

Finite element for beams

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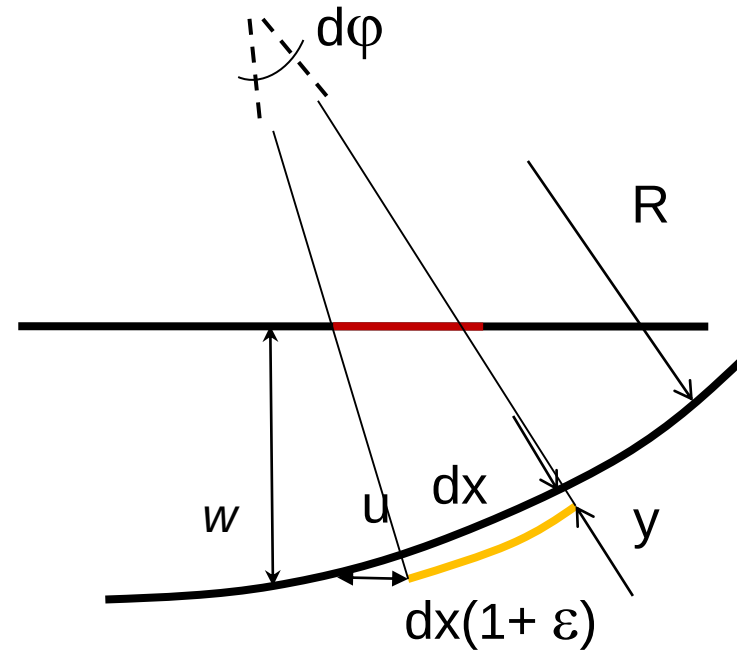
Beam theory: Kinematics

Consider an Euler-Bernoulli beam under some quasi-static distributed load

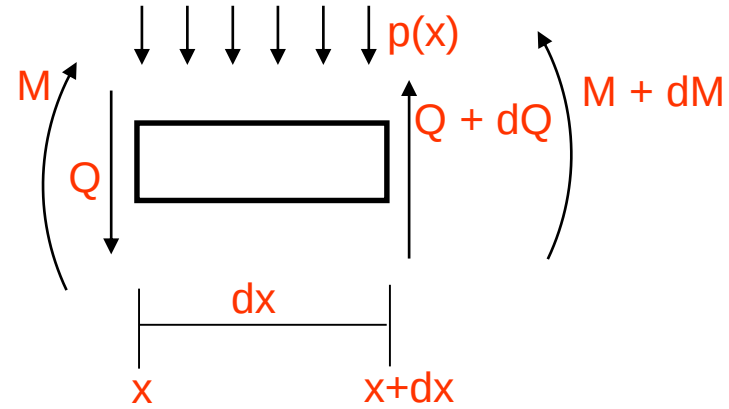
- $\sigma = 0$ in the neutral axis
- Plane sections remains plain
- Shear strain not considered
- Small strains and displacement

$$\left. \begin{aligned} dx &= R d\varphi \\ dx(1 + \varepsilon) &= (R + y) d\varphi \end{aligned} \right\} \therefore \kappa = \frac{d\varphi}{dx} = \frac{\varepsilon}{y} = -w''$$

$$\kappa = \frac{w''}{(1 + w'^2)^{3/2}}$$



Equilibrium



$$Q(x) = -\frac{dM}{dx} \quad \therefore \quad p(x) = \frac{dQ}{dx}$$

Material behaviour law

$$\sigma = E\varepsilon$$

$$M = \int_A \sigma y \, dA = \int_A E\varepsilon y \, dA = \int_A Eky^2 \, dA = EI\kappa$$

$$\therefore M = -EIw''$$

Governing equation

$$p(x) = \frac{dQ}{dx} \quad \therefore Q(x) = EIw'''$$

$$p(x) = EI \frac{\partial^4 w}{\partial x^4}$$

The solution of the *consistent* governing equations, using boundary and initial conditions, gives a complete picture of the beam behaviour.

The kinematic and static field used here can be proved to be consistent by using the principle of virtual work or the principle of virtual velocity.

Consistent set of equilibrium equation and geometrical relation for beams

$$\int_l M \kappa dx = \int_l p w dx + [Qw - Mw']$$

$$\kappa = -w''$$

$$-\int_l M w'' dx = \int_l p w dx + [Qw - Mw']$$

$$-[Mw'] + [wM'] - \int_l M'' w dx = \int_l p w dx + [Qw - Mw']$$

$$\int_l M'' + p - w dx = 0$$

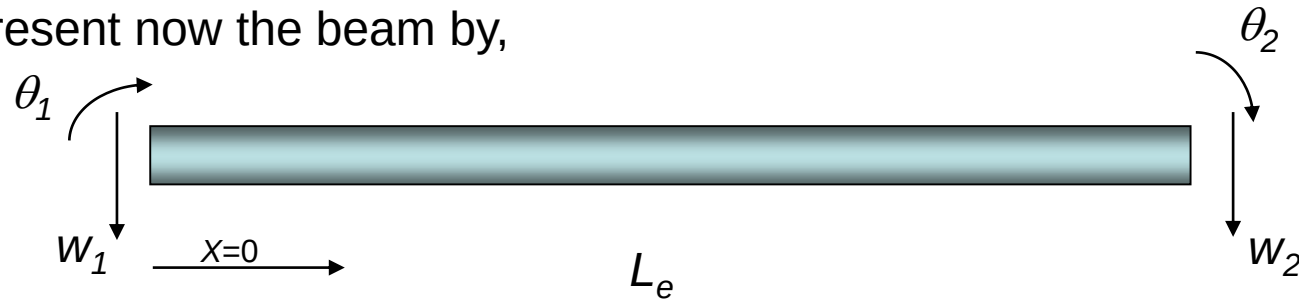
$$M'' + p = 0 \quad \text{and} \quad Q = M'$$

Discretisation of the solution

$$p(x) = EI \frac{\partial^4 w}{\partial x^4}$$

$$w = a_1 + a_2 x + a_3 x^2 + a_4 x^3 = [1 \ x \ x^2 \ x^3] [a_1 \ a_2 \ a_3 \ a_4]^T = [X] \{a\}$$

Represent now the beam by,



where w and θ are displacement and slope. Since $\theta = \frac{dw}{dx} = w'$

$$\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L_e & L_e^2 & L_e^3 \\ 0 & 1 & 2L_e & 3L_e^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [A] \{a\}$$

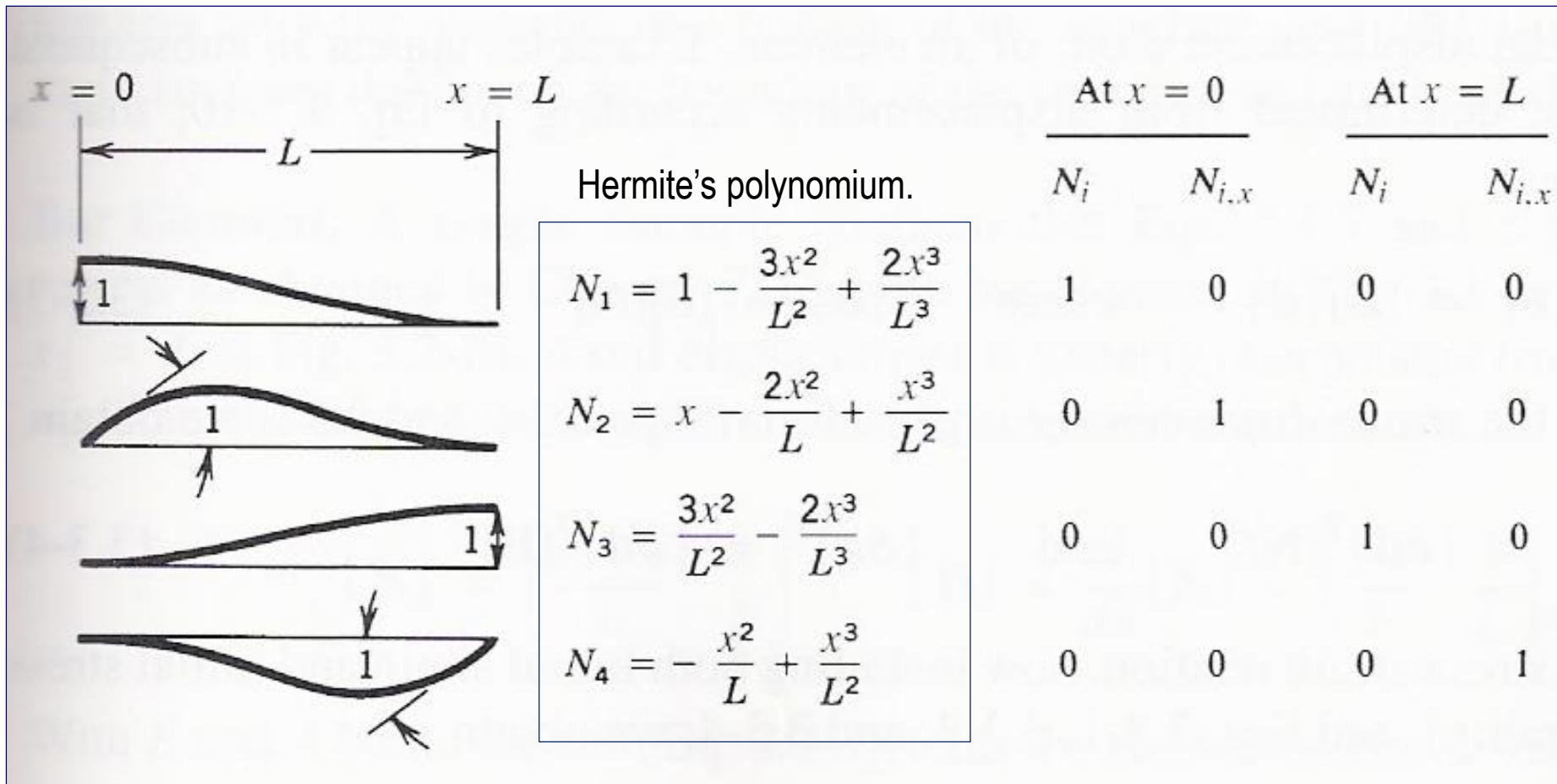
Exercise in class

$$w = [X]\{a\}$$

$$w_e = [A]\{a\}$$

$$w = [X][A]^{-1}w_e$$

$$w = [N]w_e$$



Let us now return to the beam equilibrium equation,

In class

Multiply by a function δw , which represents a virtual displacement, and integrate the product along the beam length

$$M'' + p = 0$$

$$-\int_{L_e} M'' \delta w dx = \int_{L_e} p \delta w dx$$

$$-\int_{L_e} M'' \delta w dx = \int_{L_e} p \delta w dx$$

$$1- > u = \delta w, dv = M'' dx,$$

$$2- > u = \delta w', dv = M' dx$$

$$(uv)' = u'v + uv'$$

$$\int (uv)' dx = \int u'v dx + \int uv' dx$$

$$uv = \int v du + \int u dv$$

$$\int u dv = [uv] - \int v du$$

$$\int_{L_e} M \delta \kappa dx = \int_{L_e} p \delta w dx + \left[Q \delta w - M \frac{d\delta w}{dx} \right]$$

Internal energy
due to change in
the curvature:
KINEMATICS

External work due
to distributed loads

External work due
to concentrated
loads:
STATICS

Functional (no concentrated loads)

$$\Pi = \int_{L_e} M \cdot \delta \kappa \, dx - \int_{L_e} p \cdot \delta w \, dx$$

$$\kappa = -w'' \qquad \delta \kappa = -N'' \delta w_e \qquad M = EI \kappa$$

$$\Pi = \int_{L_e} (-EI N'' w_e) \cdot (-N'' \delta w_e) \, dx - \int_{L_e} (p) \cdot (N \delta w_e) \, dx = 0$$

$$\Pi = \int_{L_e} (-N'' \delta w_e)^T (-EI N'' w_e) \, dx - \int_{L_e} (N \delta w_e)^T p \, dx = 0$$

$$\Pi = \int_{L_e} \delta w_e^T N''^T EI N'' w_e \, dx - \int_{L_e} \delta w_e^T N^T p \, dx = 0$$

$$\int_{L_e} N''^T EI N'' \, dx w_e - \int_{L_e} N^T p \, dx = 0$$

$$| \quad k_e w_e = p_e \quad \text{Compare with } p(x) = EI \frac{\partial^4 w}{\partial x^4}$$

$$(A) \cdot (B) = (B)^T (A) \quad \therefore (AB)^T = B^T A^T$$

Remarks

- The integral $\int_{L_e} N^T p \, dx$

transforms the distributed load into nodal ones.

- The differential equation

$$M'' + p = 0$$

was transformed in an algebraic system of equations.

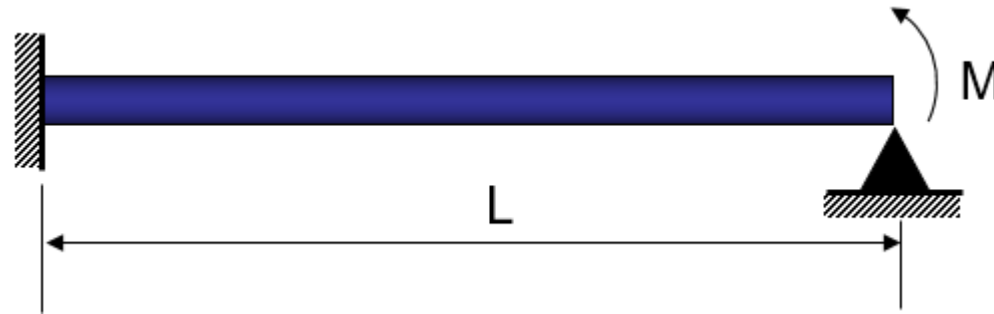
- The above transformation can be seen as the greatest advantage of FE method.

Exercise in class

$$\mathbf{k}_e = \int_{L_e} \mathbf{N}^T \mathbf{E} \mathbf{N} \, dx$$

$$\mathbf{k}_e = \frac{EI}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ \text{symm.} & 4L_e^2 & -6L_e & 2L_e^2 \\ & & 12 & -6L_e \\ & & & 4L_e^2 \end{bmatrix}$$

Let us now solve the problem below using ONE finite element.



How many FE one needs?

$$Kw = 0$$

no distributed load

$$\frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Geometrical (essential) boundary conditions $\leftrightarrow$ displacements and rotations

Natural boundary conditions $\leftrightarrow$ forces and moments

$w_1=0 \longrightarrow K_{11}=1$ e $K_{1j}=K_{j1}=0$ para $j \neq 1$

$$\frac{EI}{L^3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4L^2 & -6L & 2L^2 \\ 0 & -6L & 12 & -6L \\ 0 & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad w_1 = 0$$

The same for θ_1 e w_2 ,

$$\frac{EI}{L^3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4L^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Concentrated loads in the functional are,

$$\left[Q_w - M \frac{dw}{dx} \right] = -MN'w_e$$

so that the load vector is, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ -M \end{bmatrix}$

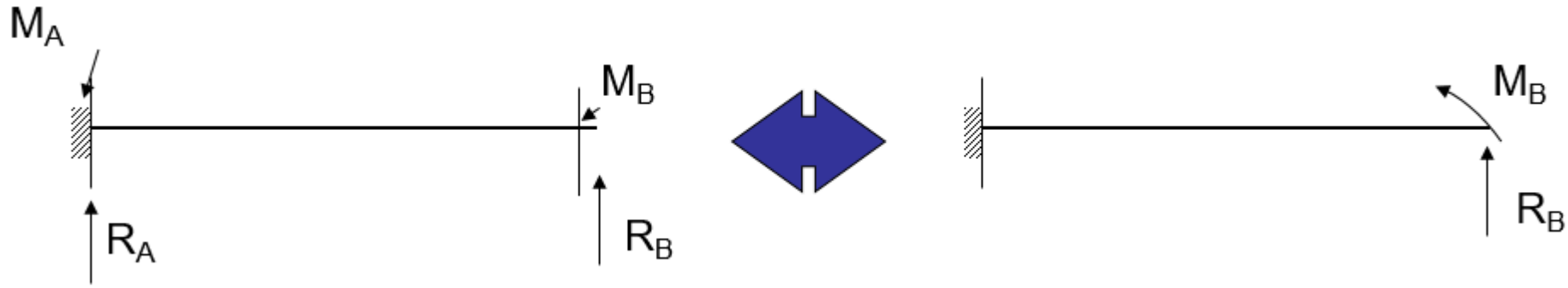
Solving for θ_2 gives,

$$\theta_2 = -\frac{ML}{4EI}$$

The final beam profile is

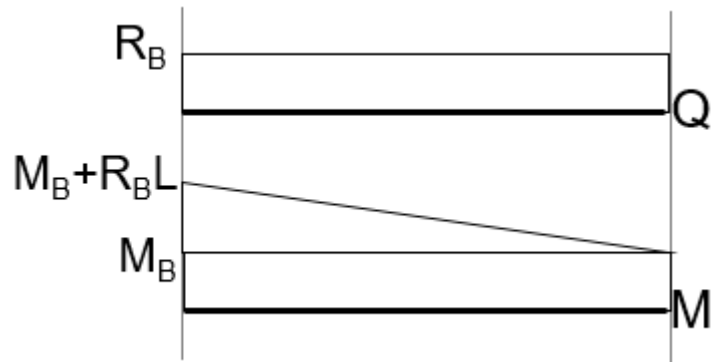
$$w = Nw_e = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_2 \end{bmatrix} = N_4 \theta_2 = \frac{M}{4EI} \left(x^2 - \frac{x^3}{L} \right)$$

Analytical solution (hyperstatic)



$$R_A = -R_B, \quad M_A = -M_B - R_B L$$

Moment-Area Method:



$$\theta_B - \theta_A = \frac{1}{EI} \left(M_B L + \frac{1}{2} R_B L^2 \right)$$

$$Q_1 = \frac{2}{3} L \frac{1}{2} \frac{R_B L^2}{EI} + \frac{L}{2} \frac{M_B L}{EI} = 0$$

$$R_B = -\frac{3M_B}{2L} \quad M_A = -M_B - R_B L = \frac{M_B}{2}$$

$$EIw^{iv} = 0$$

$$EIw = c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4$$

$$EIw'(0) = 0$$

$$c_1 = \frac{3M_B}{2L} \quad c_2 = -\frac{M_B}{2}$$

$$EIw'(L) = c_1 \frac{L^2}{2} + c_2 L = \theta_B$$

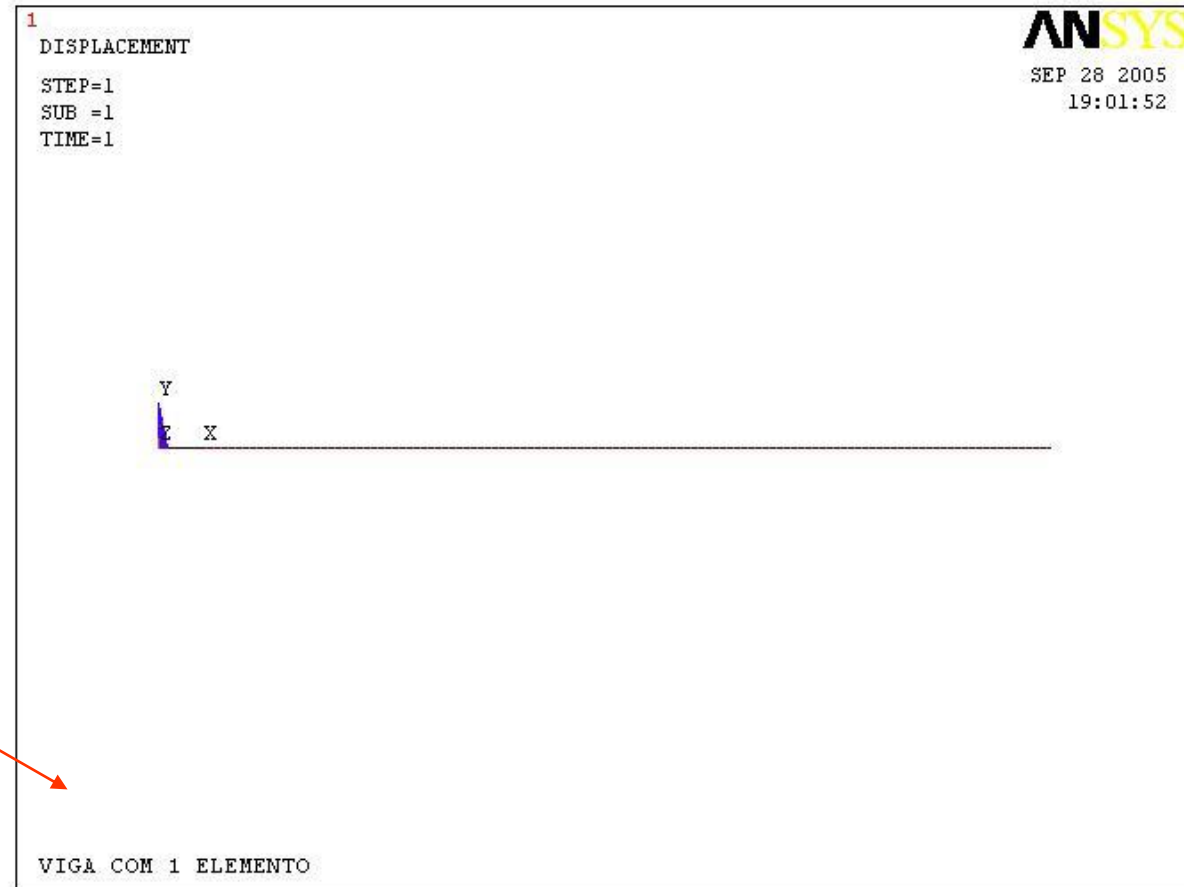
$$w = \frac{M_B}{EIL} x^2 (x - L)$$

$$w' = \theta_2 = \frac{M}{4EIL} (3x^2 - 2Lx)$$

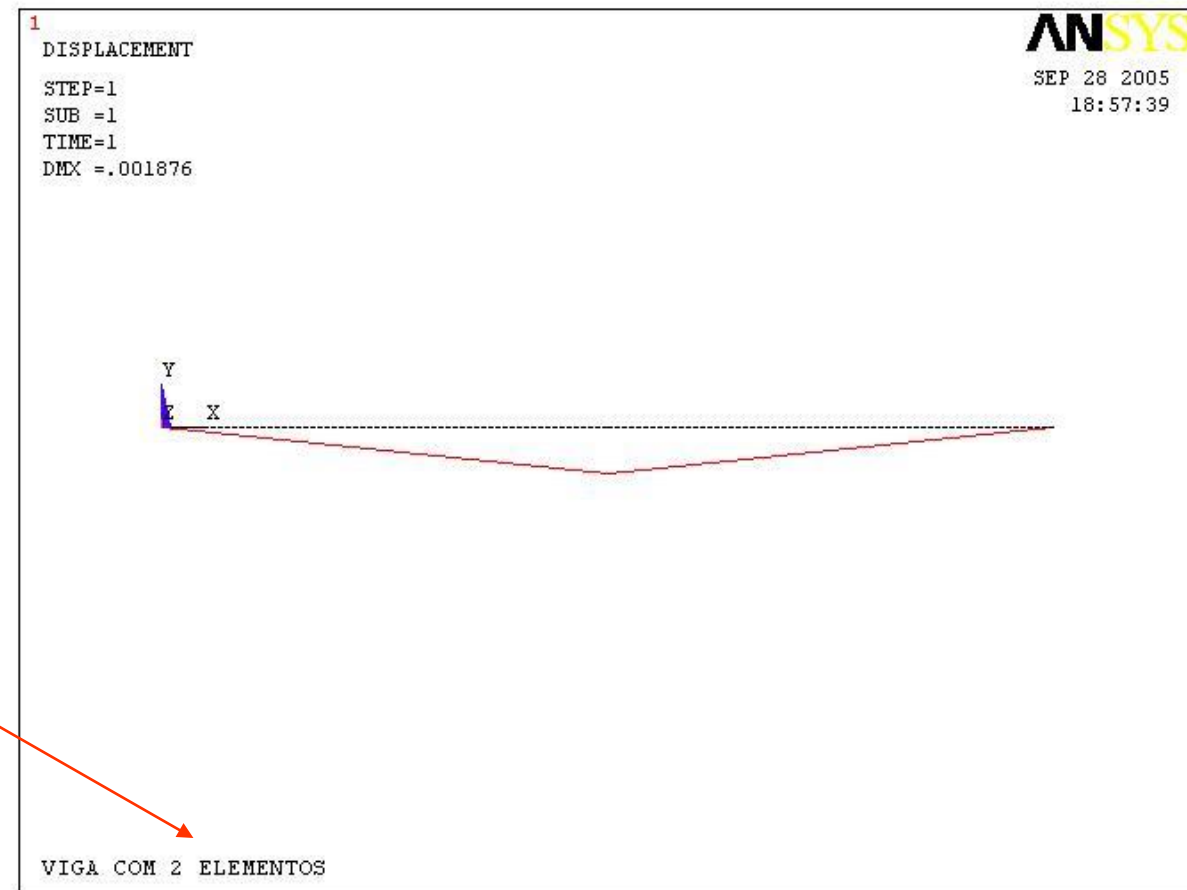
which is identical to
the FE solution

Note that we obtained the exact solution using only one element...

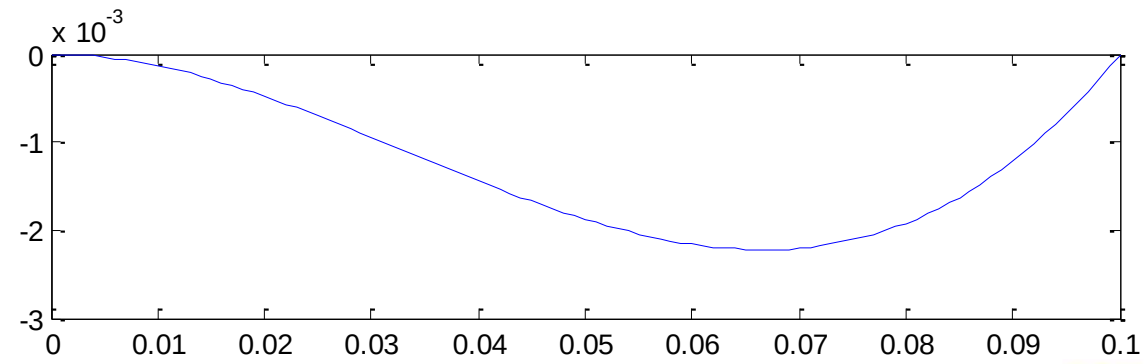
ANSYS



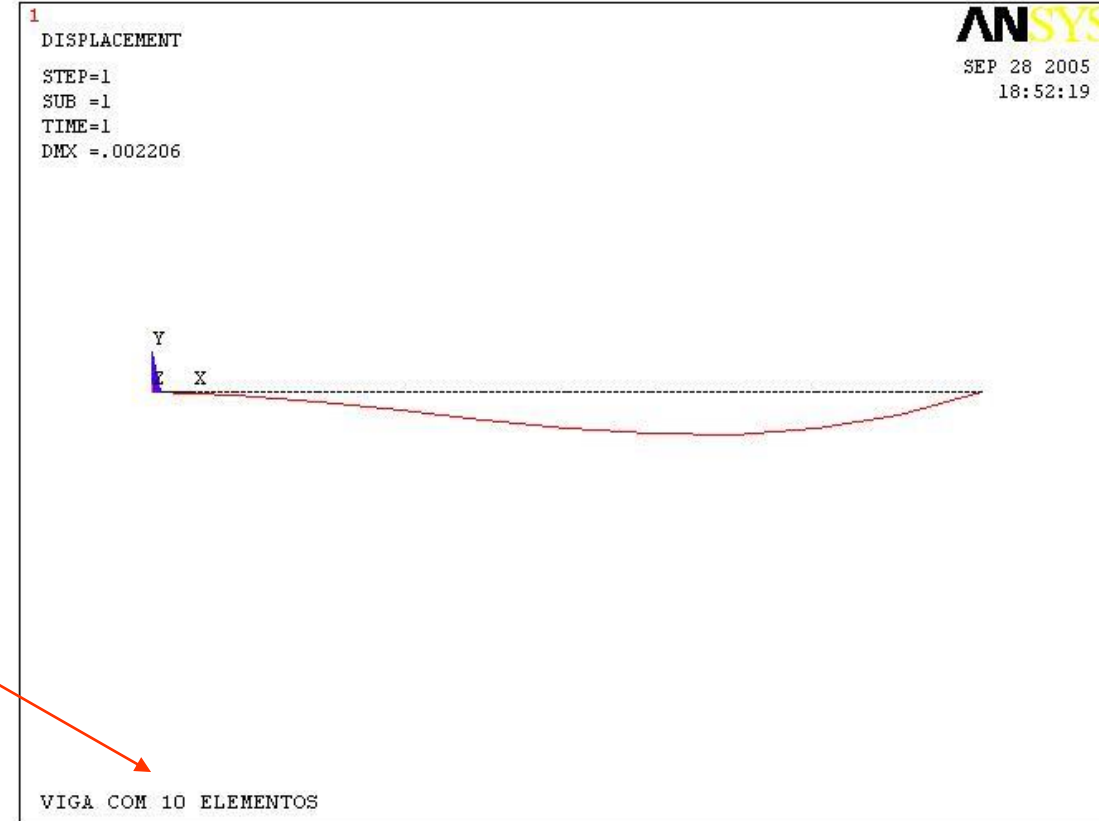
ANSYS



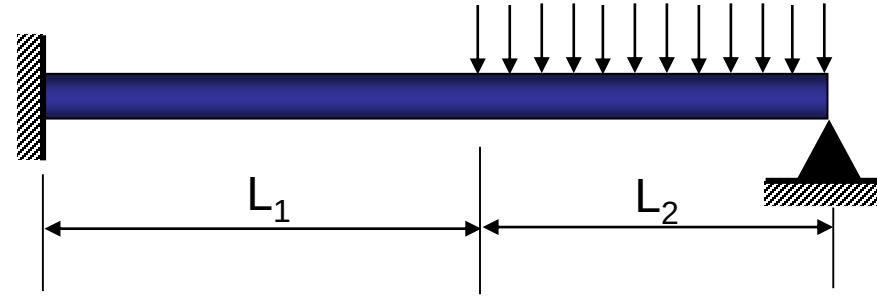
beam.m



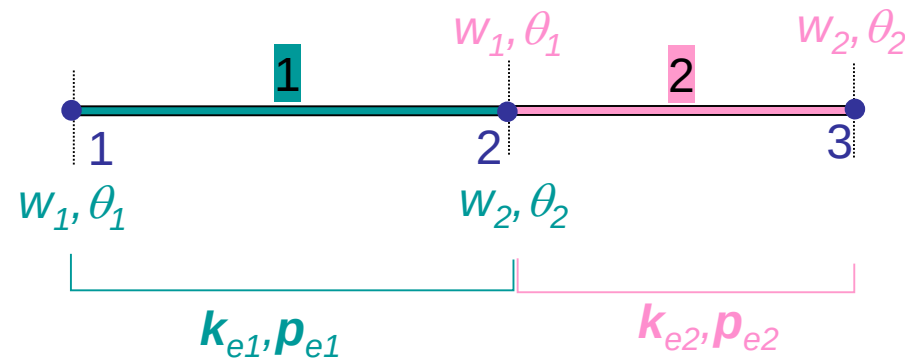
ANSYS



Consider now:



One element yields a poor solution.



In the previous example, the stiffness matrix was of order:

$$\text{DOFN} \times \text{NN} = 2 \times 2 = 4$$

$$\text{now: DOFN} \times \text{NN} = 2 \times 3 = 6$$

DOFN: degrees of freedom per node
NN: number of nodes

	1	2	3	4	5	6
1						
2						
3			X	X		
4			X	X		
5						
6						

- Element 1
- Element 2

The local d.o.f. of element 1, w_2 , θ_2 , should be added to the local d.o.f. of element 2, w_1 , θ_1

$$\mathbf{K} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 12+12 & 6L+6L & -12 & 6L \\ 6L & 2L^2 & 6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

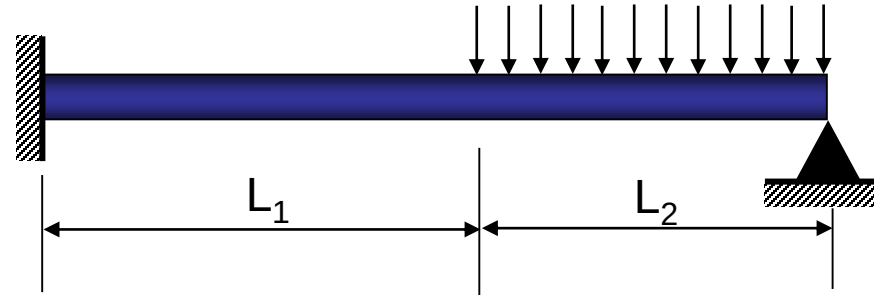
Assuming $L_1=L_2=L$ and $E_1=E_2$, $I_1=I_2$.

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \int \mathbf{N}^T \mathbf{p} \, dx = \int_0^L \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \mathbf{p} \, dx = \begin{bmatrix} pL/2 \\ pL^2/12 \\ pL/2 \\ -pL^2/12 \end{bmatrix}$$

$$\mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 + pL/2 \\ 0 + pL^2/12 \\ pL/2 \\ -pL^2/12 \end{bmatrix} = \frac{pL}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ L/6 \\ 1 \\ -L/6 \end{bmatrix}$$

Geometrical boundary conditions: $w(0)=0$, $w(L)=0$ and $w'(0)=0$

$$Kw = p$$



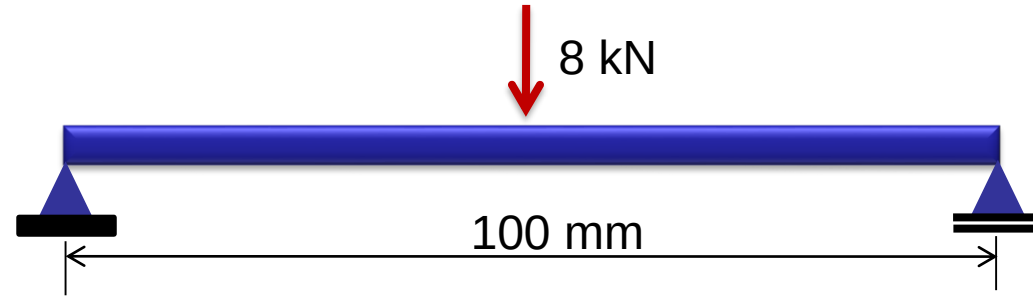
$$\frac{EI}{L^3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 12+12 & 6L+6L & -12 & 6L \\ & & & 4L^2+4L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ & & & & & 4L^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{bmatrix} = \frac{pL}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ L/6 \\ 1 \\ -L/6 \end{bmatrix}$$

$$\mathbf{w}^T = \frac{pL^4}{EI} \begin{bmatrix} 0 & 0 & \frac{L+1}{24L} & \frac{L-1}{12L^2} & \frac{7L-2}{24L} & \frac{2L-1}{6L^2} \end{bmatrix}$$

Is this correct?

Example

Obtain the mid displacement of the aluminium beam shown in the figure. Assume a diameter of 19 mm and $E=72$ GPa.



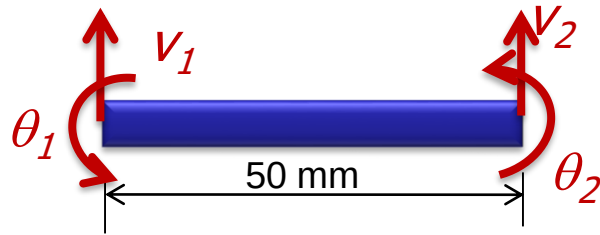
Analytical solution

$$y_{\max} = \frac{PL^3}{48EI} \quad EI = 72000 \frac{\pi 19^4}{64} = 460592433 \text{ Nmm}^2$$

$$y_{\max} = \frac{8000 \times 100^3}{48 \times 460592433} \cong 0.362 \text{ mm}$$

Solution by Finite Elements

2 elements,



$$\mathbf{K}_1 = \mathbf{K}_2 = 460592433 \begin{bmatrix} 12/50^3 & 6/50^2 & -12/50^3 & 6/50^2 \\ 6/50^2 & 4/50 & -6/50^2 & 2/50 \\ -12/50^3 & -6/50^2 & 12/50^3 & -6/50^2 \\ 6/50^2 & 2/50 & -6/50^2 & 4/50 \end{bmatrix}$$

Global stiffness matrix

Boundary conditions: $v_1=0$ e $v_3=0$

$$\mathbf{K} = \begin{bmatrix} 44216,87 & 1105422 & 44216,9 & 1105421,84 & 0 & 0 \\ 1105422 & 36847395 & -1105422 & 18423697,3 & 0 & 0 \\ -44216,9 & -1105422 & 88433,75 & 0 & -44216,9 & 1105422 \\ 1105422 & 18423697 & 0 & 73694789,3 & -1105422 & 18423697 \\ 0 & 0 & 44216,9 & 1105421,84 & 44216,87 & 1105422 \\ 0 & 0 & 1105422 & 18423697,3 & -1105422 & 36847395 \end{bmatrix}$$

Another way to
apply the BC

$$\mathbf{K}^{-1} = \begin{bmatrix} 7,24\text{E-}08 & 1,36\text{E-}06 & -9\text{E-}09 & -3,6\text{E-}08 \\ 1,36\text{E-}06 & 4,52\text{E-}05 & -1,6\text{E-}21 & -1,4\text{E-}06 \\ -9\text{E-}09 & 0 & 1,81\text{E-}08 & -9\text{E-}09 \\ -3,6\text{E-}08 & -1,4\text{E-}06 & -9\text{E-}09 & 7,24\text{E-}08 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 \\ -8000 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} -0,01086 \\ -0,36185 \\ 0 \\ 0,010856 \end{bmatrix}$$

θ_1 (points to -0,01086)
 $\mathbf{v}_2$ (points to -0,36185)
 θ_2 (points to 0)
 θ_3 (points to 0,010856)

To compare with

$$y_{\max} \cong 0.362 \text{ mm}$$

Normal force consideration: frame elements

Axial displacement:

Unidimensional-linear field

$$u = a_5 + a_6 x = [X] \{a\}$$

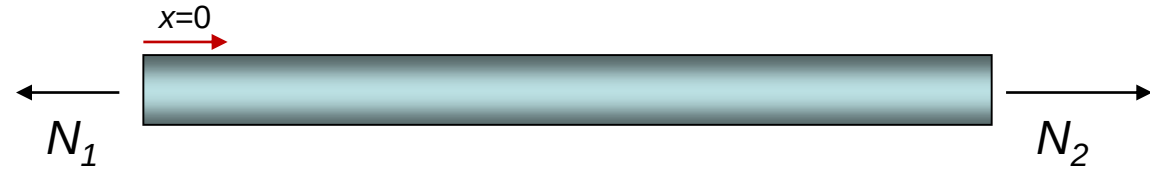
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} = [A] \{a\}$$

$$u = [X][A]^{-1} [u_1 \ u_2]^T$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix}$$

$$N = [X][A]^{-1} = [1 - x/L \quad x/L]$$

$$u = [N] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



$$\Pi = \int_0^L \sigma \varepsilon A dx$$

$$\varepsilon = \frac{\partial u}{\partial x} = [N]' \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\Pi = EA \int_0^L [u_1 \ u_2] [N]'^T [N]' [u_1 \ u_2] dx$$

$$\delta \Pi = \delta W$$

$$EA \begin{bmatrix} 1/L & -1/L \\ -1/L & 1/L \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -N_1 \\ N_2 \end{Bmatrix}$$

$$[K] \{u\} = \{F\}$$

Axial force influence on K



$$\begin{bmatrix} F_{x1} \\ F_{y1} \\ M_{z1} \\ F_{x2} \\ F_{y2} \\ M_{z2} \end{bmatrix} = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Coordinate transformation

Consider now the next figure. There, the vector v has the components

$$v_X = v \cos \beta \quad \text{and} \quad v_Y = v \sin \beta.$$

It has also the components

$$v_x = v \cos(\beta - \alpha) \quad \text{and} \quad v_y = v \sin(\beta - \alpha),$$

or

$$v_x = v(\cos \beta \cos \alpha + \sin \beta \sin \alpha)$$

and

$$v_y = v(\sin \beta \cos \alpha - \cos \beta \sin \alpha),$$

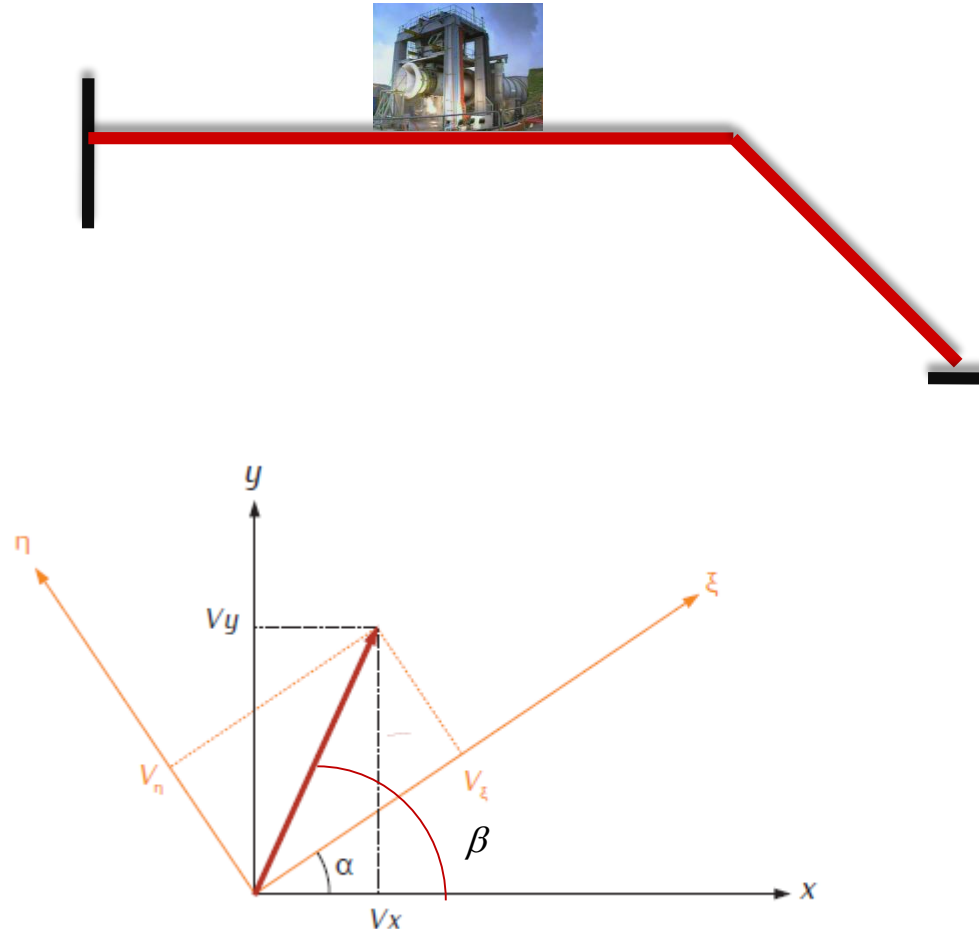
or, in matrix form,

$$\{v_{xy}\} = [T]\{v_{XY}\} \quad \text{and} \quad \{v_{XY}\} = [T]^{-1}\{v_{xy}\},$$

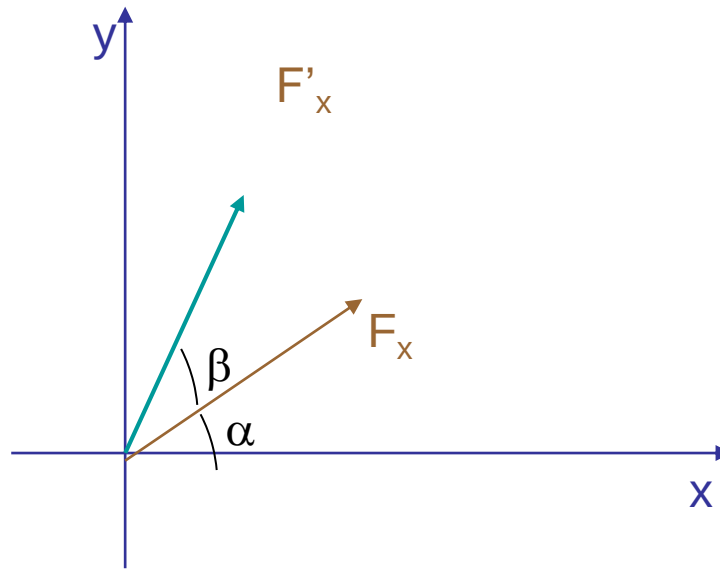
with

$$T = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}.$$

By inspection, $[T]^{-1} = [T]^T$, *ie* the coordinate transformation matrix $[T]$ is orthogonal.



A vector represented by two different coordinate systems.



$$F_x = F \cos \alpha$$

$$F_y = F \sin \alpha$$

$$F_x' = F \cos(\alpha + \beta) = F \cos \alpha \cos \beta - F \sin \alpha \sin \beta = F_x \cos \beta - F_y \sin \beta$$

$$F_y' = F \sin(\alpha + \beta) = F \sin \alpha \cos \beta + F \cos \alpha \sin \beta = F_x \sin \beta + F_y \cos \beta$$

$$\begin{bmatrix} F_x' \\ F_y' \end{bmatrix} = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\vec{i} \cdot \vec{j}' = |\vec{i}| |\vec{j}'| \cos(90^\circ + \alpha) = -\sin \alpha$$

$$\vec{i} \cdot \vec{i}' = |\vec{i}| |\vec{i}'| \cos \alpha = \cos \alpha$$

$$\vec{v} = v_x \vec{i} + v_y \vec{j} = v_\xi \vec{i}' + v_\eta \vec{j}'$$

$$\vec{i} \cdot \vec{j}'$$

$$-v_x \sin \alpha + v_y \cos \alpha = v_\eta$$

$$v_x \vec{i} \cdot \vec{i}' + v_y \vec{j} \cdot \vec{i}' = v_\xi \vec{i} \cdot \vec{i}' + v_\eta \vec{j} \cdot \vec{i}'$$

$$\vec{i} \cdot \vec{i}'$$

$$v_x \cos \alpha + v_y \sin \alpha = v_\xi$$

To transform from local to global coordinate system,

$$Kw = p \qquad K'w' = p'$$

$$Kw = T^T p'$$

$$Kw = T^T K'w'$$

$$Kw = T^T K' T w$$

$$K = T^T K' T$$

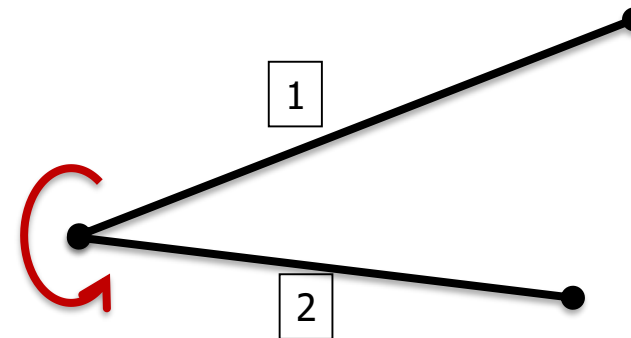
$$\mathbf{k} = \begin{bmatrix} a\lambda^2 + 12b\mu^2 & (a-12b)\lambda\mu & -6bL\mu & 6bL\lambda & 4bL^2 & 0 & 0 \\ 0 & a\mu^2 + 12b\lambda^2 & -6bL\lambda & 6bL\mu & 2bL^2 & 0 & 0 \\ 0 & 0 & -a\lambda^2 - 12b\mu^2 & -(a-12b)\lambda\mu & -6bL\lambda & -6bL\mu & -6bL^2 \\ 0 & 0 & -(a-12b)\lambda\mu & -a\mu^2 - 12b\lambda^2 & 6bL\mu & 6bL\lambda & 4bL^2 \\ 0 & 0 & 6bL\lambda & 6bL\mu & -a\lambda^2 - 12b\mu^2 & -(a-12b)\lambda\mu & -6bL\lambda \\ 0 & 0 & -6bL\mu & -6bL\lambda & -(a-12b)\lambda\mu & -a\mu^2 - 12b\lambda^2 & -6bL\mu \\ 0 & 0 & 4bL^2 & 2bL^2 & -6bL\lambda & -6bL\mu & -a\lambda^2 - 12b\mu^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$$a = \frac{EA}{L} \quad b = \frac{EI}{L^3}$$

$$\lambda = \cos \alpha \quad \mu = \sin \alpha$$

For a 3D portic [spacial portic]

- 12 degrees of freedom
- It should consider:
 - Bending
 - Axial forces
 - Torsion



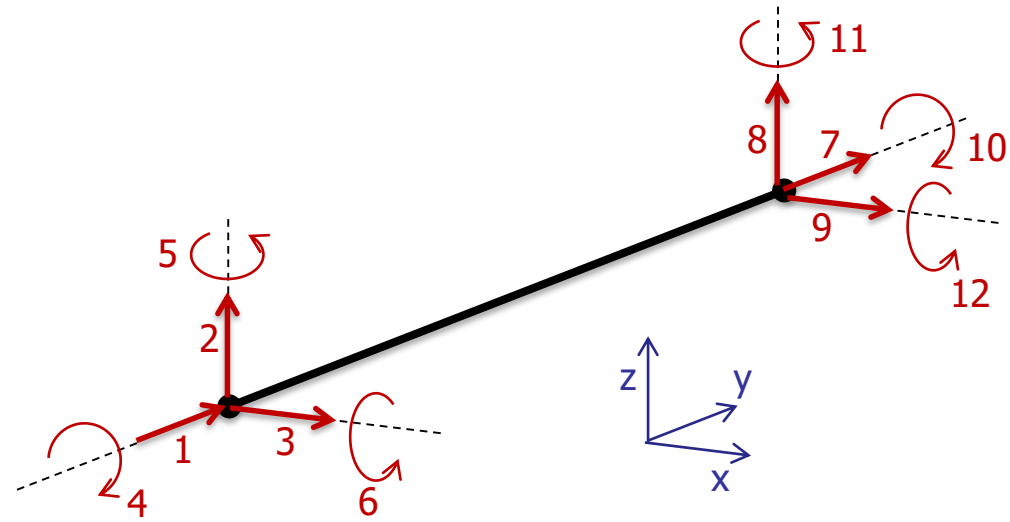
Bending in bar 1 yields
torsion in bar 2

Stiffness matrix for torsion

It is similar to the truss matrix:

- the later working in compression or traction, EA/L
- the former in torsion, GJ/L
- G is the shear modulus and J is the polar moment of inertia

$$\mathbf{k}_{torsional} = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$a = \frac{EA}{L} \quad b_y = \frac{EI_y}{L^3} \quad b_z = \frac{EI_z}{L^3}$$

$$t = \frac{GJ}{L}$$

$$\mathbf{k} = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 12b_z & 0 & 0 & 0 & 0 & 0 & -12b_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 12b_y & 0 & 0 & 0 & 0 & 0 & -12b_y & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & -6b_yL & 0 & 4b_yL^2 & 0 & 0 & 0 & -6b_yL & 0 & 2b_yL^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4b_zL^2 & 0 & 0 & 0 & 0 & 0 & 2b_zL^2 \\ -a & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 12b_z & 0 & 0 & 0 & 0 & 0 & 12b_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 12b_y & 0 & 0 & 0 & 0 & 0 & 12b_y & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & -6b_yL & 0 & 2b_yL^2 & 0 & 0 & 0 & 6b_yL & 0 & 4b_yL^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2b_zL^2 & 0 & -6b_zL & 0 & 0 & 0 & 4b_zL^2 \end{bmatrix}$$

symmetric

Timoshenko beams

- For short beams
- Shear cannot be disregarded

$$a = \frac{EA}{L} \quad b_y = \frac{EI_y}{(1 + \phi_z)L^3} \quad b_z = \frac{EI_z}{(1 + \phi_y)L^3}$$

$$t = \frac{GJ}{L} \quad \phi_y = \frac{12EI_z}{GA_{sy}L^2}$$

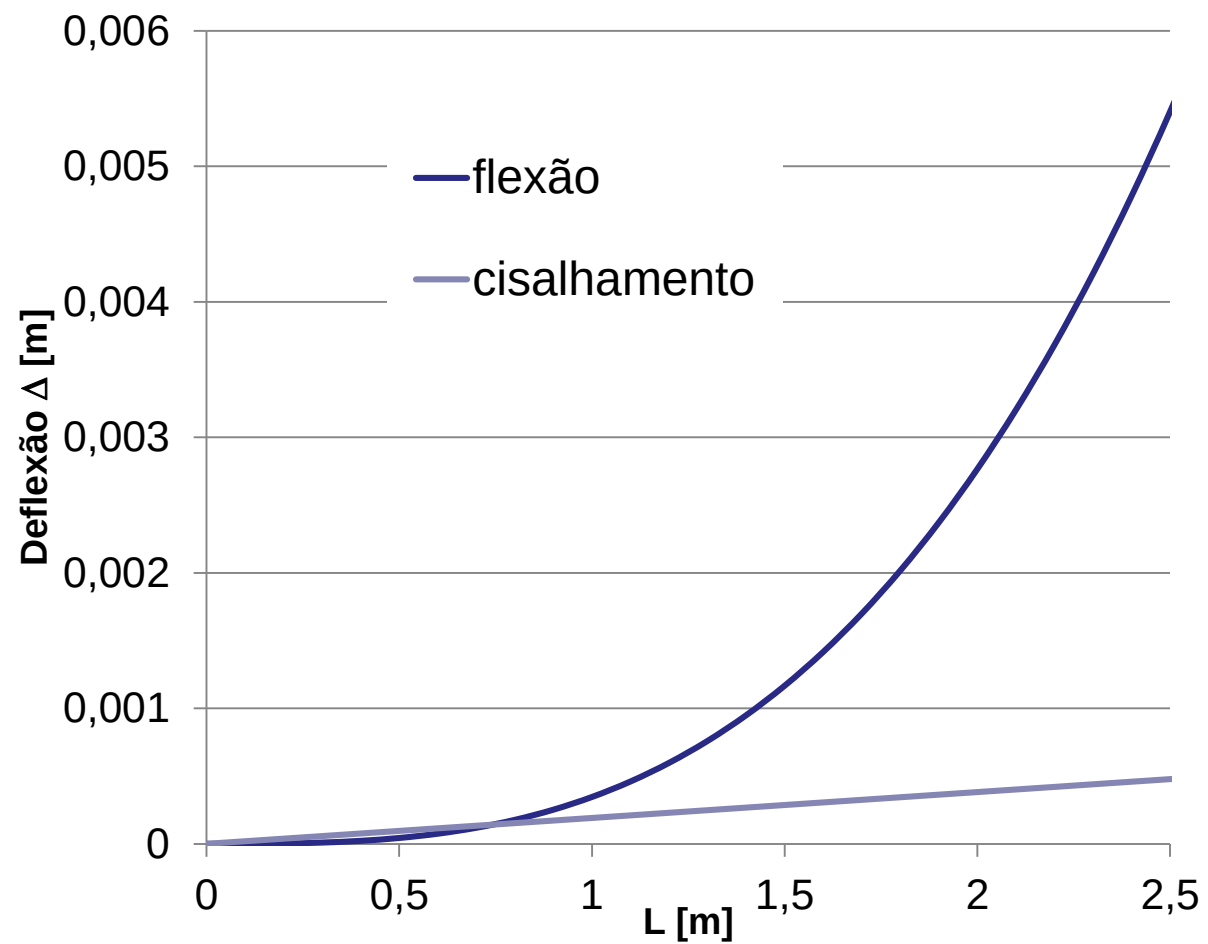
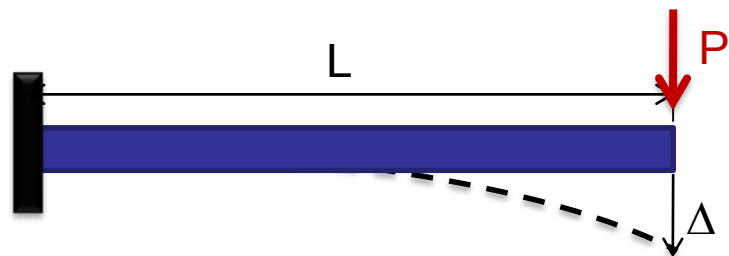
$$\phi_z = \frac{12EI_y}{GA_{sz}L^2}$$

$$\mathbf{k} = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12b_z & 0 & 0 & 0 & 0 & 0 & -12b_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12b_y & 0 & 0 & 0 & 0 & 0 & -12b_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & -t & 0 & 0 & 0 \\ 0 & 0 & -6b_yL & 0 & (4 + \phi_z)b_yL^2 & 0 & 0 & 0 & -6b_yL & 0 & (2 - \phi_z)b_yL^2 & 0 & 0 \\ 0 & 6b_zL & 0 & 0 & 0 & (4 + \phi_y)b_zL^2 & 0 & 0 & 0 & 0 & 0 & (2 - \phi_y)b_zL^2 & 0 \\ -a & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12b_z & 0 & 0 & 0 & -6b_zL & 0 & 12b_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12b_y & 0 & -6b_yL & 0 & 0 & 0 & 12b_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & -6b_yL & 0 & (2 - \phi_z)b_yL^2 & 0 & 0 & 0 & 6b_yL & 0 & (4 + \phi_z)b_yL^2 & 0 & 0 \\ 0 & 6b_zL & 0 & 0 & 0 & (2 - \phi_y)b_zL^2 & 0 & -6b_zL & 0 & 0 & 0 & (4 + \phi_y)b_zL^2 & 0 \end{bmatrix}$$

ϕ_y for different height to length beam ratios

h/L	0.05	0.1	0.2	0.3	0.4	0.5	0.6
ϕ_y	0.0065	0.026	0.104	0.234	0.416	0.65	0.936
$1+\phi_y$	1.0065	1.026	1.104	1.234	1.416	1.65	1.936
$1/(1+\phi_y)$	0.994	0.975	0.906	0.810	0.706	0.606	0.517

From
Elementos Finitos: A base da tecnologia CAE
Avelino Alves Filho
5ª edição – Editora Érica



Programming the Finite Element Method

- Fortran
- C++
- Mathematica
- Mapple
- Matlab
- Scilab

FE Programme in MatLab for beams

```
% A FE programme for beams
clear all
clc
format long
rho=7800;      % density
b=0.0283;     % beam depth
h=0.0012;     % beam height
A=b*h;        % area
E=200e9;      % elastic modulus
I=b*h^3/12;   % inertia
L=0.55;       % beam length
nel=3;        % number of elements
nno=nel+1;    % node numbers
dofg=3*nno;   % degrees of freedom
le=L/nel;     % element length  ???
```

Beam material and
geometry properties:
Pre-processor

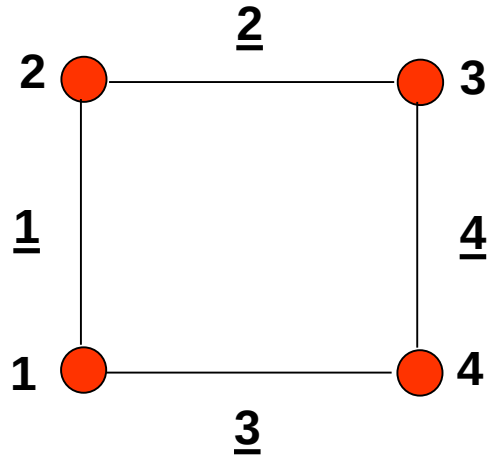
```
%x,y global coordinates
gc(1,1)=0;
for i=2:nno
    gc(i,1)=gc(i-1,1)+le; % x
    gc(i,2)=0;           % y
end
```

Nodes coordinates: 1D

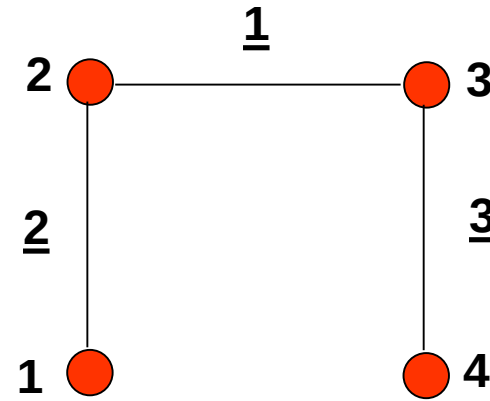
```
%connective matrix for sequential numbering
%row n stores nodes of element n
cm(1,1)=1;    cm(1,2)=2;
if nel>1
    for i=2:nel
        cm(i,1)=cm(i-1,2);
        cm(i,2)=cm(i,1)+1;
    end
end
```

Connective matrix:
in line elements

Example of connective matrixes



$$\mathbf{cm} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \\ 3 & 4 \end{bmatrix}$$



$$\mathbf{cm} = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

```
%nodal forces: fx,fy,mz
%f(nno,mz)=moment value
f=zeros(nno,3);
f(1,2)=-5000;
f(2,2)=-5000;
f(1,3)=-100000;
```

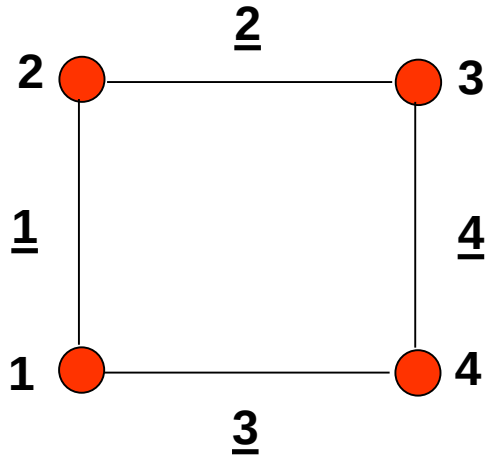
Load matrix

```
%assembly of the local stiffness matrix
k_global=zeros(dofg,dofg);
for el_i=1:nel
    nd(1)=cm(el_i,1);%node 1 of element el_1
    nd(2)=cm(el_i,2);%node 2 of element el_1
    x1=gc(nd(1),1);%x global node 1 of element el_i
    .....
    c1=b*h*Em/le(el_i);
    c2=Em*b*h^3/12/le(el_i)^3;
    k_e(1,1)=c1;          k_e(1,4)=-c1;
    k_e(2,2)=12*c2;       k_e(2,3)=6*c2*le(el_i);
    .....
    k_e=T'*k_e*T;
```

Local stiffness matrix

The connective matrix should be used to put the local stiffness elements in the right position in the global stiffness matrix

Local to global matrix

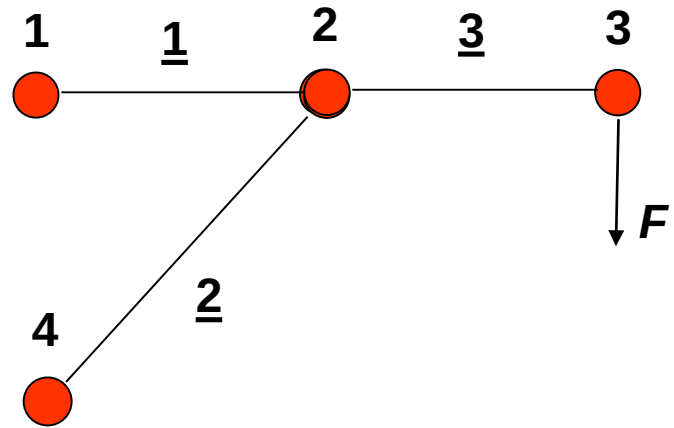


$$\mathbf{cm} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 4 \\ 3 & 4 \end{bmatrix}$$

Loop n_element:

- for each element find its position in the global matrix using cm
- add this contribution, $K_g = K_g + K_{aux}$

Example of assembling



$$\mathbf{cm} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{matrix} \underline{1} \\ \underline{2} \\ \underline{3} \end{matrix}$$

$$\mathbf{K} = \begin{matrix} & \begin{matrix} u_1 & \theta_1 & u_2 & \theta_2 & u_3 & \theta_3 & u_4 & \theta_4 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 123 & 123 & 3 & 3 \\ 1 & 1 & 123 & 123 & 3 & 3 \\ & & 3 & 3 & 3 & 3 \\ & & 3 & 3 & 3 & 3 \\ & & & & ? & & 2 & 2 \\ & & & & & & 2 & 2 \end{bmatrix} \end{matrix}$$

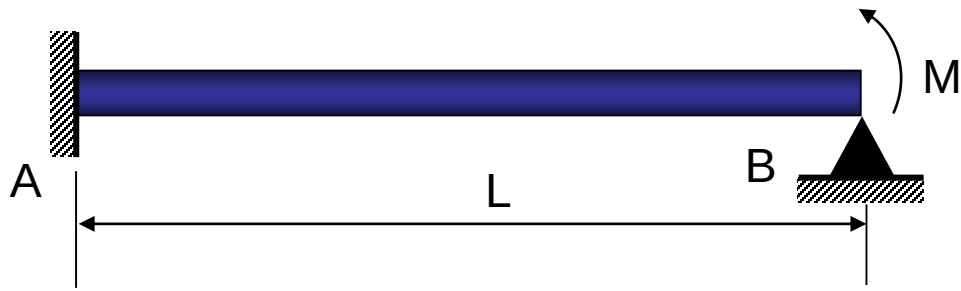

```
%boundary conditions
for i=1:dofg
    if bca(i)==1
        k_global(i,:)=0;
        ....
    end
end
%solving for displacement
u=k_global\fa;
```

Apply bc and
solve the system

Explore the results, eg
by plotting the deformed profile and the
bending moment distribution

How to calculate M, N and Q

- Use each local stiffness matrix and their corresponding displacement and rotation and operate $F = k x$
- Note that the 1 and 0 used in BC are now not retained
- Add now the nodal loads
- Be aware that one node may have the contribution of two spans
- Bear in mind the sign convention for internal forces and reactions



$$\frac{EI}{L^3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4L^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -M \end{bmatrix}$$

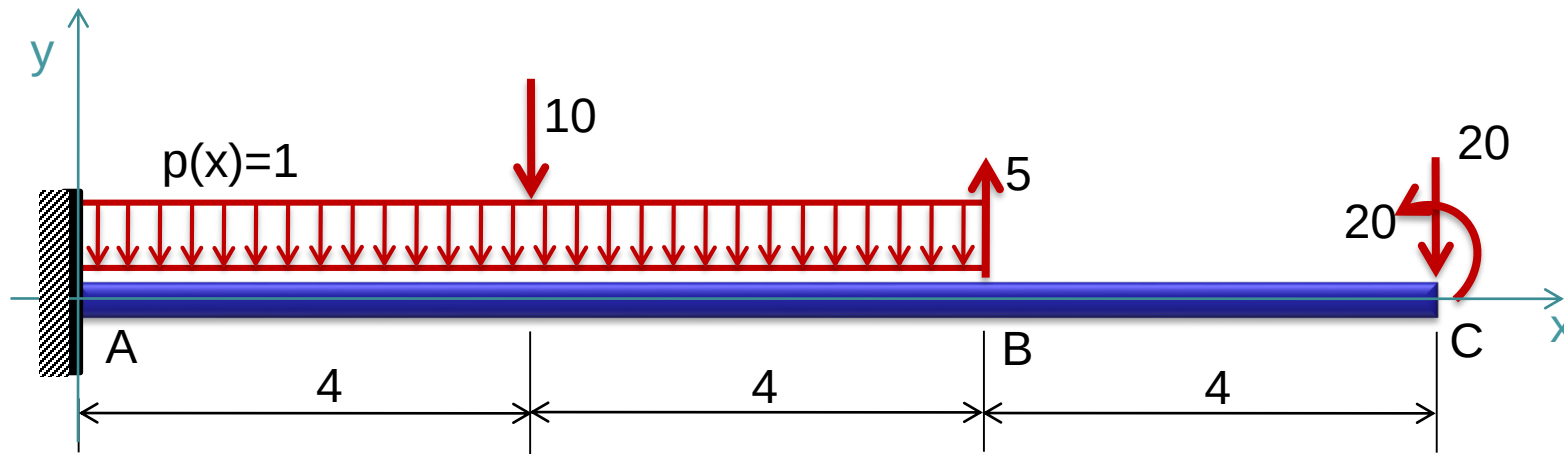
- Are the reaction signals corrected when compared to the analytical solution?
- How about the calculation of the reactions in a node belonging to two elements?
- How can the transverse shear load be calculated?

$$\frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ & & & 4L^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{ML}{4EI} \end{bmatrix} = \begin{bmatrix} R_A \\ M_A \\ R_B \\ -M \end{bmatrix}$$

$$R_A = -\frac{3M}{2L}, \quad R_B = \frac{3M}{2L}, \quad M_A = -\frac{M}{2}$$

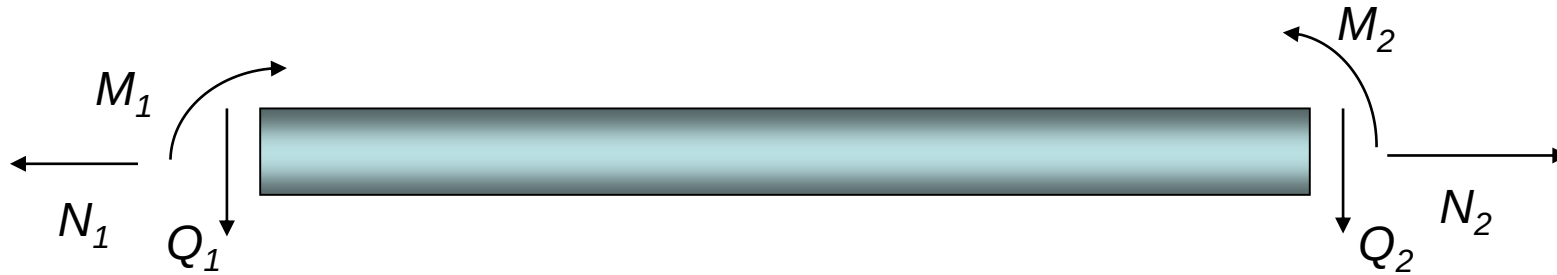
Exercises: In a report form!

1. Derive in details, using PVW, the stiffness matrix for a truss element
2. Derive in details, using PVW, the weak form of a 2D beam governing equation (M, N, T)
3. Evaluate “by hand”, adopting a FE approach, the support reactions and the transverse displacement, bending moment and transverse shear force for the beam shown in the figure. Adopt $EI=10^4$ N.m², with the variables in m and N.



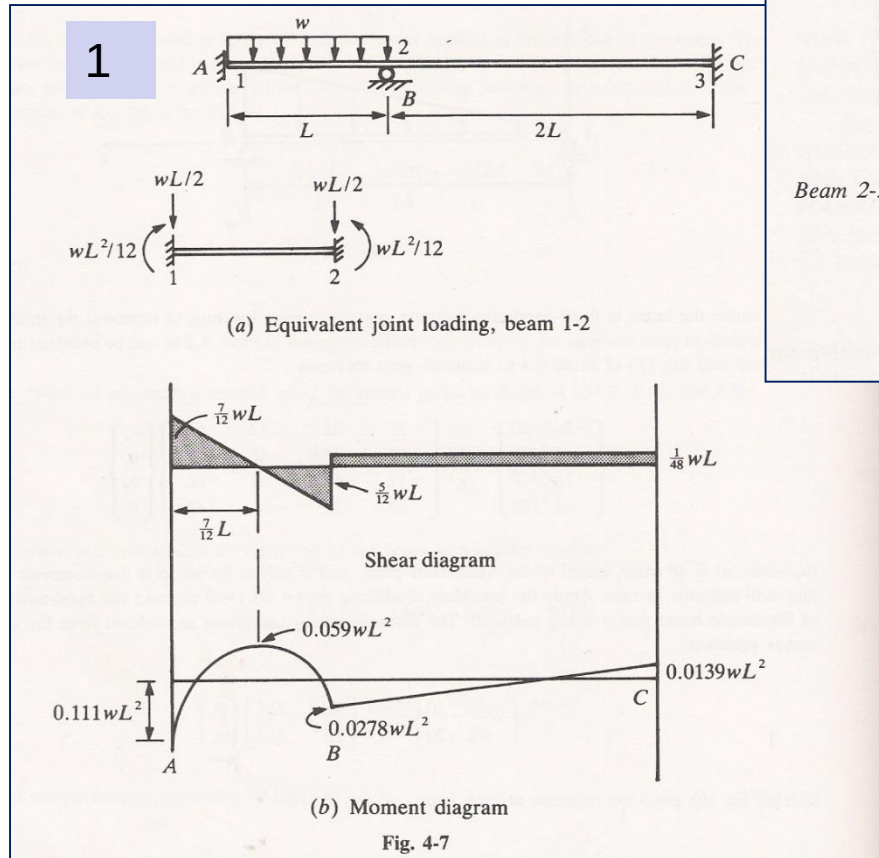
Exercises:

4. Write a FE programme to analyse a set of beams [portic] in the plane x-y.
Consider the presence of normal forces.



5. Use your programme to solve the next problems; answers below (by G R Buchanan).
6. Plot the displacement, transverse shear, bending moment along the beams for these problems.

Exemplos: (Finite Element Analysis, G R Buchanan)

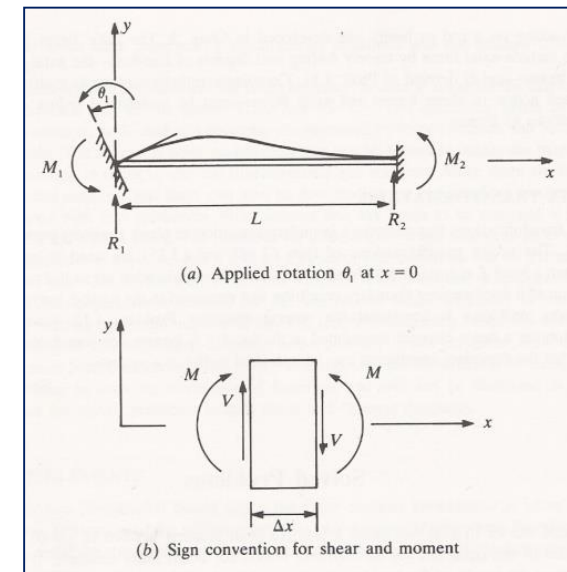


Beam 1-2:

$$\begin{Bmatrix} -wL/2 \\ -wL^2/12 \\ -wL/2 \\ wL^2/12 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \quad (a)$$

Beam 2-3:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{(2L)^3} \begin{bmatrix} 12 & 6(2L) & -12 & 6(2L) \\ 6(2L) & 4(2L)^2 & -6(2L) & 2(2L)^2 \\ -12 & -6(2L) & 12 & -6(2L) \\ 6(2L) & 2(2L)^2 & -6(2L) & 4(2L)^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} \quad (b)$$



The results for each span must be computed individually using the local stiffness matrix for that span. The reactions for span 1-2 are V_A , V_B , M_A , and M_B . The stiffness matrix is the same as Eq. (a), and the computation appears as

$$\begin{Bmatrix} V_A \\ M_A \\ V_{B1} \\ M_B \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ wL^3/72EI \end{Bmatrix} + \begin{Bmatrix} wL/2 \\ wL^2/12 \\ wL/2 \\ -wL^2/12 \end{Bmatrix} \quad (e)$$

Multiplication gives

$$V_A = R_A = \frac{7wL}{12} \quad V_{B1} = \frac{5wL}{12} \quad M_A = \frac{wL^2}{9} \quad M_B = -\frac{wL^2}{36}$$

The notation V_{B1} requires some additional explanation. The total reaction at support B is composed of the end shear at point 2 of beam 1-2 plus the end shear at point 2 of beam 2-3. Then, $R_B = V_{B1} + V_{B2}$.

Similarly, the stiffness matrix of Eq. (b) is used to compute the shears and moments for beam 2-3.

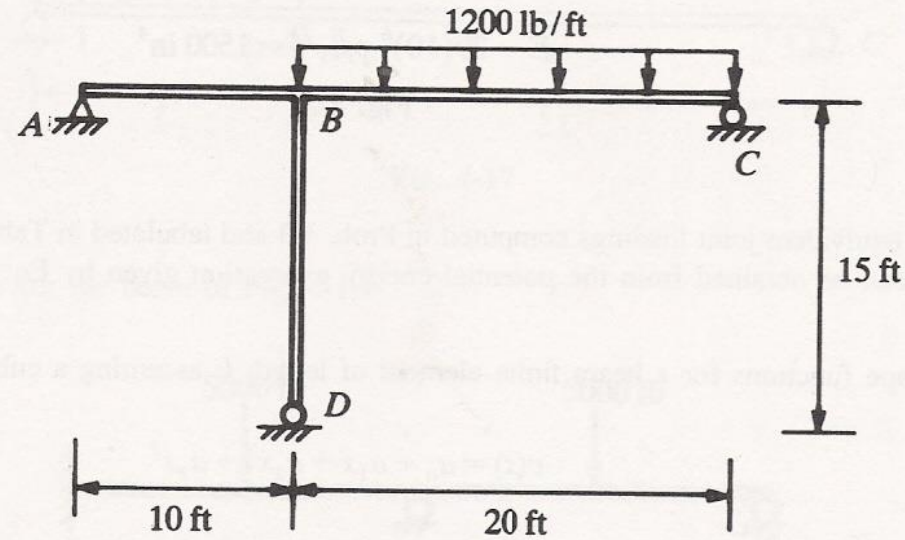
$$\begin{Bmatrix} V_{B2} \\ M_B \\ V_C \\ M_C \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 1.5 & 1.5L & -1.5 & 1.5L \\ 1.5L & 2L^2 & -1.5L & L^2 \\ -1.5 & -1.5L & 1.5 & -1.5L \\ 1.5L & L^2 & -1.5L & 2L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ wL^3/72EI \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (f)$$

$$V_{B2} = \frac{wL}{48} \quad V_C = R_C = -\frac{wL}{48} \quad M_B = \frac{wL^2}{36} \quad M_C = \frac{wL^2}{72} \quad R_B = V_{B1} + V_{B2}$$

Internal supports such as joint C should always have the same numerical value of bending moment but with opposite signs, thus indicating equilibrium at the joint. The bending moments obtained from the stiffness analysis

2

Compute the reactions at supports A , C , and D and rotations for joints A , B , C , and D along with the horizontal displacements of supports C and D for the frame of Fig. 4-24. Assume $E = 29(10)^6$ psi, $I = 1800$ in⁴, and $A = 20$ in² for all members.



$$E = 29(10)^6 \text{ psi}, I = 1800 \text{ in}^4, A = 20 \text{ in}^2$$

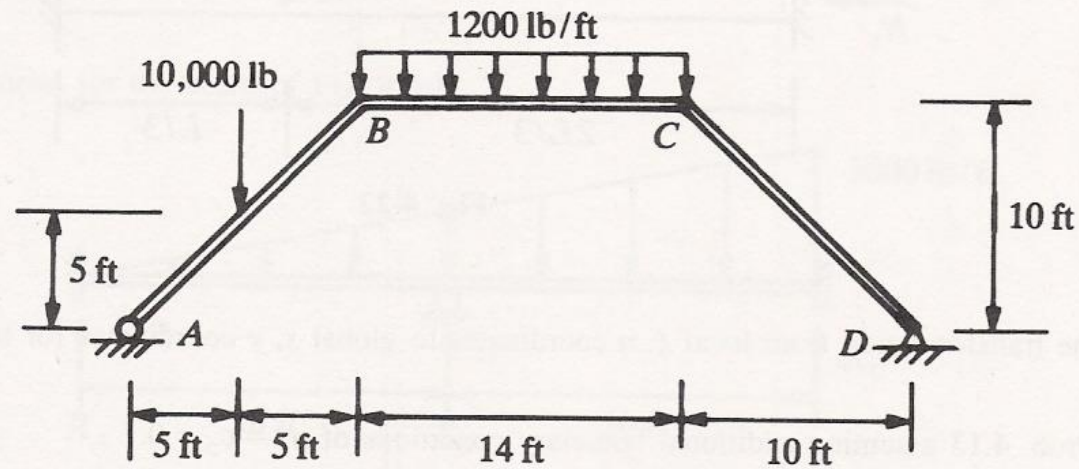
Fig. 4-24

2

$M_A = M_C = M_D = 0$, $M_B = 37,520$ ft·lb (acting on member $B-C$); vertical reactions $V_A = -3750$ lb, $V_C = 10,120$, $V_D = 17,630$; $\theta_A = 1.296(10)^{-4}$, $\theta_B = \theta_D = -3.906(10)^{-4}$, $\theta_C = 7.812(10)^{-4}$; horizontal displacement at D , $u_D = -0.0703$ in, $u_C = 0$.

3

Compute reactions, joint rotations, and the horizontal displacement of joint A for the frame of Fig. 4-25. Assume $E = 29(10)^6$ psi, $I = 1800$ in⁴, and $A = 20$ in² for all members.



$$E = 29(10)^6 \text{ psi}, I = 1800 \text{ in}^4, A = 20 \text{ in}^2$$

Fig. 4-25

3

$M_A = M_D = 0$, $M_B = -108,960$ ft·lb (acting on member B-C), $M_C = 94,400$ (acting on member B-C), $V_A = 17,370$ lb, $V_D = 9450$, $\theta_A = -4.187(10)^{-3}$, $\theta_B = -2.246(10)^{-3}$, $\theta_C = 2.438(10)^{-3}$, $\theta_D = 4.279(10)^{-3}$, $u_A = -0.90$ in.

Experiments

