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Super-Stability of Symmetric Prismatic Tensegrity Structures based on Group Representation Theory

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Abstract

Tensegrity structures are prestressed pin-jointed structures composed of struts and cables. To check stability of tensegrity structures, it's preferable to check super-stability since it is the strongest stability criterion. In this paper, we discuss the super-stability of prismatic tensegrity structures which have dihedral symmetry. In previous studies, these structures were proved to be super-stable if horizontal cables connect adjacent nodes. However, these proofs are incomplete. Therefore, we complement the proof for super-stability of symmetric prismatic tensegrity structures in this study.

Keywords: tensegrity, super-stability, dihedral symmetry, prismatic, self-equilibrium

1. Introduction

A tensegrity is a prestressed pin-jointed structure composed of struts and cables, possessing purely compression and tension, respectively. To check stability of a general structure, we can examine whether or not its strain energy has a local minimum. But in the case of stability investigation of prestressed pin-jointed structures, it's preferable that we check super-stability since it is the strongest stability criterion. Super-stable structures are always stable without considering material property and level of prestresses. It can be partially confirmed by checking positive semi-definiteness of force density matrix [1].

In this paper, we discuss the super-stability of prismatic tensegrity structures, in particular, having dihedral symmetry as shown in Figure 1. Connelly and Terrell [2] verified that these structures are super-stable if every horizontal cable connects adjacent nodes as in Figure 2. Moreover, Zhang and Ohsaki [3] confirmed this conclusion by using symmetry-adapted force density matrix, that is, singularity of a certain block of the block-diagonalized force density matrix. However, the singularity of other blocks has not been considered in the previous studies. In this study, we aimed at conducting a general qualification for super-stability of prismatic tensegrity structures by the block-diagonalization method based on group representation theory.

2. Dihedral Symmetry and Connectivity

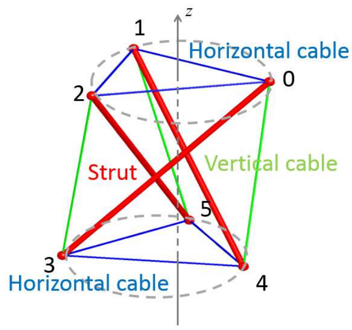
Dihedral group is a group to express symmetry of prisms composed of two regular polygons of the same form [4]. The prismatic tensegrity structures considered in this study have dihedral symmetry. We denote dihedral group by $\mathbf{D}_N$, where N means the number of vertices of the regular polygon. The dihedral group $\mathbf{D}_N$ has three types of symmetry operations as shown in Figure 3.

Symmetry operations of the dihedral group $\mathbf{D}_N$:

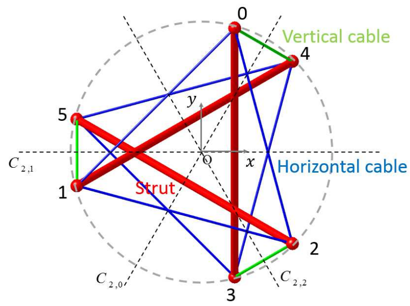
1. identity.

2. *N*-fold rotations which rotate an angle $2i\pi / N(i = 1, 2, ..., N-1)$ around *z* axis in Euclidean space.

3. *N* two-fold rotations which rotate an angle π around the axes which pass the origin and perpendicular to the *z* axis.

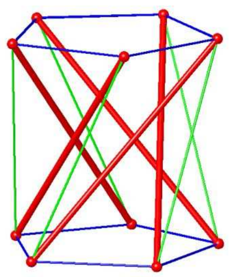


(a) perspective

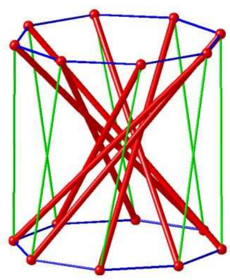


(b) plan

Figure 1: Prismatic tensegrity structure $\mathbf{D}_3^{1,1}$

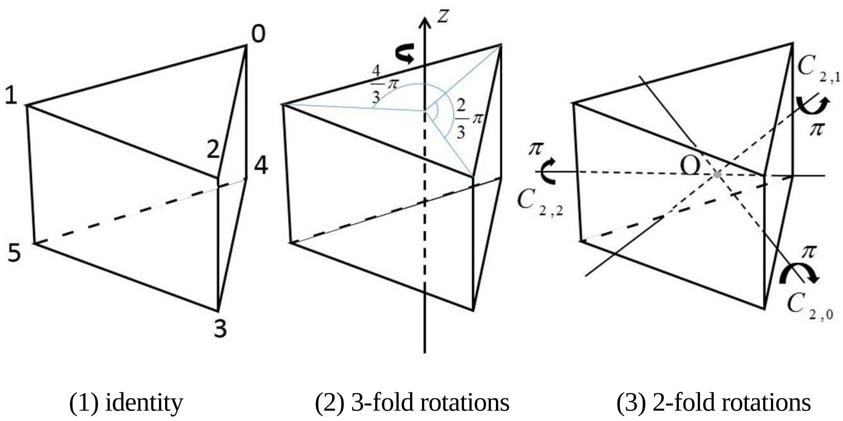


(a) $\mathbf{D}_5^{1,1}$



(b) $\mathbf{D}_8^{1,3}$

Figure 2: Super-stable structures



(1) identity (2) 3-fold rotations (3) 2-fold rotations

Figure 3: Symmetry operation of dihedral group $\mathbf{D}_3$

Furthermore, we denote connectivity of a prismatic tensegrity structure by $\mathbf{D}_N^{h,v}$, where h and v indicate connectivity of horizontal cables and vertical cables, respectively. If node $i (0 \le i < N)$ is a reference node, connectivity of prismatic tensegrity structures $\mathbf{D}_N^{h,v}$ is defined as follows [3].

- Struts

  A strut connects node $i(0 \le i < N)$ on the upper plane to node $N + i$ on the lower plane.

- Horizontal cables

  On the upper plane, a horizontal cable connects node $i(0 \le i < N)$ to node $h+i$, or node $h+i-N$ when $h+i \ge N$. On the lower plane, a horizontal cable connects node $i(N \le i < 2N)$ to node $h+i$ or node $h+i-N$ when $h+i \ge 2N$. Because of symmetry, node $i(0 \le i < N)$ is also connected to node $N-h+i$. Thus, we restrict $1 \le h \le \frac{N}{2}$.

- Vertical cables

  A vertical cable connects node $i(0 \le i < N)$ on the upper plane to node $N+v+i$, or $v+i$ when $v+i \ge N$, on the lower plane.

For example, if $h = v = 2$, connectivity of horizontal cables and vertical cables is shown in Figure 4.

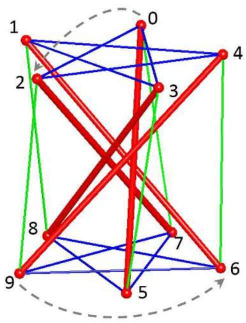


Figure 4: Connectivity of the structure $\mathbf{D}_5^{2,2}$

3. General Super-Stability Conditions

In general, a tensegrity structure is free-standing and pin-jointed without fixed nodes. In other words, a tensegrity structure is in a state of self-stress, that is, a set of member forces which are in statical equilibrium with zero external load. We consider a tensegrity structure which consists of *m* members and *n* free nodes. The connectivity of members is described by *connectivity matrix* $\mathbf{C} \in \mathbb{R}^{m \times n}$. In each row of the connectivity matrix, only two nonzero entries, +1 and −1, exist. These two entries refer to the two nodes connected by the member as well as direction of the member.

If a member numbered as $k(k = 1, 2, ..., m)$ connects node *i* to node *j*, the *k*th row $\mathbf{C}_{(k,r)}(r = 1, 2, ..., n)$ is defined as follows[3]

$$\mathbf{C}_{(k,r)} = \begin{cases} \mathrm{sign}(j-r) & \text{if } r = i \\ \mathrm{sign}(i-r) & \text{if } r = j \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where

$$\mathrm{sign}(j-i) = \begin{cases} +1 & \text{if } j > i \\ -1 & \text{if } j < i \end{cases} \tag{2}$$

A force density q_k is defined as a ratio of the force s_k of member *k* to its length l_k:

$$q_k = \frac{s_k}{l_k} \tag{3}$$

Let $\mathbf{q} \in \mathbb{R}^m$ be the force density vector and $\mathbf{Q} \in \mathbb{R}^{m \times m}$ be its diagonal version. Then, the force density matrix $\mathbf{E} \in \mathbb{R}^{n \times n}$ is defined as follows:

$$\mathbf{E} = \mathbf{C}^{\mathrm{T}}\mathbf{QC} \tag{4}$$

Self-equilibrium equations are defined by using the force density matrix and nodal coordinate vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z} (\in \mathbb{R}^n)$:

$$\mathbf{Ex} = \mathbf{Ey} = \mathbf{Ez} = \mathbf{0} \tag{5}$$

According to equation (5), nodal coordinates lies in the null space of **E**. So, in the three-dimensional space, if a structure is *non-degenerate*, rank deficiency of the force density matrix is at least 4. Here, if the following three conditions are satisfied at the same time, it is guaranteed that a three-dimensional tensegrity structure is super-stable [3].

(a). The force density matrix has 4 zero eigenvalues.

(b). The force density matrix is positive semi-definite.

(c). The geometry matrix is full-rank.

The condition (a) is the non-degeneracy condition. Furthermore, condition (c) is satisfied if a prismatic tensegrity structure is indivisible.

4. Block Diagonalization by Group Representation Theory

Force densities of each type of members are equality because prismatic tensegrity structures are of dihedral symmetry. Thus, we denote force densities of struts, horizontal cables and vertical cables by q_s, q_h and q_v, respectively. Stability is investigated by checking positive semi-definiteness of the force density matrix, but that takes a high cost, especially, in the case of a complicated structure. For the purpose of presenting analytical stability condition, the block-diagonalized force density matrix $\tilde{\mathbf{E}} \in \mathbb{R}^{2N \times 2N}$ of a prismatic tensegrity structure $\mathbf{D}_N^{h,v}$ given as follows:

$$\underset{2N \times 2N}{\tilde{\mathbf{E}}} = \begin{pmatrix} \underset{1\times1}{\tilde{\mathbf{E}}^{A_1}} & & & & & & & & & \\ & \underset{1\times1}{\tilde{\mathbf{E}}^{A_2}} & & & & & & \mathrm{O} & & \\ & & \left(\underset{1\times1}{\tilde{\mathbf{E}}^{B_1}}\right) & & & & & & & \\ & & & \left(\underset{1\times1}{\tilde{\mathbf{E}}^{B_2}}\right) & & & & & & \\ & & & & \underset{2\times2}{\tilde{\mathbf{E}}^{E_1}} & & & & & \\ & & & & & \underset{2\times2}{\tilde{\mathbf{E}}^{E_1}} & & & & \\ & \mathrm{O} & & & & & \ddots & & & \\ & & & & & & & \underset{2\times2}{\tilde{\mathbf{E}}^{E_p}} & & \\ & & & & & & & & \underset{2\times2}{\tilde{\mathbf{E}}^{E_p}} \end{pmatrix} \tag{6}$$

where A_1, A_2, B_1, B_2 and $E_k\,(k = 1, 2, \ldots, p)$ are irreducible representations of dihedral group. *p* means

$$p = \begin{cases} (N-1)/2 & N \text{ is odd} \\ (N-2)/2 & N \text{ is even} \end{cases} \tag{7}$$

B_1 and B_2 exist if N is even. Eigenvalues of the original matrix are derived by calculating these simple matrices.

The block $\tilde{\mathbf{E}}^{\mu}$ corresponding to the representation μ of force density matrix $\tilde{\mathbf{E}}$ is formulated by using representation matrix $\mathbf{R}_i^{\mu}$ [3].

$$\tilde{\mathbf{E}}^{\mu} = q\mathbf{R}_0^{\mu} - q_h\mathbf{R}_h^{\mu} - q_h\mathbf{R}_{N-h}^{\mu} - q_s\mathbf{R}_N^{\mu} - q_v\mathbf{R}_{N+v}^{\mu} \tag{8}$$

where q equals to the sum of force densities of members connected to a node:

$$q = 2q_h + q_s + q_v \tag{9}$$

According to equation (8) the block $\tilde{\mathbf{E}}^{A_1}$ corresponding to the irreducible representation A_1 is usually zero, because all representation matrix $\mathbf{R}_i^{A_1}$ is one. Hence,

$$\tilde{\mathbf{E}}^{A_1} = 0 \tag{10}$$

The block $\tilde{\mathbf{E}}^{A_2}$ corresponding to the irreducible representation A_2 is

$$\tilde{\mathbf{E}}^{A_2} = 2(q_s + q_v) \tag{11}$$

If N is even, the blocks $\tilde{\mathbf{E}}^{B_1}$ and $\tilde{\mathbf{E}}^{B_2}$ corresponding to the irreducible representations B_1 and B_2 are

$$\tilde{\mathbf{E}}^{B_1} = [2 - 2(-1)^h]q_h + [1 - (-1)^v]q_v \tag{12}$$

$$\tilde{\mathbf{E}}^{B_2} = [2 - 2(-1)^h]q_h + 2q_s + [1 - (-1)^{v+1}]q_v \tag{13}$$

The 2-dimentional blocks $\tilde{\mathbf{E}}^{E_k}\,(k = 1, 2, ..., p)$ are calculated as follows:

$$\tilde{\mathbf{E}}^{E_k} = \begin{pmatrix} 2(1 - C_{hk})q_h + (1 - C_{vk})q_v & -S_{vk}q_v \\ -S_{vk}q_v & 2(1 - C_{hk})q_h + 2q_s + (1 + C_{vk})q_v \end{pmatrix} \tag{14}$$

To satisfy condition (a) for a three-dimensional tensegrity structure, $\tilde{\mathbf{E}}$ of equation (6) should have 4 zero eigenvalues. $\tilde{\mathbf{E}}^{A_1}$ is always zero, and therefore, its eigenvalue is zero by equation (10), so, $\tilde{\mathbf{E}}^{A_2}$ and any one of the two-dimensional block $\tilde{\mathbf{E}}^{E_k}\,(k = 1, 2, ..., p)$ should be singular. Therefore, the determinants of $\tilde{\mathbf{E}}^{A_2}$ and $\tilde{\mathbf{E}}^{E_k}$ are zero:

$$\left\|\tilde{\mathbf{E}}^{A_2}\right\| = \left\|\tilde{\mathbf{E}}^{E_k}\right\| = 0 \Leftrightarrow \begin{cases} q_v = -q_s \\ t = q_h / q_v = \sqrt{2 - 2C_{vk}} \,/\, (2 - 2C_{hk}) \end{cases} \tag{15}$$

where $C_{vk} = \cos(2vk\pi / N)$, $C_{hk} = \cos(2hk\pi / N)$. Using equation (15), $\tilde{\mathbf{E}}^{E_i}\,(i = 1, 2, ..., p) \in \mathbb{R}^{2N \times 2N}$ can be written as

$$\frac{1}{q_v}\tilde{\mathbf{E}}^{E_i} = \begin{pmatrix} 2t(1 - C_{hi}) + 1 - C_{vi} & -S_{vi} \\ -S_{vi} & 2t(1 - C_{hi}) - (1 - C_{vi}) \end{pmatrix} \tag{16}$$

5. Super-Stability Conditions

Zhang and Ohsaki [3] confirmed that prismatic tensegrity structures are super-stable if horizontal cables connect adjacent nodes; i.e. $h=1$ when $k=1$, by using the block-diagonalized force density matrix. Because in equation (6), there are two zero eigenvalues in $\tilde{\mathbf{E}}^{\mathrm{A}_1}$ and $\tilde{\mathbf{E}}^{\mathrm{A}_2}$, and two zero eigenvalues in total for the positive semi-definiteness of $\tilde{\mathbf{E}}^{\mathrm{E}_1}$. Furthermore, $\tilde{\mathbf{E}}^{\mathrm{E}_i}\,(i=2,3,...,p)$ are positive definiteness if $h=1$. So, super-stability is guaranteed if $h=k=1$. Examples of super-stable structures are shown in Figure 2.

However, it has not been considered that the singularity of the block $\tilde{\mathbf{E}}^{\mathrm{E}_i}\,(i=2,3,...,p)$ in the previous studies. In equation (16), a structure is super-stable in the case of a specific block $\tilde{\mathbf{E}}^{\mathrm{E}_k}$ is positive semi-definiteness and the others are positive definiteness. In this study, we target at $\tilde{\mathbf{E}}^{\mathrm{E}_k}$ which is $k\neq 1$ in such a way as to conduct a general qualification for super-stability of prismatic tensegrity structures.

From equation (16), the two eigenvalues of $\tilde{\mathbf{E}}^{\mathrm{E}_i}$ are calculated as follows:

$$\frac{\lambda_1^{\mathrm{E}_i}}{q_v} = 2t(1-C_{hi}) + \sqrt{2(1-C_{vi})} \tag{17}$$

$$\frac{\lambda_2^{\mathrm{E}_i}}{q_v} = 2t(1-C_{hi}) - \sqrt{2(1-C_{vi})} \tag{18}$$

In equation (17), $\lambda_1^{\mathrm{E}_i} > 0$ always holds because $1-C_{hi}\geq 0$, $1-C_{vi}\geq 0$, and $t>0$ always hold since cables carry tension. So, we are to find the condition that let $\lambda_2^{\mathrm{E}_i}\geq 0$ always hold.

We consider the case of $hk = jN\pm\alpha$ $(j=1,2,...;\alpha=1,2,...,\frac{N}{2})$ and $hi = j'N\pm\beta$ $(j'=1,2,...;$ $\beta=1,2,...,\frac{N}{2})$. Then, the values of C_{hk} and C_{hi} are exchanged as follows:

$$C_{jN\pm a} = \cos\left(\frac{2\pi(jN\pm\alpha)}{N}\right) = \cos\left(2j\pi\pm\frac{2\alpha\pi}{N}\right) = \cos\left(\frac{2\alpha\pi}{N}\right) \tag{19}$$

$$C_{j'N\pm\beta} = \cos\left(\frac{2\pi(j'N\pm\beta)}{N}\right) = \cos\left(2j'\pi\pm\frac{2\beta\pi}{N}\right) = \cos\left(\frac{2\beta\pi}{N}\right) \tag{20}$$

Here, $\alpha=0$ implies each member does not carry any force from equation (15). Moreover, in equaton (18), $\lambda_2^{\mathrm{E}_i}<0$ if $\beta=0$ and $v=1$. Therefore, we don't consider the case of $\alpha=0$ or $\beta=0$.

The ratio of *i* and *k* is

$$\frac{i}{k} = \frac{hi}{hk} = \frac{j'N\pm\beta}{jN\pm\alpha} = \frac{vi}{vk} \tag{21}$$

Therefore, $vk:vi = \gamma(jN\pm\alpha):\gamma(j'N\pm\beta)$ $(\gamma=1,2,...,\frac{N}{2})$ always holds. Thus, C_{vk} and C_{vi} are calculated as follows:

$$C_{vk} = \cos\left(\frac{2\alpha\gamma\pi}{N}\right), \quad C_{vi} = \left(\frac{2\beta\gamma\pi}{N}\right) \tag{22}$$

♦ Theorem 1. *If* $\alpha = 1$, i.e. $hk = jN \pm 1$, *the prismatic tensegrity structure* $\mathbf{D}_N^{h,v}$ *is super-stable.*

Proof. When $\lambda_2^{\mathrm{E}_i} \geq 0$ always holds, equation (18) can be arranged as follows:

$$\frac{1-C_{hi}}{1-C_{hk}} \geq \sqrt{\frac{1-C_{vi}}{1-C_{vk}}} \tag{23}$$

Besides, if $hk = jN \pm 1$, the following equation is true because equation (24) has been verified in the paper of Connelly and Terrell [2].

$$\frac{1-\cos\left(\frac{2\beta\pi}{N}\right)}{1-\cos\left(\frac{2\pi}{N}\right)} \geq \frac{1-\cos\left(\frac{2\beta\gamma\pi}{N}\right)}{1-\cos\left(\frac{2\gamma\pi}{N}\right)} \tag{24}$$

Using equations (19), (20) and (22), equation (24) can be rearranged as follows:

$$\frac{1-C_{hi}}{1-C_{hk}} \geq \frac{1-C_{vi}}{1-C_{vk}} \tag{25}$$

Because C_{hk} has the largest value when $hk = jN \pm 1$, $\frac{1-C_{hi}}{1-C_{hk}} \geq 1$ always holds for $hk = jN \pm 1$. Therefore, equation (23) is true when $hk = jN \pm 1$ such that $\lambda_2^{\mathrm{E}_i} \geq 0$ is true.

If N is even, the blocks $\tilde{\mathbf{E}}^{\mathrm{B}_1}$ and $\tilde{\mathbf{E}}^{\mathrm{B}_2}$ exist. Moreover, h is always odd because hk is odd. Therefore, $\tilde{\mathbf{E}}^{\mathrm{B}_1}$ and $\tilde{\mathbf{E}}^{\mathrm{B}_2}$ are positive, since

$$\begin{aligned} \tilde{\mathbf{E}}^{\mathrm{B}_1} &= [2-2(-1)^h]t + 1 - (-1)^v = 4t + 1 - (-1)^v \\ &> 0 \end{aligned} \tag{26}$$

$$\begin{aligned} \tilde{\mathbf{E}}^{\mathrm{B}_2} &\geq 2\left(\frac{\sqrt{2-2C_v}}{1-C_1} - 1\right) \\ &\geq 2\frac{\sqrt{2-2C_1}}{1-C_1} - 2 = \frac{2\sqrt{2}}{\sqrt{1-C_1}} - 2 \\ &> 0 \end{aligned} \tag{27}$$

In this case, rank deficiency is 4 since $\tilde{\mathbf{E}}^{\mathrm{A}_1}$, $\tilde{\mathbf{E}}^{\mathrm{A}_2}$ and two copies of $\tilde{\mathbf{E}}^{\mathrm{E}_k}$ are all singular. Futhermore, the eigenvalues in other blocks are positive definite with $\lambda_2^{\mathrm{E}_i} > 0$. Therefore, the force density matrix is positive semi-definiteness and satisfies the non-degeneracy condition. Hence, the prismatic tensegrity structure $\mathbf{D}_N^{h,v}$ is super-stable if $hk = jN \pm 1$. ■

♦ Theorem 2. *If* $\alpha \neq 1$, i.e. $hk = jN \pm \alpha (\alpha = 2, 3, ..., N/2)$, *the prismatic tensegrity structure* $\mathbf{D}_N^{h,v}$ *is not super-stable.*

Proof. We consider the following two cases.

- Case 1: $hk = jN \pm 2$ as well as N and h are even.

  The block $\tilde{\mathbf{E}}^{\mathrm{B}_2}$ that exist when N is even:

$$\begin{aligned}\tilde{\mathbf{E}}^{\mathrm{B}_2} &= [2-2(-1)^h]t - 1 - (-1)^{v+1} \\ &= -1-(-1)^{v+1} \\ &\leq 0\end{aligned} \tag{28}$$

Therefore, the sufficient conditions for super-stability are not satisfied, because equation (6) has negative eigenvalue or its rank deficiency is more than 4. Hence, the prismatic tensegrity structure $\mathbf{D}_N^{h,v}$ is not super-stable in Case 1.

- Case 2: other cases for $\alpha \neq 1$ except Case 1.

When $\lambda_2^{\mathrm{E}_i} < 0$ holds, equation (18) can be arranged as follows:

$$\frac{1-C_{hi}}{1-C_{hk}} < \sqrt{\frac{1-C_{vi}}{1-C_{vk}}} \tag{29}$$

In equation (29), we convert cosine to sine and substitude $hi = j'N \pm 1$:

$$\left(\frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\alpha\pi}{N}\right)}\right)^2 < \left|\frac{\sin\left(\frac{\gamma\pi}{N}\right)}{\sin\left(\frac{\alpha\gamma\pi}{N}\right)}\right| \tag{30}$$

$\sin(\frac{\pi}{N})/\sin(\frac{\alpha\pi}{N}) < 1$ is true because $\sin(\frac{\pi}{N}) < \sin(\frac{\alpha\pi}{N})$ when $\alpha = 2,3,...,\frac{N}{2}$. To ensure that equation (30) is always true, we need to guarantee the following condition:

$$\left(\frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\alpha\pi}{N}\right)}\right)^2 \leq \left|\frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\alpha\pi}{N}\right)}\right| < \left|\frac{\sin\left(\frac{\gamma\pi}{N}\right)}{\sin\left(\frac{\alpha\gamma\pi}{N}\right)}\right| \tag{31}$$

We have the following two cases for $\alpha\gamma$.

▲ $\alpha\gamma \geq \frac{N\pi}{4}$

When $0 \leq \theta \leq \frac{\pi}{2}$, a range of sine is $\frac{\pi}{2}\theta \leq \sin\theta \leq \theta$. From the fact that $\alpha = 2,3,...,\frac{N}{2}$ and $\gamma = 1,2,...,\frac{N}{2}$:

$$\left|\frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\alpha\pi}{N}\right)}\right| < \frac{\frac{\pi}{N}}{\frac{2}{\pi}\cdot\frac{\alpha\pi}{N}} \quad \text{and} \quad \frac{\frac{2}{\pi}\cdot\frac{\gamma\pi}{N}}{1} < \left|\frac{\sin\left(\frac{\gamma\pi}{N}\right)}{\sin\left(\frac{\alpha\gamma\pi}{N}\right)}\right| \tag{32}$$

As a result, when $\frac{2\gamma}{N} \geq \frac{\pi}{2\alpha}$ i.e. $\alpha\gamma \geq \frac{N\pi}{4}$, equation (31) always holds.

▲ $\alpha\gamma < \frac{N\pi}{4}$

By using the formula $\sin\theta = \theta\prod_{x=1}^{\infty}\left(1-\frac{\theta^2}{x^2\pi^2}\right)$:

$$\frac{\sin\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\alpha\pi}{N}\right)} = \frac{\frac{\pi}{N}\prod_{x=1}^{\infty}\left(1-\frac{1}{N^2x^2}\right)}{\frac{\alpha\pi}{N}\prod_{x=1}^{\infty}\left(1-\frac{\alpha^2}{N^2x^2}\right)} = \frac{1}{\alpha\prod_{\substack{x=1 \\ \alpha\nmid x}}^{\infty}\left(1-\frac{\alpha^2}{N^2x^2}\right)} \tag{33}$$

$$\frac{\sin\left(\frac{\gamma\pi}{N}\right)}{\sin\left(\frac{\alpha\gamma\pi}{N}\right)}=\frac{\frac{\gamma\pi}{N}\prod_{x=1}^{\infty}\left(1-\frac{\gamma^2}{N^2x^2}\right)}{\frac{\alpha\gamma\pi}{N}\prod_{x=1}^{\infty}\left(1-\frac{\alpha^2\gamma^2}{N^2x^2}\right)}=\frac{1}{\alpha\prod_{\substack{x=1\\ \alpha\nmid x}}^{\infty}\left(1-\frac{\alpha^2\gamma^2}{N^2x^2}\right)} \tag{34}$$

where x is an arbitary positive integer. $\alpha\nmid x$ means x is not divisible by α.

Therefore, the following equation is true if $\alpha\gamma<N$:

$$\frac{1}{\left(1-\frac{\alpha^2\gamma^2}{N^2x^2}\right)}\geq\frac{1}{\left(1-\frac{\alpha^2}{N^2x^2}\right)}\geq 0 \tag{35}$$

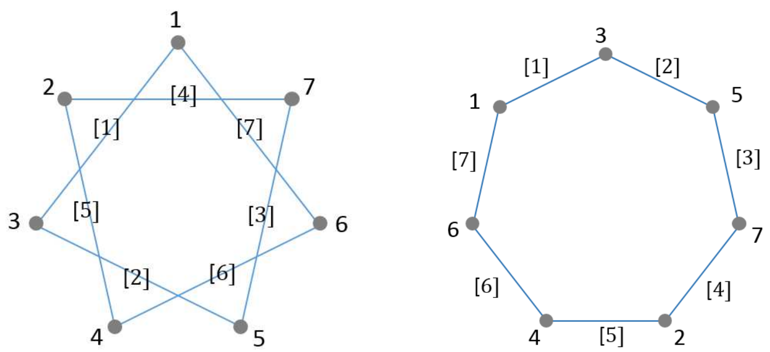
From equation (35), equation (31) is always holds when $\alpha\gamma<\frac{N\pi}{4}<N$.

Thus, $\lambda_2^{\mathrm{E}_i}<0$ holds if $hi=j'N\pm 1$. Hence, equation (6) is not positive semi-definiteness.

Consequently, the prismatic tensegrity structure $\mathbf{D}_N^{h,v}$ is not super-stable if $hk=jN\pm\alpha$ $(\alpha=2,3,...,\frac{N}{2})$. Because a certain eigenvalue holds $(\lambda_2^{\mathrm{E}_i}<0)$ or $\tilde{\mathbf{E}}^{\mathrm{B}_2}$ is non-positive. ■

Note that if the value h with $k=1$ is equal to α with $hk=jN\pm\alpha$, these structures have the same eigenvalues by equation (17) and (18). In other words, the position of a block $\tilde{\mathbf{E}}^{\mathrm{E}_k}$ in equation (6) is exchanged with another block. Therefore, their nodal coordinates and their connectivity are the same because nodal coordinates lie in the null-space of $\tilde{\mathbf{E}}$ by equation (5). For example about connectivity of horizontal cables, the example (a): a structure $\mathbf{D}_7^{2,3}$ with $k=1$ and the example (b): a structure $\mathbf{D}_7^{2,3}$ with $k=3$, i.e. $\alpha=1$ are illustrated in Figure 5. For the example (a), because $h=2$, horizontal cables connect a node to the second node such as node 1 is connected to node 3. For the example (b), the value of hk is $hk=2\times 3=6=7-1$, that is, $\alpha=1$. Then, because nodal coordinates are exchanged, horizontal cables apparently connect adjacent nodes without connectivity that connect node 1 to node 3 don't changed.

Thus, if $hk=jN\pm 1$, the horizontal cables of the prismatic structure connect adjacent nodes.



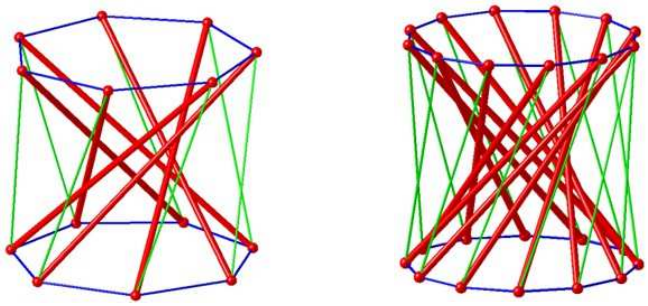
(a) $\mathbf{D}_7^{2,3}$, k = 1 (b) $\mathbf{D}_7^{2,3}$, k = 3

Figure 5: Connectivity of k =1 or k =3 of $\mathbf{D}_7^{2,3}$

In summary of the above conclusions:

- If $hk = jN \pm 1$, the prismatic tensegrity structure $\mathbf{D}_N^{h,v}$ is super-stable. Then, they have exactly the same shapes as $h = k = 1$. For instance, the structures in Figure 6 have the same shape as in the case of $h = k = 1$.

- If $hk \neq jN \pm 1$, then, the prismatic tensegrity structure $\mathbf{D}_N^{h,v}$ is not super-stable. Because a certain eigenvalue holds ($\lambda_2^{\mathrm{E}_i} < 0$) except for if $hk = jN \pm 2$ as well as N and h are even. However, the eigenvalue of $\tilde{\mathbf{E}}^{\mathrm{B}_2}$ is non-positive.



(a) $\mathbf{D}_7^{2,3}$, $k = 3$ (b) $\mathbf{D}_{12}^{5,3}$, $k = 5$

Figure 6: (a) and (b) are super-stable structures: (a) and (b) is same shape of $\mathbf{D}_7^{1,2}$, $k = 1$ and $\mathbf{D}_{12}^{1,3}$, $k = 1$

6. Conclusion

We verified that super-stable symmetric prismatic tensegrity structures exist if and only if $hk = jN \pm 1$, and in that case, they have exactly the same shapes as those proved by Connelly and Terrell; i.e., their horizontal cables connect adjacent nodes. Therefore, we complement the proof for super-stability of symmetric prismatic tensegrity structures.

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