,x = \frac{\pi}{2}$

$\displaystyle\lim_{x\to-\infty} \text{arc tg}\,x = -\frac{\pi}{2}$



$x \to 0,\ u \to 0$

$x \to t,\ u \to \text{arc tg}\,t$

$$\lim_{t\to+\infty}\int_0^t \frac{\text{arc tg}\,x}{x^2+1}\,dx = \lim_{t\to+\infty}\int_0^{\text{arc tg}\,t} \frac{u}{x^2+1}\,(x^2+1)\,du =$$

$$= \lim_{t\to+\infty}\int_0^{\text{arc tg}\,t} u\,du$$

Usando o 1º Teorema Fundamental do Cálculo, temos que

$$\lim_{t\to+\infty}\int_0^{\text{arc tg}\,t} u\,du = \lim_{t\to+\infty}\left[\frac{u^2}{2}\right]_0^{\text{arc tg}\,t} = \lim_{t\to+\infty}\left(\frac{\text{arc tg}^2\,t}{2} - \frac{0^2}{2}\right) = \lim_{t\to+\infty}\frac{\text{arc tg}^2\,t}{2} =$$

$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$ $\therefore$ a integral é convergente

c) $\int_{-\infty}^{0} e^{10x}\,dx$

$$\lim_{t \to -\infty} \int_t^0 e^{10x}\,dx$$

Pelo método de integração por substituição de variáveis, temos:

Fazendo $u = 10x$; $du = 10dx \Rightarrow dx = \frac{du}{10}$ $x \to t,\ u \to 10t$; $x \to 0,\ u \to 0$

$$\lim_{t \to -\infty} \int_t^0 e^{10x}\,dx = \lim_{t \to -\infty} \int_{10t}^0 e^u \frac{du}{10} = \frac{1}{10} \lim_{t \to -\infty} \int_{10t}^0 e^u\,du =$$

$$= -\frac{1}{10} \lim_{t \to -\infty} \int_0^{10t} e^u\,du$$

Utilizando o 1º Teorema Fundamental do Cálculo, vem:

$$-\frac{1}{10} \lim_{t \to -\infty} \int_0^{10t} e^u\,du = -\frac{1}{10} \lim_{t \to -\infty} \left[e^u\right]_0^{10t} = -\frac{1}{10} \lim_{t \to -\infty} \left(e^{10t} - e^0\right) =$$

$$= -\frac{1}{10}\left(\cancel{\lim_{t \to -\infty} e^{10t}}^{\,0} - \lim_{t \to -\infty} 1\right) = -\frac{1}{10}(0-1) = \frac{1}{10}$$

$\therefore$ a integral é convergente

d) $\int_1^{+\infty} \ln x\,dx$

$$\lim_{t \to +\infty} \int_1^t \ln x\,dx$$

Pelo método de integração por partes, temos que:

Fazendo $u = \ln x$ $dv = dx$

$du = \frac{1}{x}\,dx$ $v = \int 1\,dx \Rightarrow v = x$

$$uv - \int v\,du = \lim_{t \to +\infty}\left(\left[x \ln x\right]_1^t - \int_1^t x \cdot \frac{1}{x}\,dx\right) = \lim_{t \to +\infty}\left(\left[x \ln x\right]_1^t - \int_1^t 1\,dx\right)$$

Aplicando o 1° Teorema Fundamental do Cálculo, chegamos a:

$$\lim_{t\to+\infty}\left([x\ln x]_1^t-\int_1^t 1\,dx\right)=\lim_{t\to+\infty}\left(t\ln t-1\ln 1-[x]_1^t\right)=\lim_{t\to+\infty}(t\ln t-t+$$

$$=\underbrace{\lim_{t\to+\infty} t\ln t}_{=[0\cdot\infty]}-\lim_{t\to+\infty} t+\lim_{t\to+\infty} 1=\underbrace{\lim_{t\to+\infty}\frac{\ln t}{\frac{1}{t}}}_{=\left[\frac{\infty}{\infty}\right]}-\lim_{t\to+\infty} t+1$$

(indeterminação)

Usando a Regra de L'Hospital, teremos que:

$$\lim_{t\to+\infty}\frac{(\ln t)'}{\left(\frac{1}{t}\right)'}-\lim_{t\to+\infty} t+1=\lim_{t\to+\infty}\frac{\frac{1}{t}}{-\frac{1}{t^2}}-\lim_{t\to+\infty} t+1=-\lim_{t\to+\infty}\frac{1}{t}t^2-$$

$$-\lim_{t\to+\infty} t+1=-\lim_{t\to+\infty} t-\lim_{t\to-\infty} t+1=-\infty$$

a integral é divergente

e) $\displaystyle\int_0^5\frac{x}{\sqrt{25-x^2}}\,dx$

$$\lim_{t\to5^-}\int_0^t\frac{x}{\sqrt{25-x^2}}\,dx$$

Usando o método de integração por substituição de variáveis, tem-se

Fazendo $u=25-x^2$; $du=-2x\,dx\Rightarrow dx=\frac{du}{-2x}$ $\qquad x\to0,\ u\to25$; $\quad x\to t,\ u\to25-t^2$

$$\lim_{t\to5^-}\int_0^5\frac{x}{\sqrt{25-x^2}}\,dx=\lim_{t\to5^-}\int_{25}^{25-t^2}\frac{x}{\sqrt{u}}\,\frac{du}{-2x}=-\frac{1}{2}\lim_{t\to5^-}\int_{25}^{25-t^2}\frac{1}{u^{1/2}}\,du=$$

$$=\frac{1}{2}\lim_{t\to5^-}\int_{25-t^2}^{25}u^{-1/2}\,du=\frac{1}{2}\lim_{t\to5^-}\left[2u^{1/2}\right]_{25-t^2}^{25}=\frac{1}{2}\lim_{t\to5^-}\left(2\cdot25^{1/2}-2(25-t^2)^{1/2}\right)=$$

$$=\lim_{t\to5^-}\left(5-\sqrt{25-t^2}\right)=\lim_{t\to5^-}5-\lim_{t\to5^-}\sqrt{25-t^2}=5-\sqrt{25-5^2}=5-\sqrt{0}=5$$

a integral é convergente

f) $\int_1^{+\infty} \frac{dx}{x^2}$

$$\lim_{t\to+\infty} \int_1^t x^{-2}dx = \lim_{t\to+\infty} \left[-x^{-1}\right]_1^t = \lim_{t\to+\infty} \left(-\frac{1}{t}+\frac{1}{1}\right) = -\lim_{t\to+\infty} \frac{1}{t} + \lim_{t\to+\infty} 1 =$$

$$= 0 + 1 = 1$$

$\therefore$ a integral é convergente

g) $\int_{-\infty}^0 xe^{-x^2}dx$

Utilizando o método de integração por substituição de variáveis, vem:

Fazendo $u = x^2$; $du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$ $\qquad x\to 0,\ u\to 0$; $\quad x\to -\infty;\ u\to +\infty$

$$\int_{-\infty}^0 xe^{-x^2}dx = \int_{+\infty}^0 xe^{-u}\frac{du}{2x} = -\frac{1}{2}\int_0^{+\infty} e^{-u}du = -\frac{1}{2}\int_0^{+\infty} u^0 e^{-u}du =$$

$$= -\frac{1}{2}\Gamma(1) = -\frac{1}{2}(1-1)! = -\frac{1}{2}\,0! = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$$

$\therefore$ a integral é convergente

h) $\int_0^{\pi/2} \cos^2 x\, dx$

$$\int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \operatorname{sen}^0 x \cos^2 x\, dx = B\left(\frac{1}{2},\frac{3}{2}\right)\frac{1}{2} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}\frac{1}{2}$$

$$= \frac{\sqrt{\pi}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}\frac{1}{2} = \frac{\frac{1}{4}\sqrt{\pi}\sqrt{\pi}}{(2-1)!} = \frac{\frac{1}{4}\pi}{1!} = \frac{\frac{1}{4}\pi}{1} = \frac{\pi}{4}$$

$\therefore$ a integral é convergente

i) $\int_0^{+\infty} x^2 dx$

$$\lim_{t\to+\infty} \int_0^t x^2dx = \lim_{t\to+\infty} \left[\frac{x^3}{3}\right]_0^t = \lim_{t\to+\infty} \left(\frac{t^3}{3}-\frac{0^3}{3}\right) = \lim_{t\to+\infty} \frac{t^3}{3} = +\infty$$

$\therefore$ a integral é divergente

j) $$\int_{-\infty}^{+\infty} \frac{dx}{9+x^2} = \int_{-\infty}^{0} \frac{dx}{9+x^2} + \int_{0}^{+\infty} \frac{dx}{9+x^2} = \int_{-\infty}^{0} \frac{dx}{9\left(1+\frac{x^2}{9}\right)} + \int_{0}^{+\infty} \frac{dx}{9\left(1+\frac{x^2}{9}\right)} =$$

$$= \frac{1}{9} \lim_{t \to -\infty} \int_{t}^{0} \frac{dx}{1+\left(\frac{x}{3}\right)^2} + \frac{1}{9} \lim_{v \to +\infty} \int_{0}^{v} \frac{dx}{1+\left(\frac{x}{3}\right)^2}$$

Pelo método de integração por substituição de variáveis, vem:

Fazendo $u = \frac{x}{3}$; $du = \frac{dx}{3} \Rightarrow dx = 3du$

$x \to 0;\ u \to 0$

$x \to t;\ u \to t/3$

$x \to v;\ u \to v/3$

$$\frac{1}{9} \lim_{t \to -\infty} \int_{t}^{0} \frac{dx}{1+\left(\frac{x}{3}\right)^2} + \frac{1}{9} \lim_{t \to +\infty} \int_{0}^{v} \frac{dx}{1+\left(\frac{x}{3}\right)^2} = \frac{1}{9} \lim_{t \to -\infty} \int_{t/3}^{0} \frac{3du}{1+u^2} +$$

$$+ \frac{1}{9} \lim_{v \to +\infty} \int_{0}^{v} \frac{3du}{1+u^2} = \frac{1}{9} \cdot 3 \lim_{t \to -\infty} \int_{t/3}^{0} \frac{du}{1+u^2} + \frac{1}{9} \cdot 3 \int_{0}^{v/3} \frac{du}{1+u^2} =$$

$$= \frac{1}{3} \lim_{t \to -\infty} \left[\operatorname{arctg} u\right]_{t/3}^{0} + \frac{1}{3} \lim_{v \to +\infty} \left[\operatorname{arctg} u\right]_{0}^{v/3} = \frac{1}{3} \lim_{t \to -\infty} \left(\operatorname{arctg} 0 - \operatorname{arctg} \frac{t}{3}\right) +$$

$$+ \frac{1}{3} \lim_{t \to +\infty} \left(\operatorname{arctg} \frac{v}{3} - \operatorname{arctg} 0\right) = \frac{1}{3}\left(-\frac{\pi}{2} - 0\right) + \frac{1}{3}\left(\frac{\pi}{2} - 0\right) = -\frac{\pi}{6} + \frac{\pi}{6} = 0$$

∴ a integral é convergente

k) $$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{1}{2}x^2} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-\frac{1}{2}x^2} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-x^2/2} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-x^2/2} dx$$

Pelo método de integração por substituição de variáveis, vem

Fazendo $u = \frac{x^2}{2}$; $du = x\,dx$; $dx = \frac{du}{x}$

$x \to 0;\ u \to 0$

$x \to -\infty;\ u \to +\infty$

$x \to +\infty;\ u \to +\infty$

$x = +\sqrt{2u}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \frac{x}{x} e^{-x^2/2} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{x}{x} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{0} \frac{x}{x} e^{-u} \frac{du}{x}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{x}{x} e^{-u} \frac{du}{x} = +\frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{1}{\sqrt{2u}} e^{-u} du + \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} \frac{1}{\sqrt{2u}} e^{-u} du$$

$$= +\frac{1}{2\sqrt{\pi}} \int_{0}^{+\infty} u^{-1/2} e^{-u} du + \frac{1}{2\sqrt{\pi}} \int_{0}^{+\infty} u^{-1/2} e^{-u} du = \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) +$$

$$+ \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{\pi}} \sqrt{\pi} + \frac{1}{2\sqrt{\pi}} \sqrt{\pi} = \frac{1}{2} + \frac{1}{2} = 1$$

l) $\displaystyle\int_0^1 \sqrt{1-x}\, dx = \int_0^1 (1-x)^{1/2} dx = \int_0^1 x^0 (1-x)^{1/2} dx = B\left(1, \frac{3}{2}\right) =$

$$= \frac{\Gamma(1) \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(1 + \frac{3}{2}\right)} = \frac{0! \; \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{1 \cdot \Gamma\left(\frac{3}{2}\right)}{\frac{3}{2} \Gamma\left(\frac{3}{2}\right)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$