Dynamic structural analysis:

Response history

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Problem:

- given a known load (in time and space) obtain the structure response (in time and space)
- or, obtain
  - displacement,
  - velocity
  - acceleration

  of a loaded structure

Source: Concepts and applications of finite element analysis, RD Cook, DS Malkus, ME Plesha, RJ Witt

Response history

Modal methods:
It is necessary to solve an eigen-problem

Ritz vectors:
More efficient than modal methods

Response spectra

Component mode synthesis (substructuring)

Usually no good
for non-linear response

Response history

$$\{D\}_{n+1} = f(\{D\}_n, \{D\}_{n+1}, \{\ddot{D}\}_{n+1}, \{\dot{D}\}_n, \{\ddot{D}\}_n)$$

Direct integration methods

Implicit direct integration

algorithms

Explicit methods

$$\{D\}_{n+1} = f(\{D\}_n, \{\dot{D}\}_n, \{\ddot{D}\}_n, \{D\}_{n-1})$$

$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + \{R^{\text{int}}\}_n = \{R^{ext}\}_n$$

$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + [K]\{D\}_n = \{R^{ext}\}_n$$

- Number of multiplications per time step implicit/explicit:
    - 2 for 1D
    - 15-150 2D
    - 4000 3D
- Because in implicit, matrix becomes less narrowly banded
- Implicit requires more storage

Explicit methods

- Conditionally stable (a critical time step must not be exceeded to avoid instability)
- Matrices can be made diagonal (uncoupling)
- Low cost per time step but many steps
- Wave propagation problem:
    - Blast and impact loads
    - High modes are important
    - Response spans over small time interval

Implicit methods

- Unconditionally stable (calculation remains stable regardless of time step but accuracy may suffer)
- Matrices cannot be made diagonal (coupling)
- High cost per time step but few steps
- Structural dynamics problem:
    - Response dominated by lower modes, eg structure vibration, earthquacke
    - Response spans over many fundamental periods

Explicit direct integration

$$\{D\}_{n+1} = \{D\}_n + \Delta t\{\dot{D}\}_n + \frac{\Delta t^2}{2}\{\ddot{D}\}_n + \dots \qquad (1)$$

$$\{D\}_{n-1} = \{D\}_n - \Delta t\{\dot{D}\}_n + \frac{\Delta t^2}{2}\{\ddot{D}\}_n - \dots \qquad (2)$$

1 - 2 $\quad \{\dot{D}\}_n = \frac{1}{2\Delta t}\left(\{D\}_{n+1} - \{D\}_{n-1}\right) \rightarrow \boxed{\{D\}_{n+1} = \{D\}_{n-1} + 2\Delta t\{\dot{D}\}_n}$

1 + 2 $\quad \boxed{\{\ddot{D}\}_n = \frac{1}{\Delta t^2}\left(\{D\}_{n+1} - 2\{D\}_n + \{D\}_{n-1}\right)}$

$$\boxed{[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + \{R^{\text{int}}\}_n = \{R^{ext}\}_n}$$

$$\boxed{\left[\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\,\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\text{ext}}\}_n - \{\mathbf{R}^{\text{int}}\}_n + \frac{2}{\Delta t^2}[\mathbf{M}]\{\mathbf{D}\}_n - \left[\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{2\,\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_{n-1}}$$

Half-step central differences

$$\{\dot{\mathbf{D}}\}_{n-1/2} = \frac{1}{\Delta t}\left(\{\mathbf{D}\}_n - \{\mathbf{D}\}_{n-1}\right) \quad \text{and} \quad \{\dot{\mathbf{D}}\}_{n+1/2} = \frac{1}{\Delta t}\left(\{\mathbf{D}\}_{n+1} - \{\mathbf{D}\}_n\right)$$

$$\{\ddot{\mathbf{D}}\}_n = \frac{1}{\Delta t}\left(\{\dot{\mathbf{D}}\}_{n+1/2} - \{\dot{\mathbf{D}}\}_{n-1/2}\right) = \frac{1}{\Delta t^2}\left(\{\mathbf{D}\}_{n+1} - 2\{\mathbf{D}\}_n + \{\mathbf{D}\}_{n-1}\right)$$

$$\{\mathbf{D}\}_{n+1} = \{\mathbf{D}\}_n + \Delta t\{\dot{\mathbf{D}}\}_{n+1/2}$$

$$\{\dot{\mathbf{D}}\}_{n+1/2} = \{\dot{\mathbf{D}}\}_{n-1/2} + \Delta t\{\ddot{\mathbf{D}}\}_n$$

$$[\mathbf{M}]\{\ddot{\mathbf{D}}\}_n + [\mathbf{C}]\{\dot{\mathbf{D}}\}_{n-1/2} + \{\mathbf{R}^{\text{int}}\}_n = \{\mathbf{R}^{\text{ext}}\}_n$$

$$\frac{1}{\Delta t^2}[\mathbf{M}]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\text{ext}}\}_n - \{\mathbf{R}^{\text{int}}\}_n + \frac{1}{\Delta t^2}[\mathbf{M}]\left(\{\mathbf{D}\}_n + \Delta t\{\dot{\mathbf{D}}\}_{n-1/2}\right) - [\mathbf{C}]\{\dot{\mathbf{D}}\}_{n-1/2}$$

$$\frac{1}{2\ \Delta t}[\mathbf{C}]\left(\{\mathbf{D}\}_{n-1} - \{\mathbf{D}\}_{n+1}\right) \approx \frac{1}{\Delta t}[\mathbf{C}]\left(\{\mathbf{D}\}_{n-1} - \{\mathbf{D}\}_n\right)$$

Use this in the bottom eq. slide 6 to obtain:

$$\boxed{\frac{1}{\Delta t^2}[\mathbf{M}]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\text{ext}}\}_n - \{\mathbf{R}^{\text{int}}\}_n + \left[\frac{2}{\Delta t^2}\mathbf{M} - \frac{1}{\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_n - \left[\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{\Delta t}\mathbf{C}\right]\{\mathbf{D}\}_{n-1}}$$

THIS IS EXPLICIT: IMPLEMENT IT IN A PROGRAM

Remarks on explicit integration

- [M] being diagonal uncouples the system
- In slide 6, the system is uncoupled only if [C] is diagonal
- The critical time step in the previous equation is

$$\Delta t \le \frac{2}{\omega_{\max}}\left(\sqrt{1-\xi^2}-\xi\right)$$

$$\xi = \text{damping ratio}$$

$$\omega_{\max} = \text{largest possible "calculated" frequency}$$

- Internal forces can be calculated from either

$$\{R^{\text{int}}\}_n = [K]\{D\}_n$$

$$\{R_{\text{int}}\}_n = \sum_{i=1}^{N_{els}} \left(\{r_{\text{int}}\}_n\right)_i \qquad \{r_{\text{int}}\}_n = \int [B]^T \{\sigma\}_n \, dV$$

It is not necessary to form [K]

Minimize n. of integration points to 1: be carefull with stress calculation though!

To start the process

$$\{\ddot{D}\}_0 = \frac{1}{\Delta t/2}\left(\{\dot{D}\}_0 - \{\dot{D}\}_{-1/2}\right) \quad \rightarrow \quad \{\dot{D}\}_{-1/2} = \{\dot{D}\}_0 - \frac{\Delta t}{2}\{\ddot{D}\}_0$$

$$\{\ddot{D}\}_0 = [M]^{-1}\left(\{R^{ext}\}_0 - [K]\{D\}_0 - [C]\{\dot{D}\}_0\right)$$

$$\{D\}_{-1} = \{D\}_0 - \Delta t\{\dot{D}\}_0 + \frac{\Delta t^2}{2}\{\ddot{D}\}_0$$

Critical time step

$$\Delta t_{cr} = \frac{L_{mesh}}{c} = \frac{L_{mesh}}{\sqrt{E/\rho}} \qquad C_n = \frac{\Delta t_{actual}}{\Delta t_{cr}}$$

- consistent mass matrix gives higher frequencies than lumped mass matrix
- lumped mass increases Δt
- higher order elements have higher frquencies than lower order ones. Use the latter

Exercise I

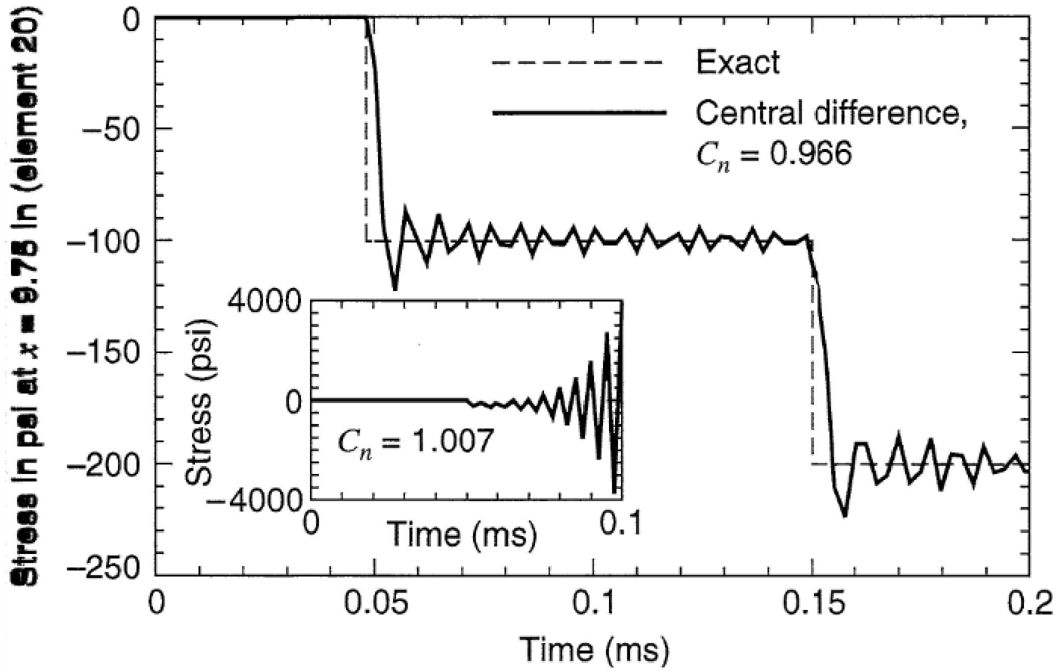


Figure 11.12-2. Axial stress versus time for a 40-element model of the bar in Fig. 11.12-1. Central difference solution with $\Delta t = 2.400(10^{-6})$ s ($C_n = 0.966$). Inset shows instability that results from too large a Δt ($\Delta t = 2.500(10^{-6})$ s, for which $C_n = 1.007$).

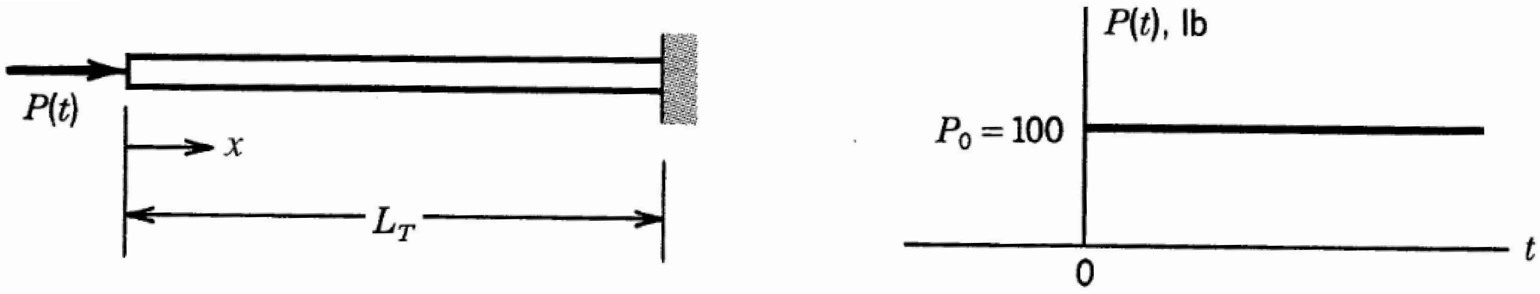


Figure 11.12-1. One-dimensional uniform bar with instantaneous axial tip loading. $A = 1.0$ in^2, $E = 30(10^6)$ psi, $\rho = 7.4(10^{-4})$ lb-s^2/in^4, $L_T = 20$ in. Load $P_0 = 100$ lb is applied at $t = 0$.

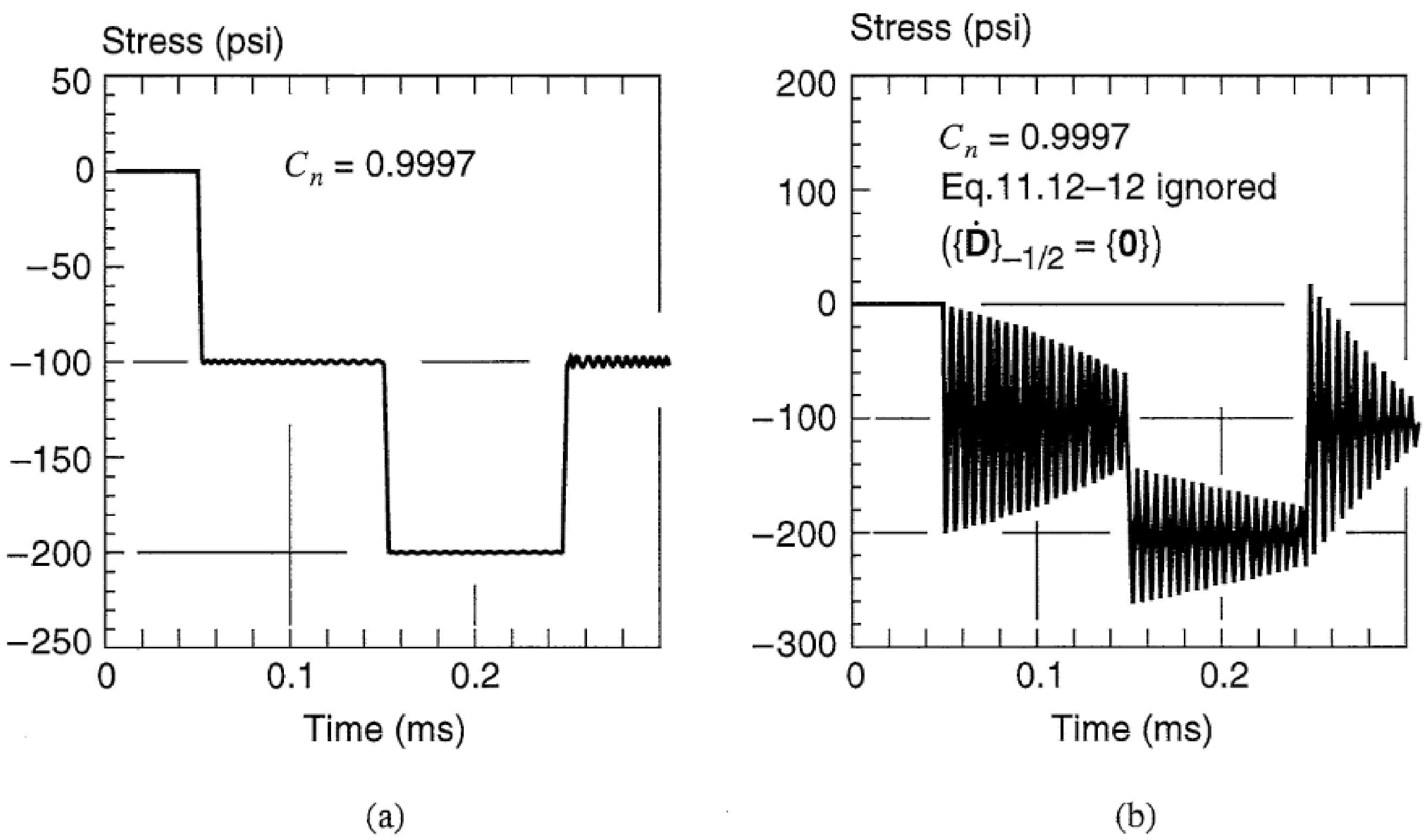


Figure 11.12-3. Central difference solutions for axial stress versus time for a 40-element model of the bar in Fig. 11.12-1. $\Delta t = 2.483(10^{-6})$ s ($C_n = 0.9997$), proper and improper initial conditions.

Implicit direct integration (Newmark family methods)

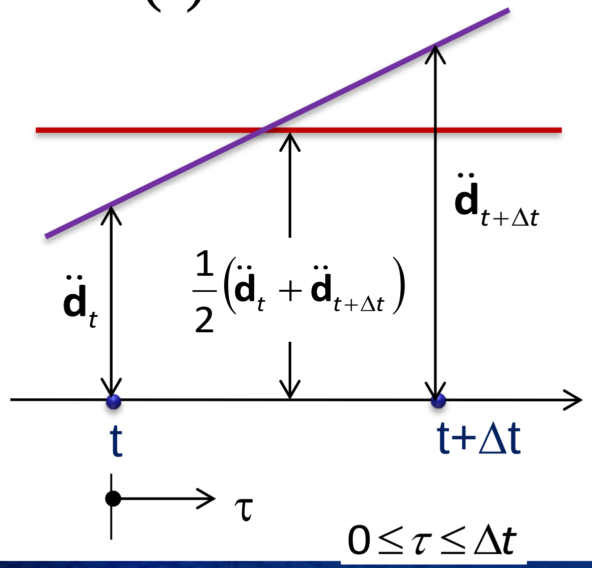
- Most of these methods are *unconditionable stable:* large time keeps the solution stable, although may compromise accuracy

$$0 \le \tau \le \Delta t$$

$$\Delta t = t_{n+1} - t_n$$

$$u = u(t)$$

Integração com aceleração cte.



Average acceleration method

$$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$\dot{u}(\tau) = \dot{u}_n + \frac{\tau}{2}(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$u(\tau) = u_n + \tau\dot{u}_n + \frac{\tau^2}{4}(\ddot{u}_{n+1} + \ddot{u}_n)$$

Linear acceleration method

$$\ddot{u}(\tau) = \ddot{u}_n + \frac{\tau}{\Delta t}(\ddot{u}_{n+1} - \ddot{u}_n)$$

$$\dot{u}(\tau) = \dot{u}_n + \tau\ddot{u}_n + \frac{\tau^2}{2\,\Delta t}(\ddot{u}_{n+1} - \ddot{u}_n)$$

$$u(\tau) = u_n + \tau\dot{u}_n + \frac{\tau^2}{2}\ddot{u}_n + \frac{\tau^3}{6\,\Delta t}(\ddot{u}_{n+1} - \ddot{u}_n)$$

Integração com aceleração linear

IMPLICIT

At step n+1 ($\tau=\Delta t$)

Average Acceleration (Eqs. 11.13-1):

$$\dot{u}_{n+1} = \dot{u}_n + \frac{1}{2}\Delta t(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$u_{n+1} = u_n + \Delta t\,\dot{u}_n + \frac{1}{4}\Delta t^2(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$u_{n+1} = u_n + \tfrac{1}{2}\Delta t(\dot{u}_{n+1} + \dot{u}_n)$$

Linear Acceleration (Eqs. 11.13-2):

$$\dot{u}_{n+1} = \dot{u}_n + \frac{1}{2}\Delta t(\ddot{u}_{n+1} + \ddot{u}_n)$$

$$u_{n+1} = u_n + \Delta t\,\dot{u}_n + \Delta t^2\left(\frac{1}{6}\ddot{u}_{n+1} + \frac{1}{3}\ddot{u}_n\right)$$

Newmark relations

$$\{\dot{\mathbf{D}}\}_{n+1} = \{\dot{\mathbf{D}}\}_n + \Delta t\left[\gamma\{\ddot{\mathbf{D}}\}_{n+1} + (1-\gamma)\{\ddot{\mathbf{D}}\}_n\right]$$

$$\{\mathbf{D}\}_{n+1} = \{\mathbf{D}\}_n + \Delta t\{\dot{\mathbf{D}}\}_n + \frac{1}{2}\Delta t^2\left[2\beta\{\ddot{\mathbf{D}}\}_{n+1} + (1-2\beta)\{\ddot{\mathbf{D}}\}_n\right]$$

$\{\ddot{D}\}_{n+1}$

TABLE 11.13-1. STABILITY AND ACCURACY OF SELECTED IMPLICIT DIRECT INTEGRATION METHODS.

Version [or references]	γ	β	Stability condition	Error in $\{\mathbf{D}\}$ for $\xi = \mathbf{0}$
		Newmark Methods		
Average acceleration	$\frac{1}{2}$	$\frac{1}{4}$	Unconditional	$O(\Delta t^2)$
Linear acceleration	$\frac{1}{2}$	$\frac{1}{6}$	$\Omega_{\text{crit}} = 3.464$ if $\xi = 0$	$O(\Delta t^2)$
Fox-Goodwin	$\frac{1}{2}$	$\frac{1}{12}$	$\Omega_{\text{crit}} = 2.449$ if $\xi = 0$	$O(\Delta t^4)$
Algorithmically damped	$\geq \frac{1}{2}$	$\geq \frac{1}{4}(\gamma+\frac{1}{2})^2$	Unconditional	$O(\Delta t)$
		Hilber-Hughes-Taylor (α-method), $-\frac{1}{3} \leq \alpha \leq 0$		
[2.13,11.55]	$\frac{1}{2}(1-2\alpha)$	$\frac{1}{4}(1-\alpha)^2$	Unconditional	$O(\Delta t^2)$

$$\{\ddot{\mathbf{D}}\}_{n+1} = \frac{1}{\beta\,\Delta t^2}\left(\{\mathbf{D}\}_{n+1} - \{\mathbf{D}\}_n - \Delta t\,\{\dot{\mathbf{D}}\}_n\right) - \left(\frac{1}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n$$

$$\{\dot{\mathbf{D}}\}_{n+1} = \frac{\gamma}{\beta\,\Delta t}\left(\{\mathbf{D}\}_{n+1} - \{\mathbf{D}\}_n\right) - \left(\frac{\gamma}{\beta} - 1\right)\{\dot{\mathbf{D}}\}_n - \Delta t\left(\frac{\gamma}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n$$

$$\boxed{[M]\{\ddot{D}\}_{n+1} + [C]\{\dot{D}\}_{n+1} + [K]\{D\}_{n+1} = \{R^{ext}\}_{n+1}}$$

$$\{\ddot{D}\}_0 = [M]^{-1}\left(\{R^{ext}\}_0 - [K]\{D\}_0 - [C]\{\dot{D}\}_0\right)$$

$$[\mathbf{K}^{\mathrm{eff}}]\{\mathbf{D}\}_{n+1} = \{\mathbf{R}^{\mathrm{ext}}\}_{n+1} + [\mathbf{M}]\left\{\frac{1}{\beta\,\Delta t^2}\{\mathbf{D}\}_n + \frac{1}{\beta\,\Delta t}\{\dot{\mathbf{D}}\}_n + \left(\frac{1}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n\right\}$$

$$+ [\mathbf{C}]\left\{\frac{\gamma}{\beta\,\Delta t}\{\mathbf{D}\}_n + \left(\frac{\gamma}{\beta} - 1\right)\{\dot{\mathbf{D}}\}_n + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)\{\ddot{\mathbf{D}}\}_n\right\}$$

$$[\mathbf{K}^{\mathrm{eff}}] = \frac{1}{\beta\,\Delta t^2}[\mathbf{M}] + \frac{\gamma}{\beta\,\Delta t}[\mathbf{C}] + [\mathbf{K}]$$

Implicit: to implement

Exercise II

1. Atribua dimensões e material realistas à ponte de pedestres abaixo e calcule suas frequências naturais e modos de vibrar. Em seguida, adicione o carregamento indicado ao nó 1 e calcule a amplitude da força para que o deslocamento do nó 2 seja L/10. Use os métodos implícitos e explícitos. Plote a resposta do nó 2 no intervalo [0..4*T*].