

Aula 7

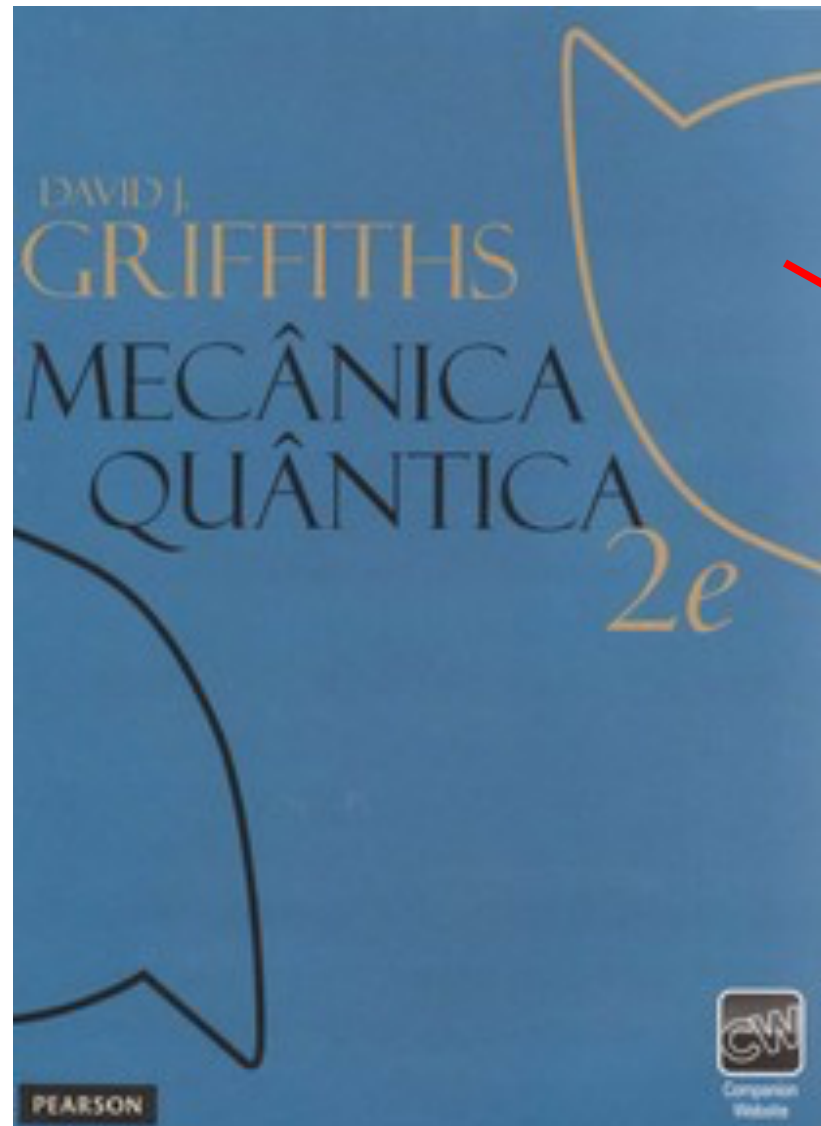
Operadores escada

Autoestados de S_y

Probabilidades e valor médio

Exemplos

Para estudar o formalismo de spin vamos usar :



gato

Operadores de spin

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

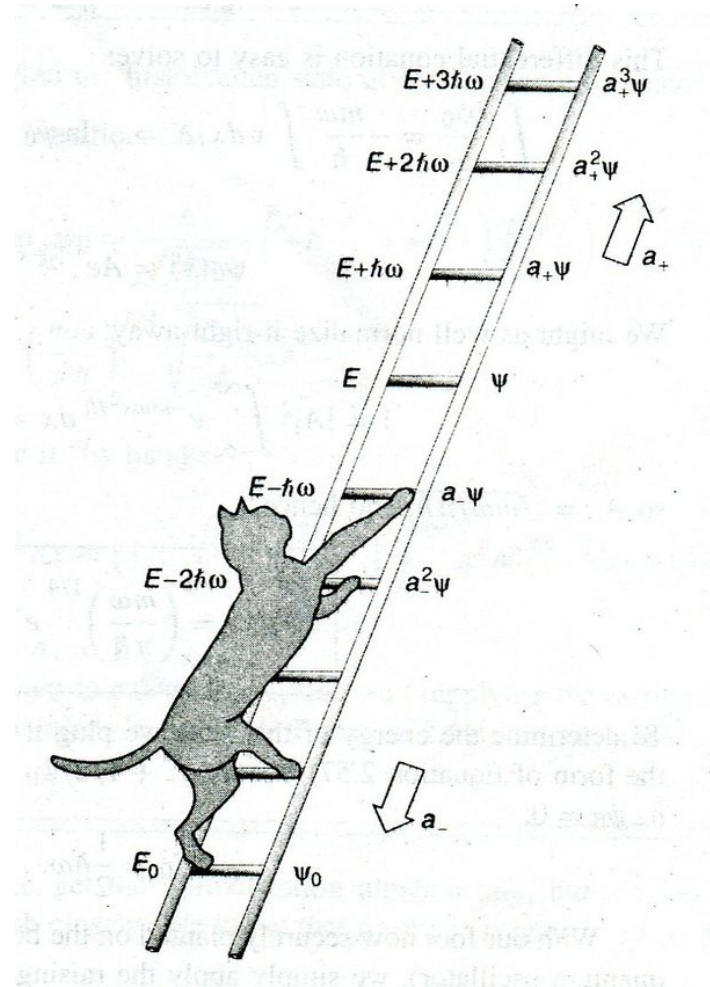
Operadores escada

$$\hat{S}_+ = \hat{S}_x + i \hat{S}_y$$

levantamento

$$\hat{S}_- = \hat{S}_x - i \hat{S}_y$$

abaixamento



$$\hat{S}_+ = \hat{S}_x + i \hat{S}_y$$

$$\hat{S}_+ = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_+ = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Levantamento

$$\hat{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\hat{S}_- = \hat{S}_x - i \hat{S}_y$$

$$\hat{S}_- = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{S}_- = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Abaixamento

$$\hat{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_+ |\downarrow\rangle = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{S}_+ |\downarrow\rangle = \hbar |\uparrow\rangle$$

$$\hat{S}_+ |\uparrow\rangle = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\hat{S}_+ |\uparrow\rangle = 0$$

$$\hat{S}_- |\uparrow\rangle = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{S}_- |\uparrow\rangle = \hbar |\downarrow\rangle$$

$$\hat{S}_- |\downarrow\rangle = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\hat{S}_- |\downarrow\rangle = 0$$

Sistema de equações lineares

$$\left\{ \begin{array}{l} A_1 \alpha + A_2 \beta = a \alpha \\ A_3 \alpha + A_4 \beta = a \beta \end{array} \right. \quad \alpha = ? \quad \beta = ?$$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad A |\chi\rangle = a |\chi\rangle$$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} A_1 - a & A_2 \\ A_3 & A_4 - a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

$$\begin{vmatrix} A_1 - a & A_2 \\ A_3 & A_4 - a \end{vmatrix} = 0$$

Para que exista
solução não-trivial !

Autoestados de S_y

$$\hat{S}_y |y\rangle = a |y\rangle$$

$$\hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad |y\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = a \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{vmatrix} 0 - a & -i \frac{\hbar}{2} \\ i \frac{\hbar}{2} & 0 - a \end{vmatrix} = 0$$

$$a^2 = \frac{\hbar^2}{4}$$



$$a = \pm \frac{\hbar}{2}$$

$$\bullet \quad a = +\frac{\hbar}{2}$$

$$\hat{S}_y |y_+\rangle = +\frac{\hbar}{2} |y_+\rangle \quad |y_+\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} -i\beta \\ i\alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \left\{ \begin{array}{l} -i\beta = \alpha \\ i\alpha = \beta \end{array} \right.$$

$$i\beta^* = \alpha^*$$

$$\beta^2 = \alpha^2$$

Normalização


$$\langle y_+ | y_+ \rangle = (\alpha^* \ \beta^*) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 1$$

$$\alpha^* \alpha + \beta^* \beta = \alpha^2 + \beta^2 = 1 \qquad \beta^2 = \alpha^2$$

$$2 \alpha^2 = 1 \qquad \alpha = \pm \frac{1}{\sqrt{2}}$$

$$\alpha = \frac{1}{\sqrt{2}} \quad \longrightarrow \quad i \alpha = \beta \quad \longrightarrow \quad \beta = \frac{i}{\sqrt{2}}$$

$$|y_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$



$$a = -\frac{\hbar}{2}$$

$$\hat{S}_y |y_{-}\rangle = -\frac{\hbar}{2} |y_{-}\rangle \quad |y_{-}\rangle = \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$$

$$\begin{pmatrix} -i\beta' \\ i\alpha' \end{pmatrix} = \begin{pmatrix} -\alpha' \\ -\beta' \end{pmatrix} \quad \begin{cases} i\beta' = \alpha' \\ i\alpha' = -\beta' \end{cases}$$

$$\beta'^2 = \alpha'^2$$

Normalização

$$\langle \chi' | \chi' \rangle = (\alpha'^* \quad \beta'^*) \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = 1$$

$$\alpha'^* \alpha' + \beta'^* \beta' = \alpha'^2 + \beta'^2 = 1 \qquad \beta'^2 = \alpha'^2$$

$$2 \alpha'^2 = 1 \qquad \alpha' = \pm \frac{1}{\sqrt{2}}$$

$$\alpha' = \frac{1}{\sqrt{2}} \quad \longrightarrow \quad i \alpha' = -\beta' \quad \longrightarrow \quad \beta' = -\frac{i}{\sqrt{2}}$$

$$|y_{-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Probabilidades

Já temos três conjuntos completos de autoestados (autovetores) :

$$|x_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |x_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|y_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad |y_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

São três bases !

Cada uma com dois estados:

$|\chi_+\rangle \quad |\chi_-\rangle$
(ortonormais)

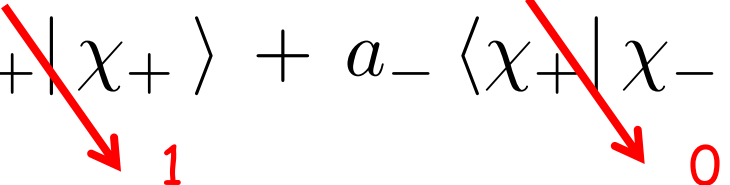
Para um estado genérico $|\chi\rangle$

$$|\chi\rangle = a_+ |\chi_+\rangle + a_- |\chi_-\rangle = \sum_i a_i |\chi_i\rangle$$

$|a_i|^2$ = probabilidade de que o estado esteja em $|\chi_i\rangle$

Para encontrar a_+ basta multiplicar o estado por $\langle\chi_+|$

$$\langle\chi_+|\chi\rangle = a_+ \langle\chi_+|\chi_+\rangle + a_- \langle\chi_+|\chi_-\rangle$$


1 0

$$\langle\chi_+|\chi\rangle = a_+$$

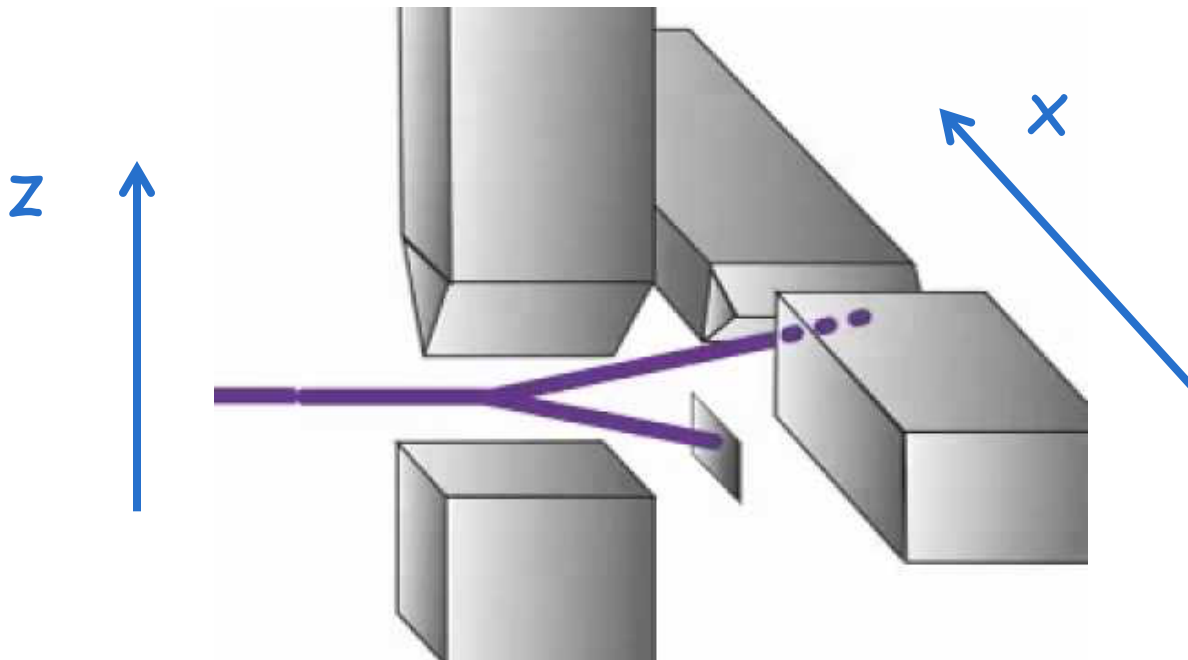
$$P_{a_+} = |a_+|^2 = |\langle\chi_+|\chi\rangle|^2$$

Valor médio de um operador

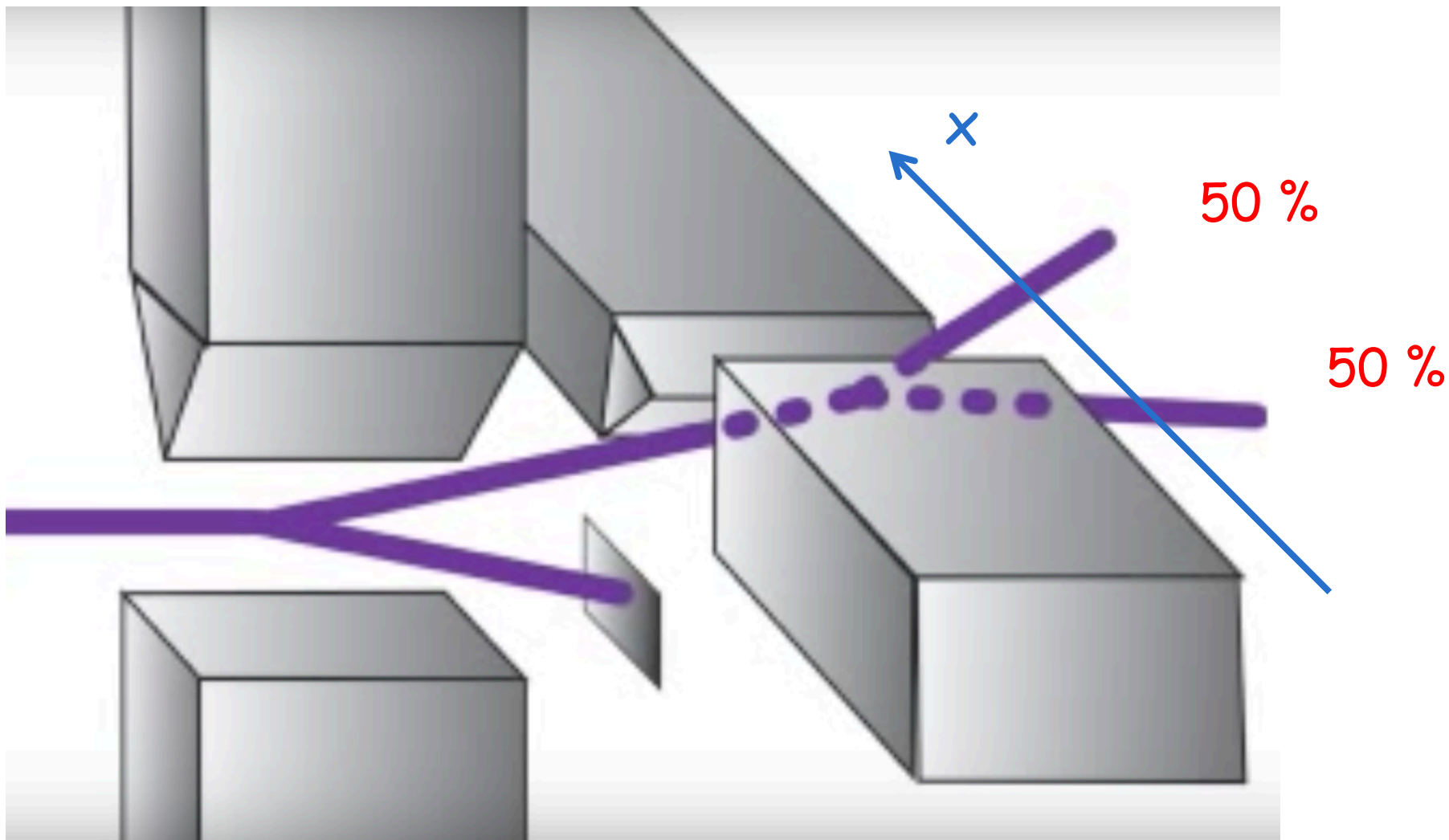
$$\langle A \rangle = \langle \chi | \hat{A} | \chi \rangle$$

Média dos valores medidos da grandeza A no estado $|\chi\rangle$

Exemplo: medidas da componente do spin na direção x quando o sistema está no estado $|\uparrow\rangle$



Na experiência vemos que :



$$\langle \hat{S}_x \rangle = \langle \uparrow | \hat{S}_x | \uparrow \rangle \quad (\text{"sanduche"})$$

$$\langle \uparrow | = (1 \quad 0) \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad | \uparrow \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\langle \hat{S}_x \rangle = \frac{\hbar}{2} (1 \quad 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

Em metade das medidas encontramos $+\frac{\hbar}{2}$

Em metade das medidas encontramos $-\frac{\hbar}{2}$

Exemplo : um elétron se encontra no estado de spin dado por

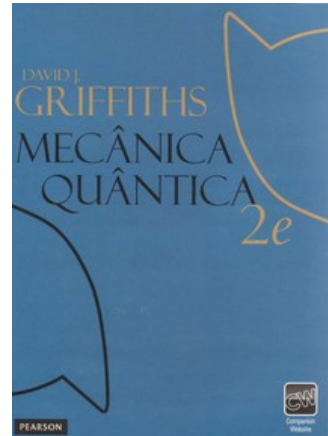
$$|\chi\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

O estado está normalizado:

$$\langle \chi | \chi \rangle = \frac{1}{6} (1-i \ 2) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{1}{6} (2+4) = 1$$

Probabilidade de medir S_x e encontrar $+\frac{\hbar}{2}$? (encontrar $|x_+\rangle$)

$$P_{x_+} = |\langle x_+ | \chi \rangle|^2$$





$$P_{x_+} = |\langle x_+ | \chi \rangle|^2$$

$$|x_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \longrightarrow \quad \langle x_+| = \frac{1}{\sqrt{2}} (1 \quad 1)$$

$$\langle x_+ | \chi \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{6}} (1 \quad 1) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{3+i}{\sqrt{12}}$$

$$P_{x_+} = \left| \frac{3+i}{\sqrt{12}} \right|^2 = \frac{5}{6}$$

Probabilidade de medir S_z e encontrar $+\frac{\hbar}{2}$? (encontrar $|\uparrow\rangle$)

- $P_{z_+} = |\langle z_+ | \chi \rangle|^2 = |\langle \uparrow | \chi \rangle|^2$

$$\langle \uparrow | \chi \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{1+i}{\sqrt{6}}$$

$$P_{z_+} = \left| \frac{1+i}{\sqrt{6}} \right|^2 = \frac{2}{6} = \frac{1}{3}$$

Probabilidade de medir S_z e encontrar $-\frac{\hbar}{2}$? (encontrar $|\downarrow\rangle$)

● $P_{z-} = |\langle z_- | \chi \rangle|^2 = |\langle \downarrow | \chi \rangle|^2$

$$\langle \downarrow | \chi \rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{2}{\sqrt{6}}$$

$$P_{z-} = \left| \frac{2}{\sqrt{6}} \right|^2 = \frac{4}{6} = \frac{2}{3}$$

Qual é o valor médio de S_z neste estado ?

$$\langle \hat{S}_z \rangle = \langle \chi | \hat{S}_z | \chi \rangle$$

$$\frac{1}{6} (1 - i \quad 2) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 + i \\ 2 \end{pmatrix} = \frac{\hbar}{12} (2 - 4) = -\frac{\hbar}{6}$$

Correção do exercício 1 da primeira prova