AIASYB2:

APLICACIÓN DE LA INTELIGENCIA ARTIFICIAL EN LOS SENSORES Y BIOSENSORES

Multiresolution Analysis Wavelets

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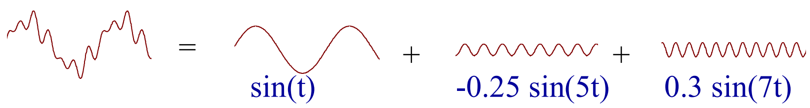
Multiresolution Analysis

- Example of Scaling and Wavelet Functions
- Nested Spaces and Complementary Spaces
- Multiresolution
- Fourier Transform versus Wavelet Transform
- DiscreteWavelets Transform
- Aplications
- Bibliography

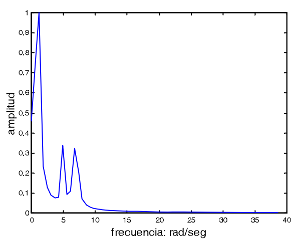
Fourier Transform versus Wavelet Transform

Fourier Analysis

- Breaks down a signal into constituent sinusoids of different frequencies



- Transform our view of the signal from from time-based to frequency-based.



- In transforming to the frequency domain, time information is lost:

  When did a determined event took place?

- If it is a *stationary* signal this drawback isn't very important.

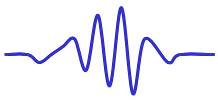
- Fourier analysis is not suited to detecting nonstationary or transitory characteristics:
  - drift,
  - trends,
  - abrupt changes: breakdown points, discontinuities in higher derivatives
  - beginnings and ends of events
  - self similarities.

Why Wavelets?

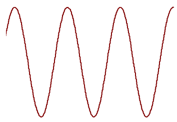
- Wavelet analysis allows the use of long time intervals where we want more precise low-frequency information, and shorter regions where we want high-frequency information (all in the same signal)
- Ability to perform *local analysis* : to analyze a localized area of a larger signal.
- Compress or de-noise a signal without appreciable degradation.

What is a Wavelet?

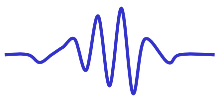
- A wavelet is a waveform of effectively limited duration that has an average value of zero.



- Sinusoids : unlimited duration, smooth and predictable.



- Wavelets: limited duration, irregular and asymetric.

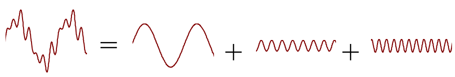


Continuos Wavelet Transform

- Fourier transform: breaks down a signal in sum of sinusoids of different frequencies → Fourier Coefficients

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-jwt}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, e^{jwt}\, dw$$



- Wavelet: breaks down a signal in sum of scaled and shifted versions of the wavelet function → Wavelet Coefficients

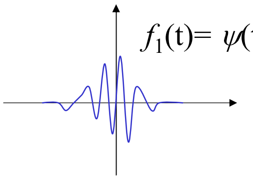
$$C(scale, position) = \int_{-\infty}^{\infty} f(t)\, \psi(scale, position)\, dt$$

$$f(t) = \iint C(scale, position)\, \psi^*(scale, position)\, de\, dp$$

C: measurement of similarity between the signal and the wavelet

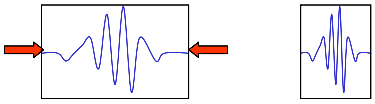
Scaling

$f_1(t) = \psi(t)$



$s = 1/a$

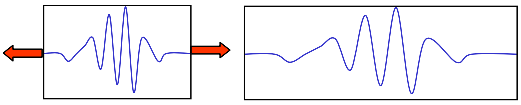
$a=2$



$f_2(t) = \psi(2t)$

A low scale compresses the signal ⇒ Fast changing ⇒ High frequencies

$a=1/2$



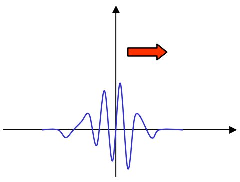
$f_3(t) = \psi(t/2)$

A high scale stretches the signal ⇒ Slow changing ⇒ Low frequencies

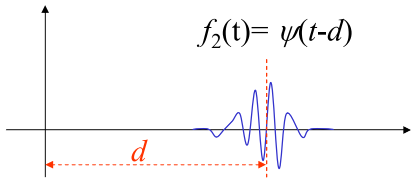
$f_1(1) = f_2(0.5) = f_3(2)$

Shifting

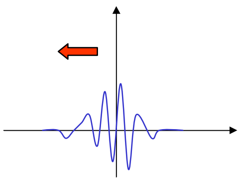
$f_1(t) = \psi(t)$



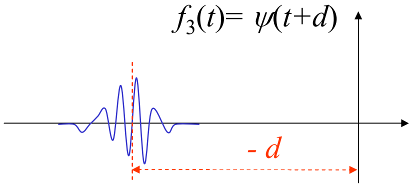
$f_2(t) = \psi(t-d)$



$f_1(t) = \psi(t)$



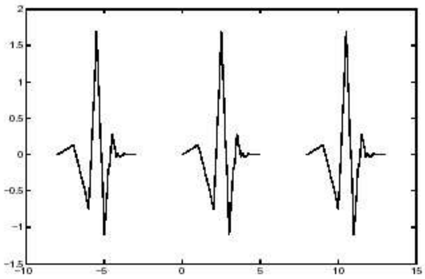
$f_3(t) = \psi(t+d)$



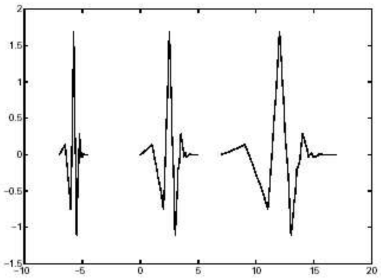
Si $d = 5$, $f_1(0) = f_2(5) = f_3(-5)$

Shifting	$\psi(t-d)$
Scaling	$\frac{1}{\sqrt{s}}\psi\left(\frac{t}{s}\right)$
Scaling and Shifting	$\frac{1}{\sqrt{s}}\psi\left(\frac{t-d}{s}\right)$

$$C(s,d) = \int_{-\infty}^{\infty} f(t)\,\frac{1}{\sqrt{s}}\psi\left(\frac{t-d}{s}\right)\,dt$$



wavelet db3(t): centered
wavelet db3(t + 8): left
wavelet db3(t - 8): right

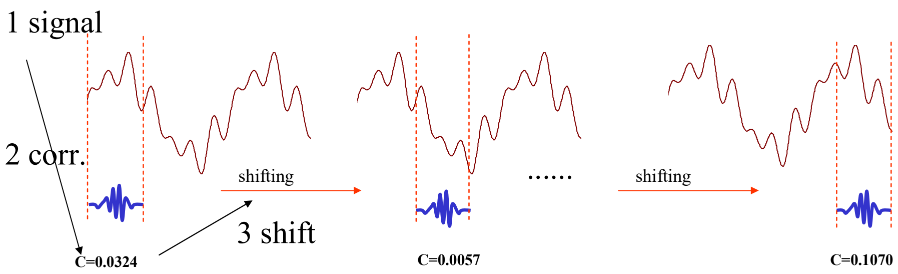


wavelet db3(t): centered
wavelet db3(2t+7): left
wavelet db3(t/2 -7): right

Wavelet Properties

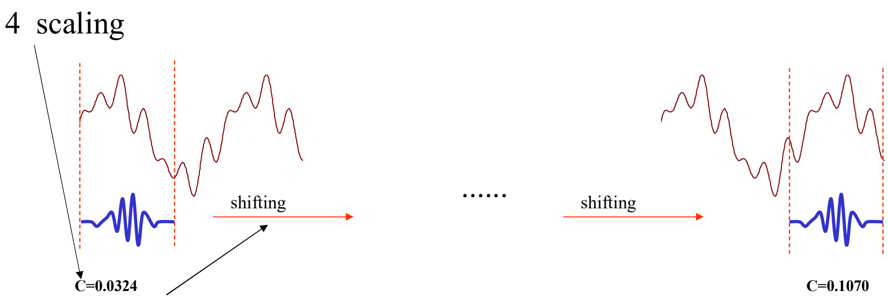
- Mother Wavelet: $\psi(t)$
- Scaling and Shifting: $\psi_{s,d}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-d}{s}\right)$
- Null mean value: $\int \psi(t)\,dt = 0$
- Admissibility condition: (wave of limited wide) $\int \frac{|\Psi(w)|^2}{|w|}\,dw < \infty, \quad \Rightarrow \quad |\Psi(0)|^2 = 0$
- Regularity condition: (concentrated in time) $\int t^p \psi(t)\,dt \approx 0, \quad p = 0, 1, \ldots, N$

Steps to Compute the Coefficients



1. Take a wavelet and compare it to a section at the start of the original signal
2. Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal
3. Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.

Steps to Compute the Coefficients



4. Scale (stretch) the wavelet and repeat steps 1 through 3
5. Repeat steps 1 through 4 for all scales

Discrete Wavelet Transform

- Scale and displacement are continuous variables
- We choose only a finite subset of scales and displacement
- *Discrete wavelet transform*:
  - Displacements and scales in powers of 2:

    $s^{-1} = 2^j, \quad d = k\,2^j = k\,s^{-1}, \quad j$ and k integers

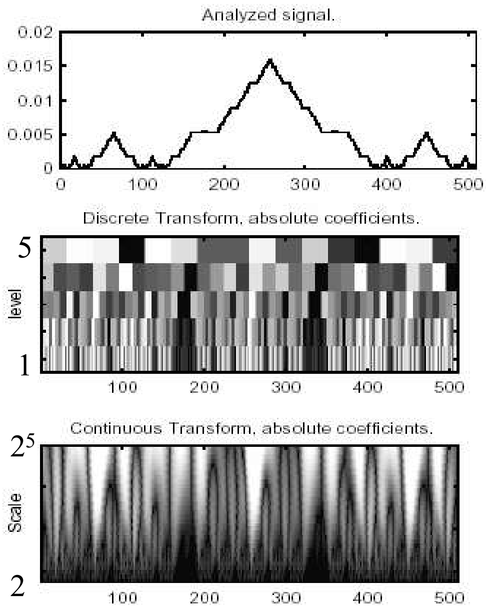
$$\psi_{j,k}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-k2^j}{2^j}\right) = 2^{-j/2}\psi\left(2^{-j}t-k\right)$$

$$C(s,d) = C(j,k) = \sum_{n=-\infty}^{\infty} f(n)2^{-j/2}\psi(2^{-j}n-k)$$

Levels and Resolution

- Scale s and level j are related by: $s = 2^j$
- Resolution : $1/s$
- The smaller is the resolution (larger scale) the higher is the level of detail than can be accessed.

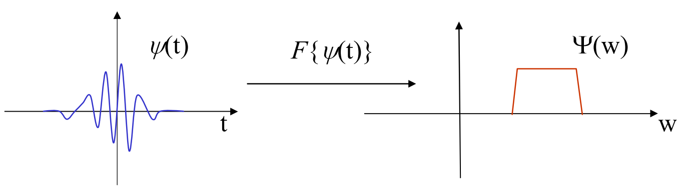
j	10	9	…	2	1	0	-1	-2
Scale	1024	512	…	4	2	1	1/2	1/4
Resolution	$1/2^{10}$	$1/2^9$	…	1/4	1/2	1	2	4



Fourier Transform of the Wavelet signal

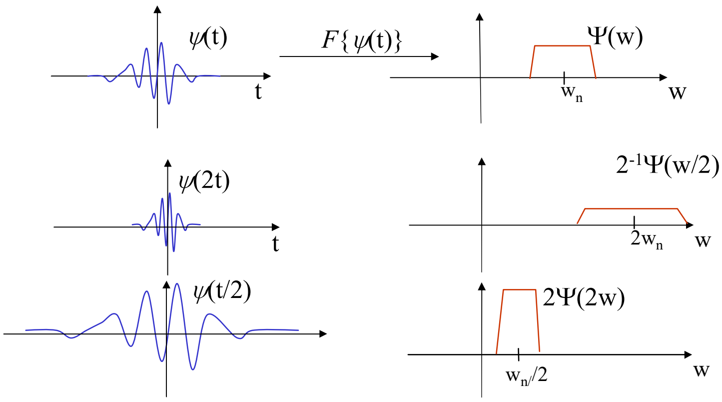
Wavelets have a bandpass structure:

$$\int \frac{|\Psi(w)|^2}{|w|}\, dw < \infty, \quad \Rightarrow \quad |\Psi(0)|^2 = 0$$



Effect of the Scaling in the FT

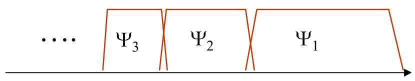
$$F\{\psi(at)\} = \frac{1}{|a|}\Psi\left(\frac{w}{a}\right)$$



The wavelet compressed by a factor of 2 extends the frequency of the wavelet spectrum by the same factor and moves the frequencies that factor

- Given a signal we can cover its full spectrum with the spectrum of scaled wavelets in the same way that we can cover the signal in the time domain with the displacement of wavelets.

- If we see a wavelet as a bandpass filter, a series of scaled wavelets can be seen as a bank of bandpass filters.



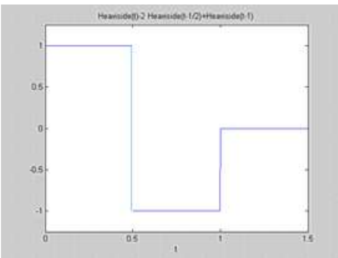
- We can see the wavelet transform of a signal as the signal passing trough a bank of filters: Filter Banks

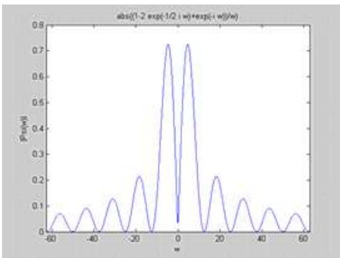
Wavelet de Haar

$$\psi(t) = \begin{cases} 1 & \text{si } \; 0 \le t < 1/2 \\ \text{-1} & \text{si } \; 1/2 \le t < 1 \\ 0 & \text{en otro caso} \end{cases}$$

Fourier Transform

$$\Psi(w) = \frac{j}{\sqrt{2}} \frac{\left(1 - 2e^{-j\frac{w}{2}} + e^{-jw}\right)}{w}$$

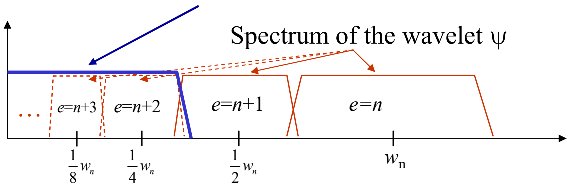




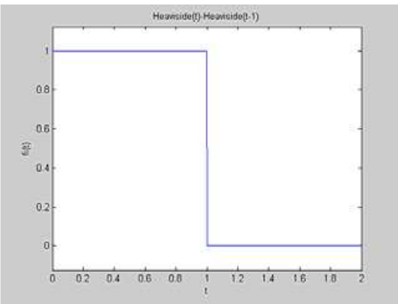

```
% Haar Wavelet
syms w t
wv=sym('Heaviside(t)')-2*sym('Heaviside(t-1/2)')…
        +sym('Heaviside(t-1)');
figure(1),ezplot(wv,[0,1.5])

% Its Fourier Transform
WV=fourier(wv); WV=simplify(WV)
figure(2),ezplot(abs(WV),[-20*pi,20*pi])
```

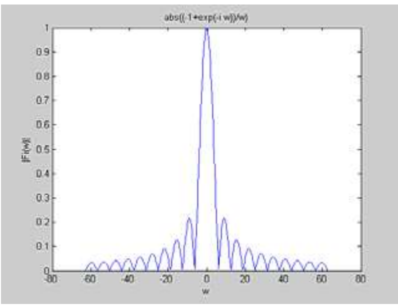
Scale Function

- Everytime we stretch the wavelet by a factor of 2, the bandwidth is halved: reduces by one half: an infinite number of Spectra is needed to reach the zero frequency.
- We make no attempt to cover the entire spectrum, but we use a low pass filter covering the hole, when this is sufficiently small: the spectrum correspond to the denominated *scale function*.



Scale Function for the Haar Wavelet






```
%Haar Scale Function
syms w t
wv=sym('Heaviside(t)')-sym('Heaviside(t-1)');
figure(1),ezplot(wv,[0,2])
% Transformada de Fourier
WV=fourier(wv); WV=simplify(WV)
figure(2),ezplot(abs(WV),[-20*pi,20*pi])
```

Solution of the Refinament Equation

$$\phi(t) = 2\sum_{k=0}^{N} h(k)\phi(2t-k)$$

$$\int_{-\infty}^{\infty} \phi(t)e^{-jwt}dt = 2\sum_{k=0}^{N} h(k)\int_{-\infty}^{\infty} \phi(2t-k)e^{-jwt}dt$$

$$= 2\sum_{k=0}^{N} h(k)\tfrac{1}{2}\int_{-\infty}^{\infty} \phi(\tau)e^{-jw(\tau+k)/2}d\tau$$

$$= \sum_{k=0}^{N} h(k)e^{-jwk/2}\int_{-\infty}^{\infty} \phi(\tau)e^{-jw\tau/2}d\tau$$

then $\Phi(w) = H(\tfrac{w}{2})\,\Phi(\tfrac{w}{2}) = H(\tfrac{w}{2})\,H(\tfrac{w}{4})\,\Phi(\tfrac{w}{4})$

$$\vdots$$

$$= \left(\prod_{j=1}^{\infty} H(\tfrac{w}{2^j})\right)\Phi(0)$$

If the scale function area is normalized to one :

$$\Phi(0) = \int_{-\infty}^{\infty} \phi(t)dt = 1$$

$$\Phi(w) = \prod_{j=1}^{\infty} H(\tfrac{w}{2^j})$$

Ralationship between Filter and Scale Function

Interesting Properties for H(w)

- H(0) = 1, so $\Phi(0) = 1$
- In order the scale function have finite energy

$$\int_{-\infty}^{\infty} |\Phi(w)|^2 \, dw < \infty$$

it must be fullfilled: $H(w) \underset{w \to \pi}{\longrightarrow} 0$

Solution of the Wavelet Equation

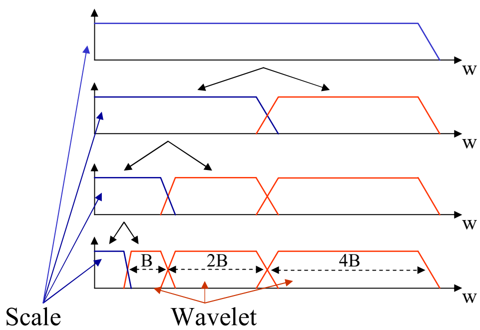
$$\psi(t) = 2\sum_{k=0}^{N} g(k)\phi(2t-k)$$

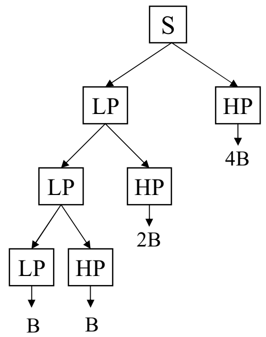
proceeding in a similar way:

$$\begin{aligned}\Psi(w) &= G(\tfrac{w}{2})\,\Phi(\tfrac{w}{2}) \\ &= G(\tfrac{w}{2})\left(\prod_{j=2}^{\infty} H(\tfrac{w}{2^j})\right)\Phi(0)\end{aligned}$$

Band Coding

- We break down the spectrum of the signal in two : a Low Pass (LP) and a High Pass (HP)
  - The HP contains the details, low scale, and the LP contains the approximations, high scale.
- The LP breaks down again in two: LP and HP
- The process continuous until obtaining the desired number of bands



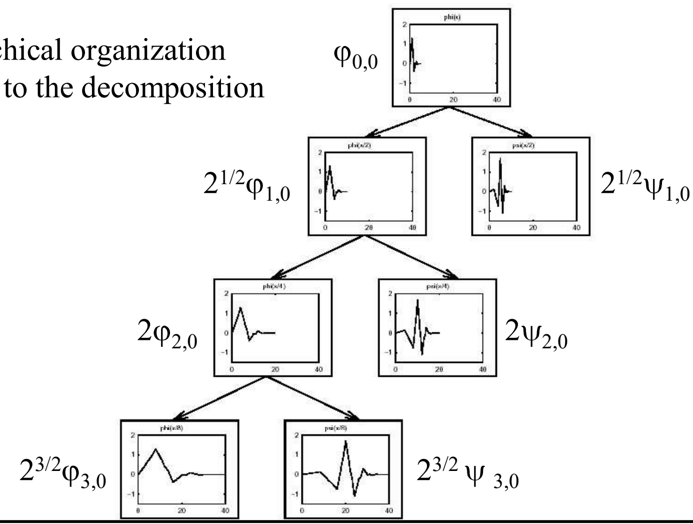


With scales in powers of 2:

$$s = 2j, \quad d = k\,2j = k\,s$$

The scale funtion becomes: $\varphi_{j,k}(t) = \frac{1}{\sqrt{2^j}}\varphi\left(\frac{t-k2^j}{2^j}\right) = 2^{-j/2}\varphi\left(2^{-j}t-k\right)$

Hierarchical organization similar to the decomposition

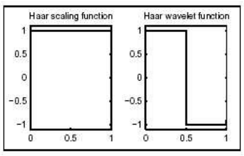


Conclusion

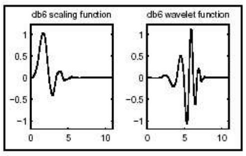
- Applying a Wavelet Transform is equivalent to passing the signal through a bandpass Filter Bank
- The Wavelet defines the details: that is, it gives the bandpass filters with a bandwidth that is reduced by half in each step.
- The scale function defines the approximations.
- If the transformation is performed in this way is not necessary to specify the wavelet explicitly.

Example of Wavelets and their Associated Scale Functions

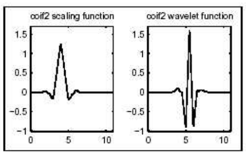
Haar



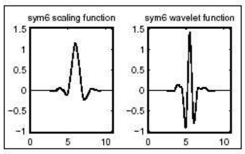
Daubechies



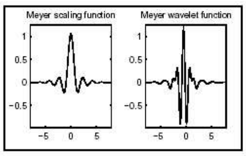
Coiflets



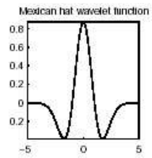
Symlets



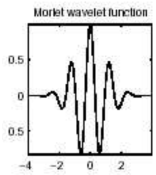
Meyer



Mexican Hat

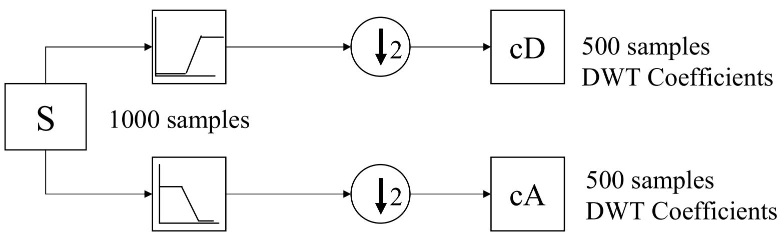


Morlet



Subsampling

- Every time that the signal is broken down into two parts, the number of samples of each part is half of the original signal. The result are the coefficients of the discrete wavelets



Downsampling: of every two samples one is eliminated

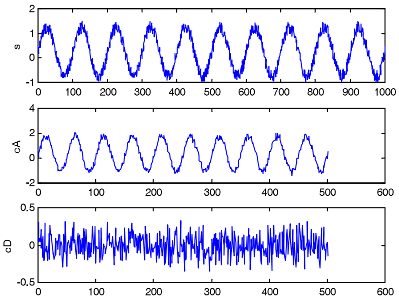
Example 1

```
t=linspace(0,pi,1000);
s=sin(20*t)+0.5*rand(1,1000);
[cA,cD]=dwt(s,'db2');
subplot(3,1,1),plot(s),ylabel('s')
subplot(3,1,2),plot(cA),ylabel('cA')
subplot(3,1,3),plot(cD),ylabel('cD')
```

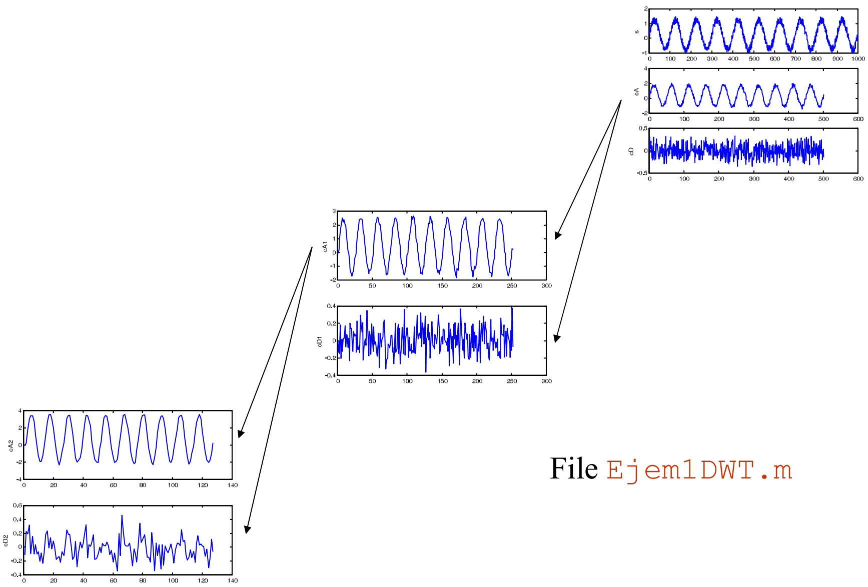
Signal

Aproximation components

Detail components



Decomposition Tree



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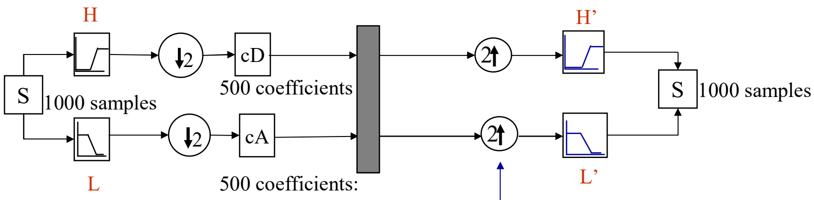
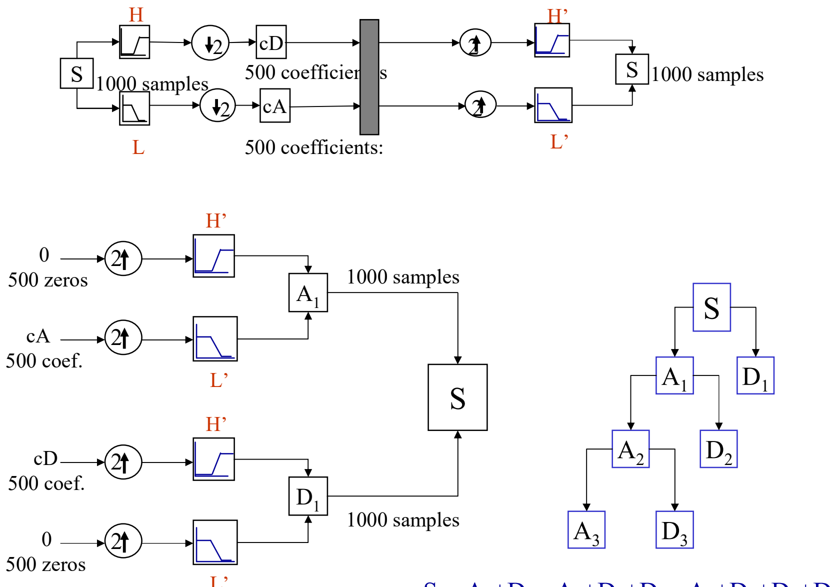
Reconstruction or Synthesis

- Inverse Discrete Wavelet Transform (IDWT)

$$s(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C(j,k)\psi_{j,k}(t)$$



Upsampling: lengthening a signal component by inserting zeros between samples.



$S = A_1+D_1= A_2+D_2+D_1= A_3+D_3+D_3+D_1$

2-Dimensional Transform

For image processing: $\psi(x,y)$

Shifting and Scaling

$$\psi_{s_1,s_2,d_1,d_2}(x,y) = \frac{1}{\sqrt{s_1 s_2}}\psi\left(\frac{x-d_1}{s_1}, \frac{y-d_2}{s_2}\right)$$

Scale values in powers of 2: $s = 2^j, \quad d = k\,2^j = k\,s$

Two-dimensional wavelet is defined as the tensor product of 1-dimensiona wavelets:

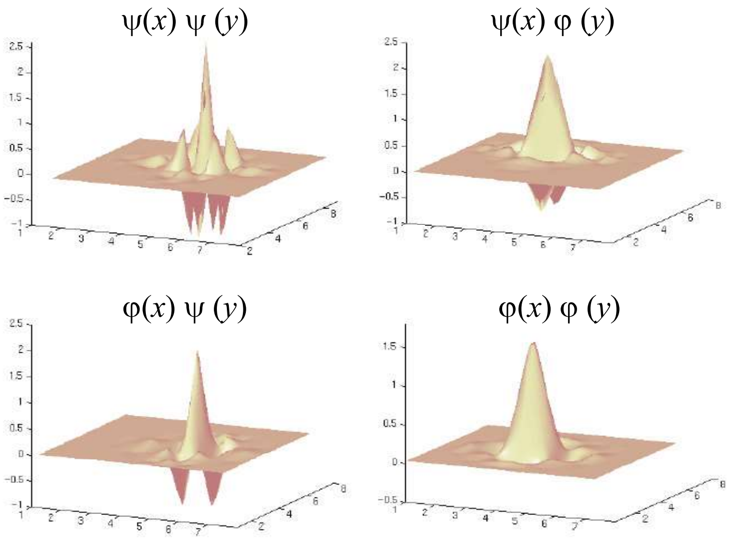
Scale Function: $\varphi(x,y) = \varphi(x)\,\varphi(y)$

Wavelets:

$\psi_1(x,y) = \varphi(x)\,\psi(y)$,

$\psi_2(x,y) = \psi(x)\,\varphi(y)$,

$\psi_3(x,y) = \psi(x)\,\psi(y)$



Two-Dimensional Coiflet Wavelet

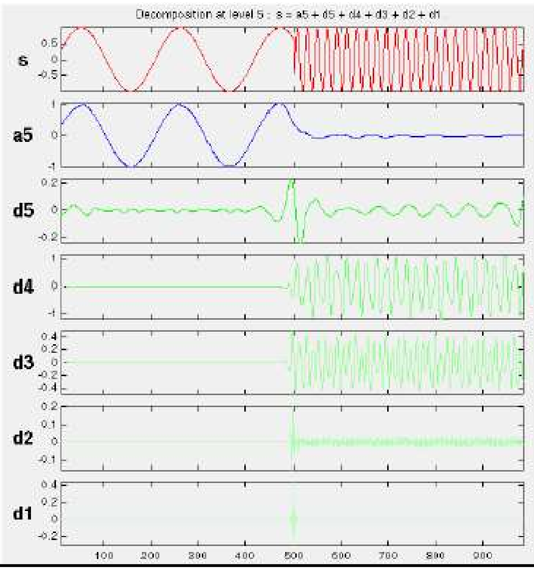
Applications

- Detecting Discontinuities
- Detecting Trends
- Detecting Self-Similarity
- Identifying Pure Frequencies
- Suppressing Signals
- De-Noising Signals
- Compressing Signals

Detecting Discontinuities

Dos sinusoides de distintas frecuencias

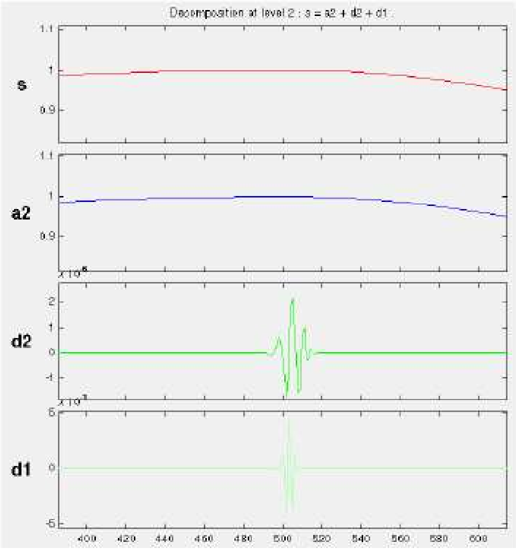
Wavelet: db5
Nivel: 5



Detecting Discontinuities

Dos exponenciales conectadas en $t = 500$

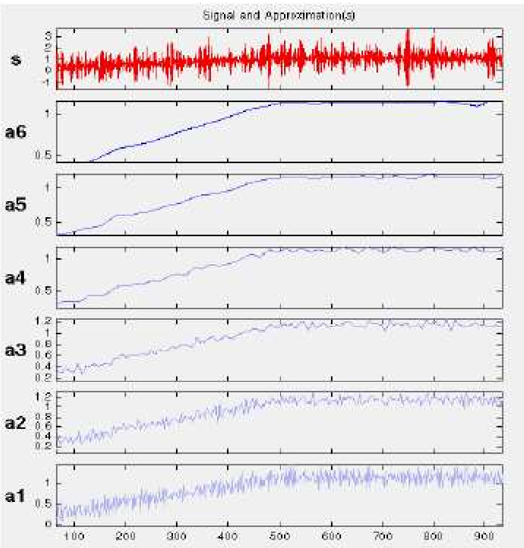
Wavelet: db4
Nivel: 2



Detecting Trends

Rampa oscurecida por ruido coloreado

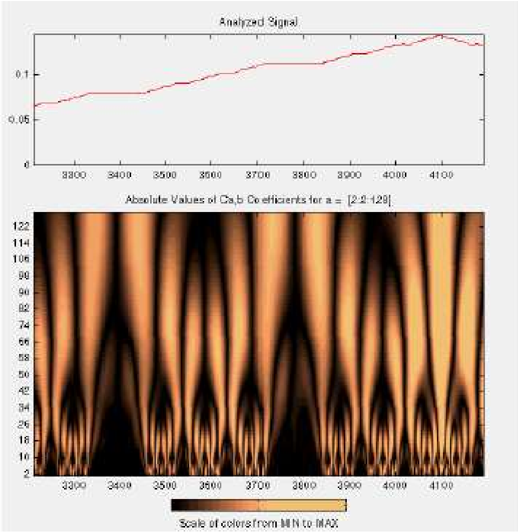
Wavelet: db3
Nivel: 6



Detecting Self-Similarity

Fractal

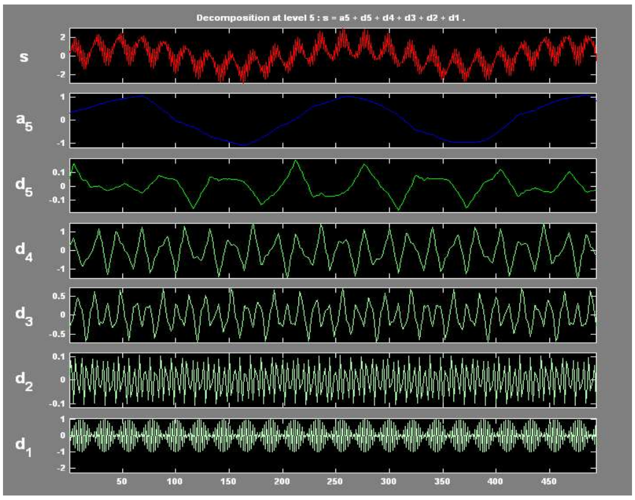
Wavelet: coif3
Nivel: 2:2:128



Detecting Frequencies

Tres sinusoides de Frecuencias: 0.005, 0.05, 0.5 (A5, D4, D1)

Wavelet: db3
Nivel: 5



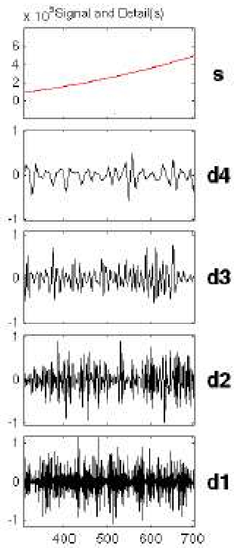
Suppressing Signals

Polinomio de 2º orden con ruido.

En los detalles se ha eliminado completamente la señal polinómica

Wavelet: db3

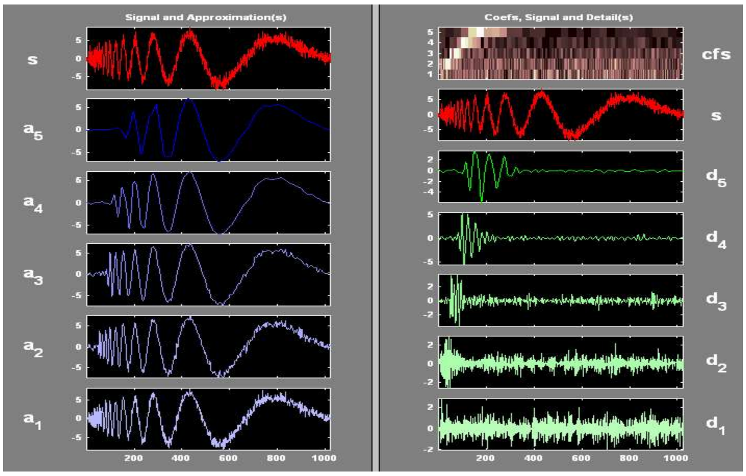
Nivel: 4



De Noising Signals

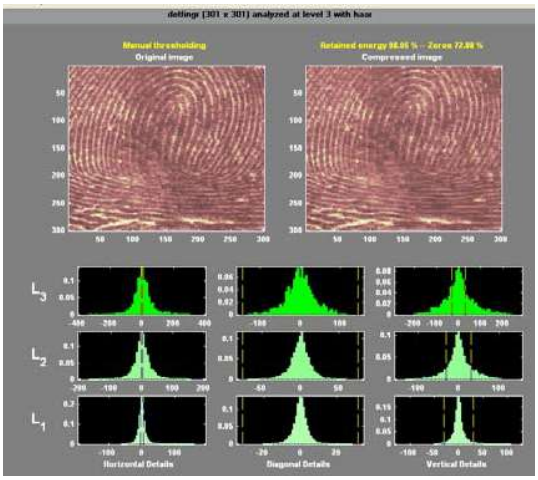
Desplazamiento Doppler de una sinusoide con adición de ruido

Wavelet: sym4, Nivel: 5



Compressing Images

Usada en la compresión de imágenes para almacenamiento de información.



Summary

The wavelets allow you to perform stationary analysis of signals, as the Fourier transform does, but also analysis of localized areas of a signal, allowing study characteristics such as shifts, trends, abrupt changes and start and end events.

Wavelet Transform can be interpreted as a Filter Banks, so its explicit specification is not required.

Twp-dimensional Wavelet Transform can be used for image processing.

Bibliography

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Strang, G., T. Nguyen (1996), "Wavelets and filter banks",Wellesley-Cambridge Press.

Mallat, S. (2001), “A wavelet tour of signal processing”. 2º Edition, Academic Press.

http://www.wavelet.org/

Wavelet Toolbox for use with MATLAB.