Vibração Forçada Qualquer: 1 Grau de Liberdade

Sistemas Dinâmicos II

Profs. Ettore de Barros, Larissa Driemeier e Marcílio Alves

Baseado no Livro:

If the forcing function is periodic but not harmonic, it can be replaced by a sum of harmonic functions using the harmonic analysis procedure discussed in Section 1.11. Using the principle of superposition, the response of the system can then be determined by superposing the responses due to the individual harmonic forcing functions.

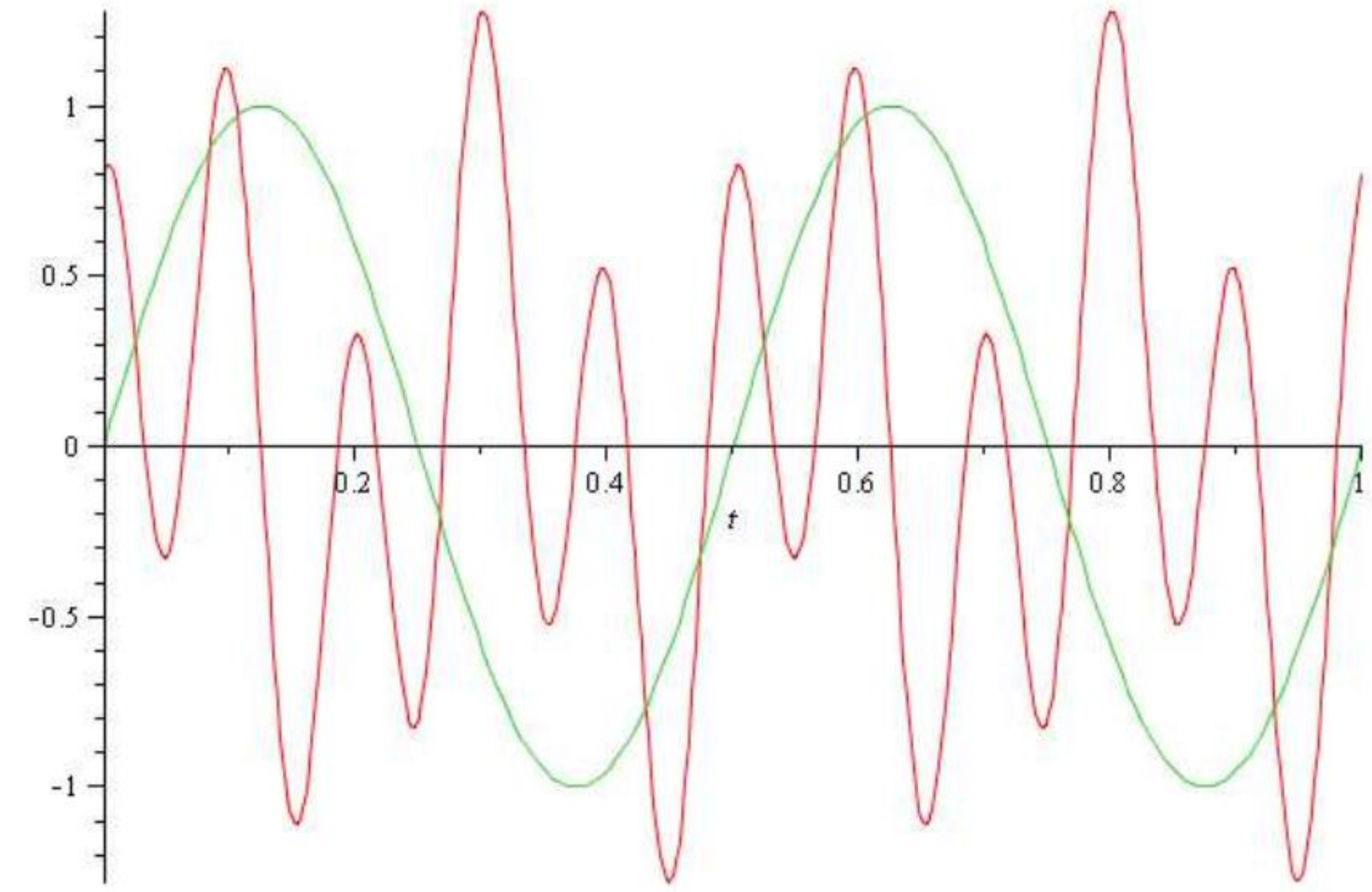
The response of a system subjected to any type of nonperiodic force is commonly found using the following methods:

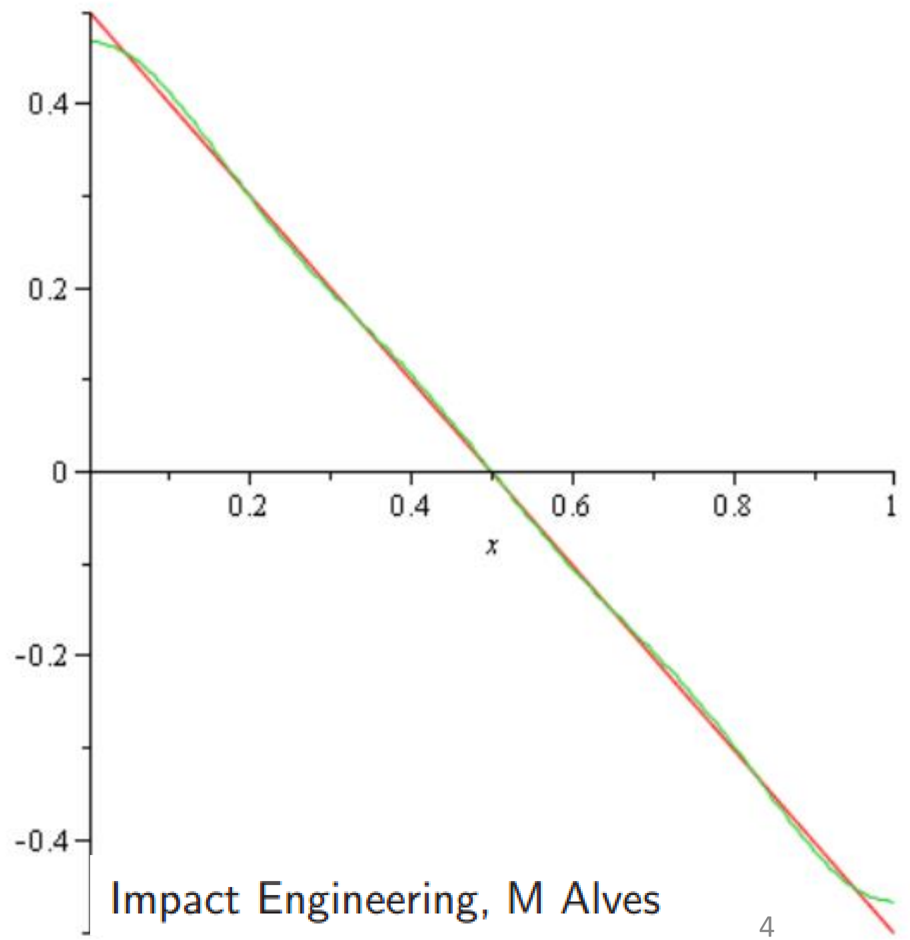
1. Convolution integral.
2. Laplace transform.
3. Numerical methods.

The first two methods are analytical ones, in which the response or solution is expressed in a way that helps in studying the behavior of the system under the applied force with respect to various parameters and in designing the system. The third method, on the other hand, can be used to find the response of a system under any arbitrary force for which an analytical solution is difficult or impossible to find. However, the solution found is applicable only for the particular set of parameter values used in finding the solution. This makes it difficult to study the behavior of the system when the parameters are varied. This chapter presents all three methods of solution.

Let us plot the function $\sin(\omega t)$ in the interval, say, $[0..1]$ with $\omega = 4\pi$, as shown in the next figure. Now, on the same plot we add a function, say $0.8\cos(5\omega t)+0.5\sin(2\omega t)$ and we observe that the period of both waves is the same. We conclude with similar examples that we can add as many trigonometric functions as we wish, of any frequency larger than ω and yet such a composed function repeats itself every ωt.

We can modify slightly this reasoning and plot the function $0.406\cos(\pi x)+0.045\cos(3\pi x)+0.016\cos(5\pi x)$ and the line $0.5-x$, as in the figure, and be convinced that the straight line can be well approximated by a sum of trigonometric functions.





Response Under a General Periodic Force

When the external force $F(t)$ is periodic with period $\tau = 2\pi/\omega$, it can be expanded in a Fourier series (see Section 1.11):

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.1}$$

where

$$a_j = \frac{2}{\tau}\int_0^{\tau} F(t) \cos j\omega t \, dt, \qquad j = 0, 1, 2, \ldots \tag{4.2}$$

and

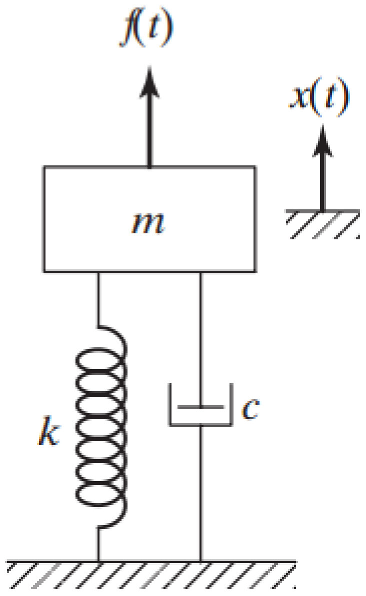
$$b_j = \frac{2}{\tau}\int_0^{\tau} F(t) \sin j\omega t \, dt, \qquad j = 1, 2, \ldots \tag{4.3}$$

Let a spring-mass-damper system, Fig. 4.2(a), be subjected to a periodic force. This is a second-order system because the governing equation is a second-order differential equation:

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{4.7}$$

If the forcing function $f(t)$ is periodic, it can be expressed in Fourier series so that the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.8}$$



$m\ddot{x} + c\dot{x} + kx = f(t)$

(a)

Determine the response of a spring-mass-damper system subjected to a periodic force with the equation of motion given by Eq. (4.8). Assume the initial conditions as zero.

Solution: The right-hand side of Eq. (4.8) is a constant plus a sum of harmonic functions. Using the principle of superposition, the steady-state solution of Eq. (4.4) is the sum of the steady-state solutions of the following equations:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \tag{E.1}$$

$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t \tag{E.2}$$

$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t \tag{E.3}$$

Noting that the solution of Eq. (E.1) is given by

$$x_p(t) = \frac{a_0}{2k} \tag{E.4}$$

and, using the results of Section 3.4, we can express the solutions of Eqs. (E.2) and (E.3), respectively, as

$$x_p(t) = \frac{(a_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j)$$

$$x_p(t) = \frac{(b_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j)$$

$$\phi_j = \tan^{-1}\left(\frac{2\zeta jr}{1 - j^2r^2}\right)$$

Thus the complete steady-state solution of Eq. (4.8) is given by

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty} \frac{(b_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta jr)^2}} \sin(j\omega t - \phi_j) \tag{E.9}$$

It can be seen from the solution, Eq. (E.9), that the amplitude and phase shift corresponding to the jth term depend on j. If $j\omega = \omega_n$, for any j, the amplitude of the corresponding harmonic will be comparatively large. This will be particularly true for small values of j and ζ. Further, as j becomes larger, the amplitude becomes smaller and the corresponding terms tend to zero. Thus the first few terms are usually sufficient to obtain the response with reasonable accuracy.

The solution given by Eq. (E.9) denotes the steady-state response of the system. The transient part of the solution arising from the initial conditions can also be included to find the complete solution. To find the complete solution, we need to evaluate the arbitrary constants by setting the value of the complete solution and its derivative to the specified values of initial displacement $x(0)$ and the initial velocity $\dot{x}(0)$. This results in a complicated expression for the transient part of the total solution.

A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period and then stops. The simplest form is the impulsive force—a force that has a large magnitude F and acts for a very short time Δt. From dynamics we know that impulse can be measured by finding the change it causes in momentum of the system [4.2]. If $\dot{x}_1$ and $\dot{x}_2$ denote the velocities of the mass m before and after the application of the impulse, we have

$$\text{Impulse} = F\Delta t = m\dot{x}_2 - m\dot{x}_1 \tag{4.12}$$

By designating the magnitude of the impulse $F\Delta t$ by $\boldsymbol{F}$, we can write, in general,

$$\boldsymbol{F} = \int_t^{t+\Delta t} F\,dt \tag{4.13}$$

A unit impulse acting at $t = 0$ ($\boldsymbol{f}$) is defined as

$$\boldsymbol{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F\,dt = F\,dt = 1 \tag{4.14}$$

It can be seen that in order for $F\,dt$ to have a finite value, F tends to infinity (since dt tends to zero).

The unit impulse, $\boldsymbol{f} = 1$, acting at $t = 0$, is also denoted by the Dirac delta function as

$$\boldsymbol{f} = \boldsymbol{f}\delta(t) = \delta(t) \tag{4.15}$$

and the impulse of magnitude $\boldsymbol{F}$, acting at $t = 0$, is denoted as[1]

$$\boldsymbol{F} = \boldsymbol{F}\,\delta(t) \tag{4.16}$$

4.5.1 Response to an Impulse

We first consider the response of a single-degree-of-freedom system to an impulse excitation; this case is important in studying the response under more general excitations. Consider a viscously damped spring-mass system subjected to a unit impulse at $t = 0$, as shown in Figs. 4.6(a) and (b). For an underdamped system, the solution of the equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4.17}$$

is given by Eq. (2.72) as

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos \omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t\right\} \tag{4.18}$$

where

$$\zeta = \frac{c}{2m\omega_n} \tag{4.19}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{4.20}$$

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.21}$$

If the mass is at rest before the unit impulse is applied ($x = \dot{x} = 0$ for $t < 0$ or at $t = 0^-$), we obtain, from the impulse-momentum relation,

$$\text{Impulse} = \boldsymbol{f} = 1 = m\dot{x}(t = 0) - m\dot{x}(t = 0^-) = m\dot{x}_0 \tag{4.22}$$

Thus the initial conditions are given by

$$x(t=0) = x_0 = 0 \tag{4.23}$$

$$\dot{x}(t=0) = \dot{x}_0 = \frac{1}{m} \tag{4.24}$$

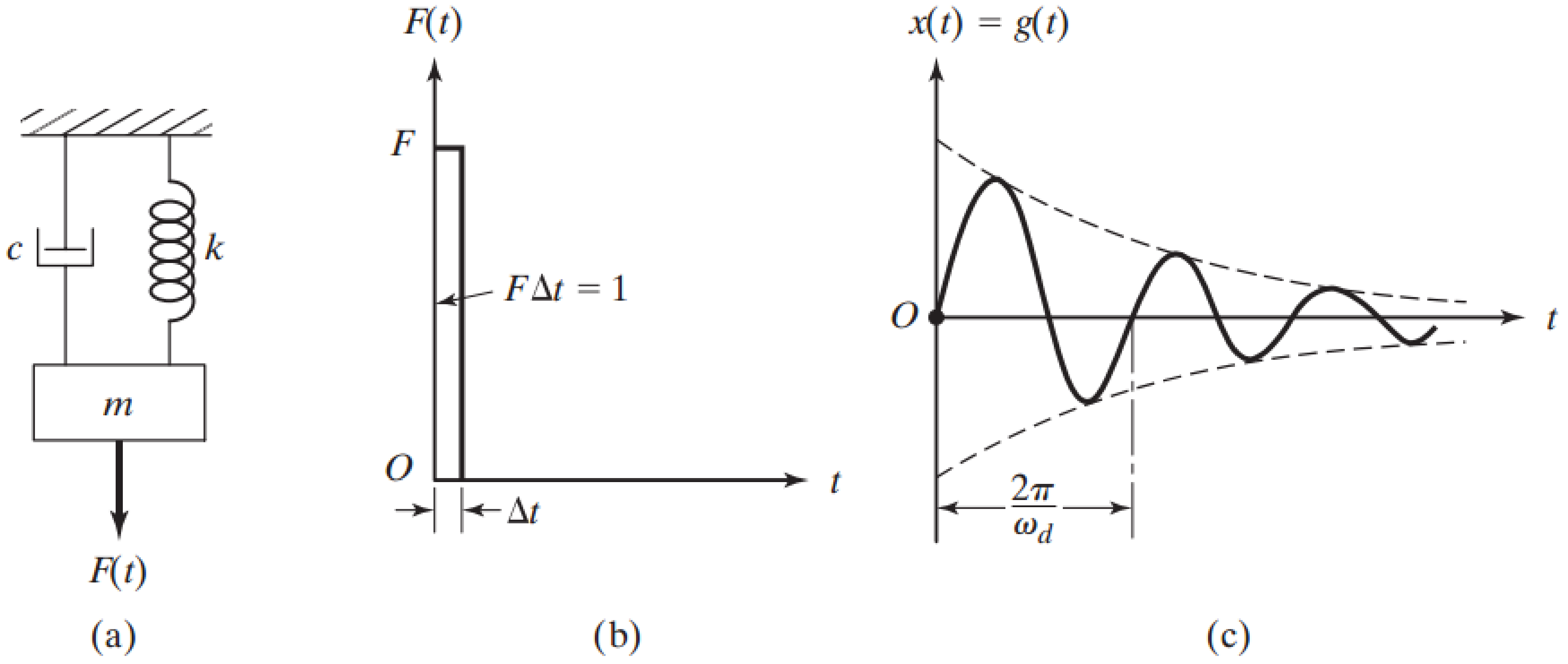


FIGURE 4.6 A single-degree-of-freedom system subjected to an impulse.

In view of Eqs. (4.23) and (4.24), Eq. (4.18) reduces to

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t \tag{4.25}$$

Equation (4.25) gives the response of a single-degree-of-freedom system to a unit impulse, which is also known as the *impulse response function*, denoted by $g(t)$. The function $g(t)$, Eq. (4.25), is shown in Fig. 4.6(c).

If the magnitude of the impulse is $\boldsymbol{F}$ instead of unity, the initial velocity $\dot{x}_0$ is $\boldsymbol{F}/m$ and the response of the system becomes

$$x(t) = \frac{\boldsymbol{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = \boldsymbol{F}g(t) \tag{4.26}$$

If the impulse $\boldsymbol{F}$ is applied at an arbitrary time $t = \tau$, as shown in Fig. 4.7(a), it will change the velocity at $t = \tau$ by an amount $\boldsymbol{F}/m$. Assuming that $x = 0$ until the impulse is applied, the displacement x at any subsequent time t, caused by a change in the velocity at time τ, is given by Eq. (4.26) with t replaced by the time elapsed after the application of the impulse—that is, $t - \tau$. Thus we obtain

$$x(t) = \boldsymbol{F}g(t - \tau) \tag{4.27}$$

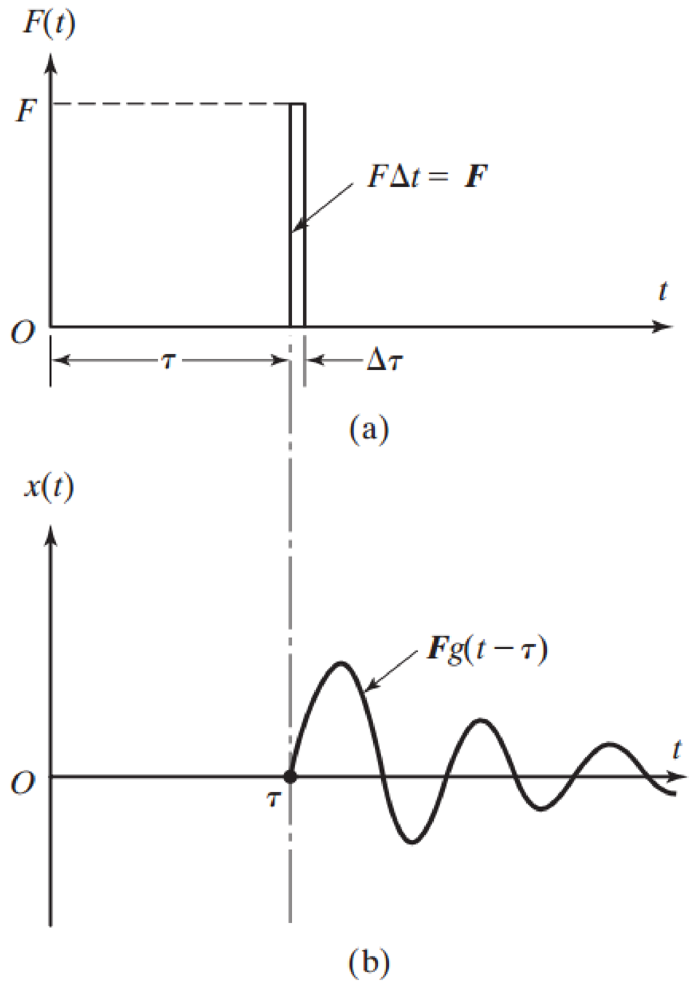


FIGURE 4.7 Impulse response.

In the vibration testing of a structure, an impact hammer with a load cell to measure the impact force is used to cause excitation, as shown in Fig. 4.8(a). Assuming $m = 5$ kg, $k = 2000$ N/m, $c = 10$ N-s/m, and $\boldsymbol{F} = 20$ N-s, find the response of the system.

Solution: From the known data, we can compute

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{5}} = 20 \text{ rad/s}, \qquad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{10}{2\sqrt{2000(5)}} = 0.05$$

$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n = 19.975 \text{ rad/s}$$

Assuming that the impact is given at $t = 0$, we find (from Eq. (4.26)) the response of the system as

$$x_1(t) = \boldsymbol{F}\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

$$= \frac{20}{(5)(19.975)}e^{-0.05(20)t}\sin 19.975t = 0.20025e^{-t}\sin 19.975t \text{ m} \qquad \text{(E.1)}$$

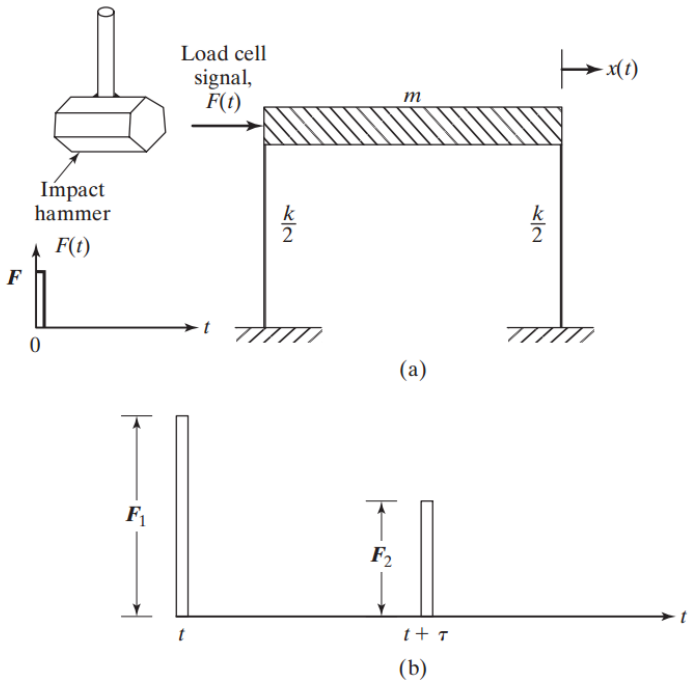


FIGURE 4.8 Structural testing using an impact hammer.

INTEGRAL DE CONVOLUÇÃO

4.5.2 Response to a General Forcing Condition

Now we consider the response of the system under an arbitrary external force $F(t)$, shown in Fig. 4.9. This force may be assumed to be made up of a series of impulses of varying magnitude. Assuming that at time τ, the force $F(\tau)$ acts on the system for a short period of time $\Delta\tau$, the impulse acting at $t = \tau$ is given by $F(\tau)\,\Delta\tau$. At any time t, the elapsed time since the impulse is $t - \tau$, so the response of the system at t due to this impulse alone is given by Eq. (4.27) with $\boldsymbol{F} = F(\tau)\,\Delta\tau$:

$$\Delta x(t) = F(\tau)\,\Delta\tau\, g(t-\tau) \tag{4.28}$$

The total response at time t can be found by summing all the responses due to the elementary impulses acting at all times τ:

$$x(t) \simeq \sum F(\tau) g(t-\tau)\,\Delta\tau \tag{4.29}$$

Letting $\Delta\tau \rightarrow 0$ and replacing the summation by integration, we obtain

$$x(t) = \int_0^t F(\tau) g(t-\tau)\, d\tau \tag{4.30}$$

By substituting Eq. (4.25) into Eq. (4.30), we obtain

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)\, d\tau \tag{4.31}$$

which represents the response of an underdamped single-degree-of-freedom system to the arbitrary excitation $F(t)$. Note that Eq. (4.31) does not consider the effect of initial conditions of the system, because the mass is assumed to be at rest before the application of the impulse, as implied by Eqs. (4.25) and (4.28). The integral in Eq. (4.30) or Eq. (4.31) is called the *convolution* or *Duhamel integral*. In many cases the function $F(t)$ has a form that permits an explicit integration of Eq. (4.31). If such integration is not possible, we can evaluate numerically without much difficulty, as illustrated in Section 4.9 and in Chapter 11. An elementary discussion of the Duhamel integral in vibration analysis is given in reference [4.6].

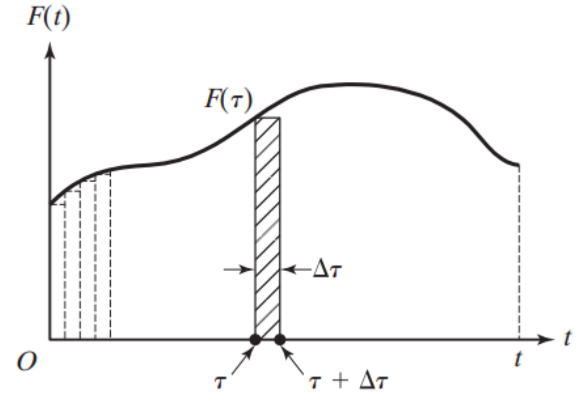
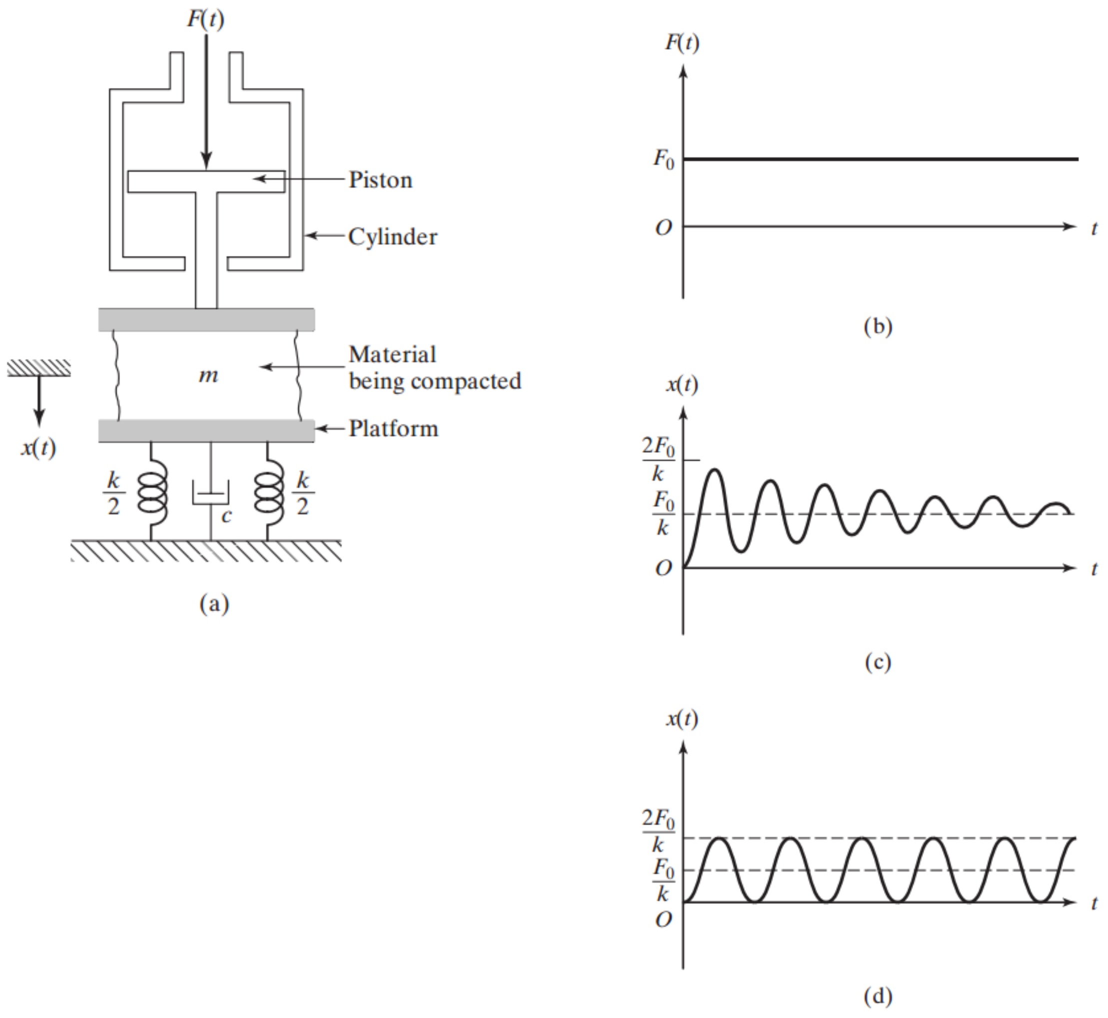


FIGURE 4.9 An arbitrary (nonperiodic) forcing function.

A compacting machine, modeled as a single-degree-of-freedom system, is shown in Fig. 4.10(a). The force acting on the mass m (m includes the masses of the piston, the platform, and the material being compacted) due to a sudden application of the pressure can be idealized as a step force, as shown in Fig. 4.10(b). Determine the response of the system.



Solution: Since the compacting machine is modeled as a mass-spring-damper system, the problem is to find the response of a damped single-degree-of-freedom system subjected to a step force. By noting that $F(t) = F_0$, we can write Eq. (4.31) as

$$
\begin{aligned}
x(t) &= \frac{F_0}{m\omega_d}\int_0^t e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau)\, d\tau \\
&= \frac{F_0}{m\omega_d}\left[e^{-\zeta\omega_n(t-\tau)} \left\{ \frac{\zeta\omega_n \sin\omega_d(t-\tau) + \omega_d \cos\omega_d(t-\tau)}{(\zeta\omega_n)^2 + (\omega_d)^2} \right\} \right]_{\tau=0}^{t} \\
&= \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \cos(\omega_d t - \phi) \right]
\end{aligned}
\tag{E.1}
$$

where

$$
\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}} \right) \tag{E.2}
$$

This response is shown in Fig. 4.10(c). If the system is undamped ($\zeta = 0$ and $\omega_d = \omega_n$), Eq. (E.1) reduces to

$$
x(t) = \frac{F_0}{k}[1 - \cos\omega_n t] \tag{E.3}
$$

Equation (E.3) is shown graphically in Fig. 4.10(d). It can be seen that if the load is instantaneously applied to an undamped system, a maximum displacement of twice the static displacement will be attained—that is, $x_{max} = 2F_0/k$.

Rectangular Pulse Load

If the compacting machine shown in Fig. 4.10(a) is subjected to a constant force only during the time $0 \leq t \leq t_0$ (Fig. 4.12a), determine the response of the machine.

Solution: The given forcing function, $F(t)$, can be considered as the sum of a step function $F_1(t)$ of magnitude $+F_0$ beginning at $t = 0$ and a second step function $F_2(t)$ of magnitude $-F_0$ starting at time $t = t_0$, as shown in Fig. 4.12(b).

Thus the response of the system can be obtained by subtracting Eq. (E.1) of Example 4.10 from Eq. (E.1) of Example 4.9. This gives

$$x(t) = \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\left[-\cos(\omega_d t - \phi) + e^{\zeta\omega_n t_0}\cos\{\omega_d(t - t_0) - \phi\}\right] \tag{E.1}$$

with

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{E.2}$$

To see the vibration response graphically, we consider the system as undamped, so that Eq. (E.1) reduces to

$$x(t) = \frac{F_0}{k}\left[\cos\omega_n(t - t_0) - \cos\omega_n t\right] \tag{E.3}$$

The response is shown in Fig. 4.12(c) for two different pulse widths of t_0 for the following data (Problem 4.90): $m = 100$ kg, $c = 50$ N-s/m, $k = 1200$ N/m, and $F_0 = 100$ N. The responses will be different for the two cases $t_0 > \tau_n/2$ and $t_0 > \tau_n/2$, where τ_n is the undamped natural time period of the system. If $t_0 > \tau_n/2$, the peak will be larger and occur during the forced-vibration era (that is, during 0 to t_0) while the peak will be smaller and occur in the residual-vibration era (that is, after t_0) if $t_0 > \tau_n/2$. In Fig. 4.12(c), $\tau_n = 1.8138$ s and the peak corresponding to $t_0 = 1.5$ s is about six times larger than the one with $t_0 = 0.1$ s.

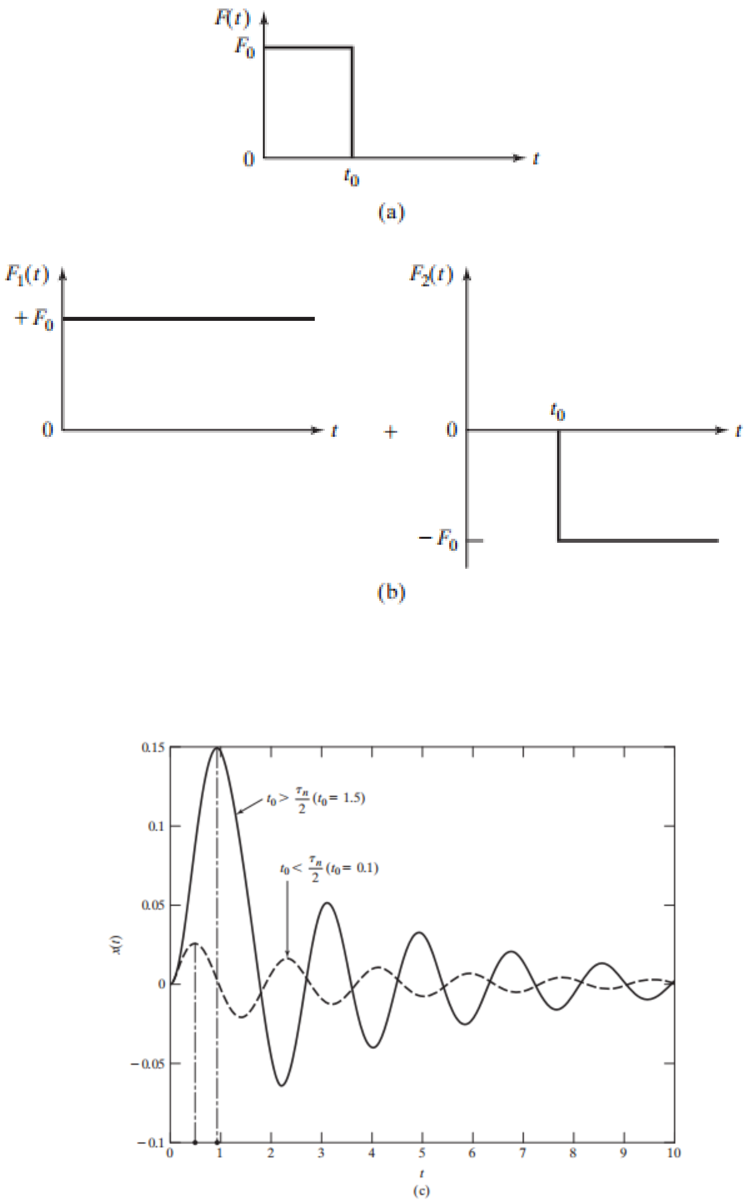
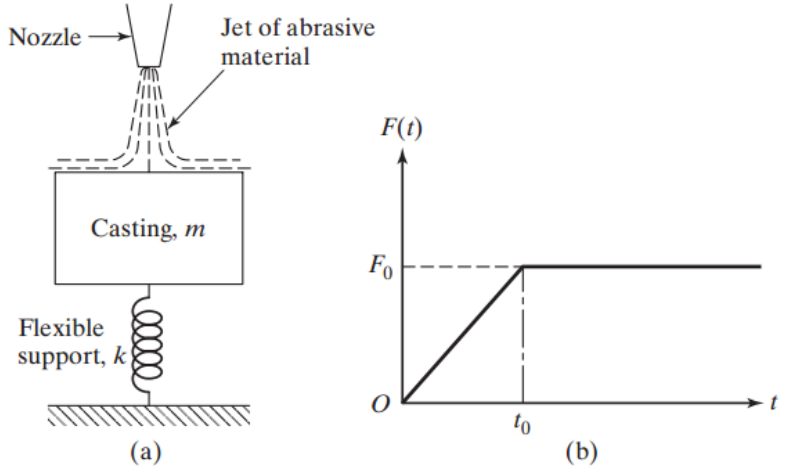


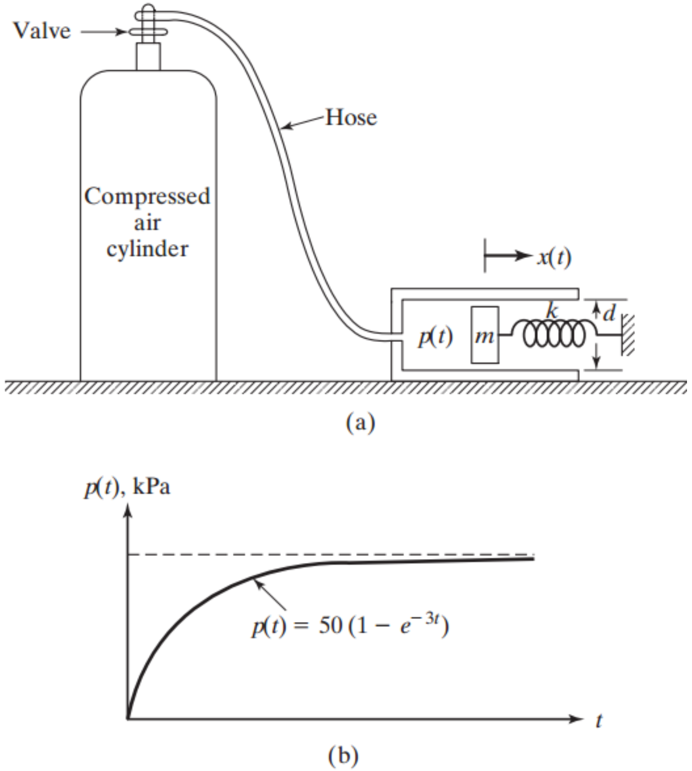
FIGURE 4.12 Response due to a pulse load.

EXERCÍCIOS

4.16 Sandblasting is a process in which an abrasive material, entrained in a jet, is directed onto the surface of a casting to clean its surface. In a particular setup for sandblasting, the casting of mass m is placed on a flexible support of stiffness k as shown in Fig. 4.44(a). If the force exerted on the casting due to the sandblasting operation varies as shown in Fig. 4.44(b), find the response of the casting.



4.18 A compressed air cylinder is connected to the spring-mass system shown in Fig. 4.45(a). Due to a small leak in the valve, the pressure on the piston, $p(t)$, builds up as indicated in Fig. 4.45(b). Find the response of the piston for the following data: $m = 10$ kg, $k = 1000$ N/m, and $d = 0.1$ m.



4.26 A camcorder of mass m is packed in a container using a flexible packing material. The stiffness and damping constant of the packing material are given by k and c, respectively, and the mass of the container is negligible. If the container is dropped accidentally from a height of h onto a rigid floor (see Fig. 4.50), find the motion of the camcorder.

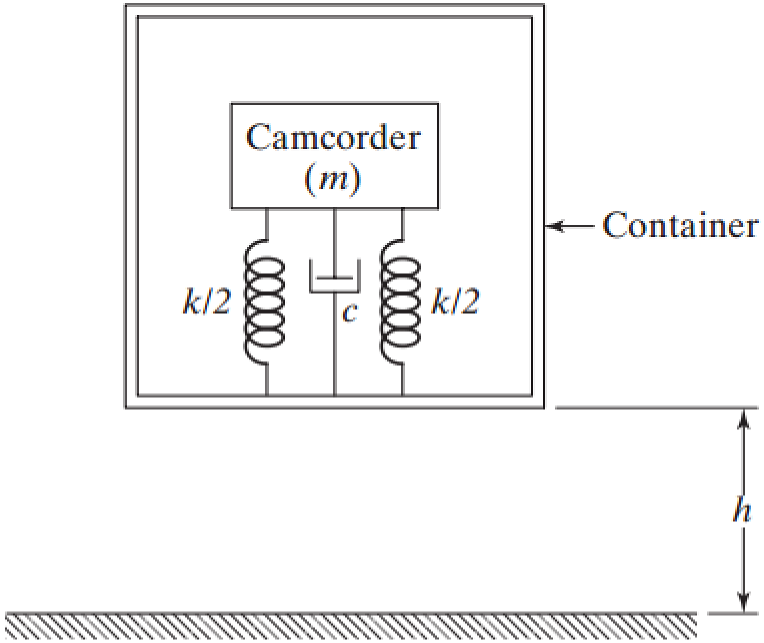
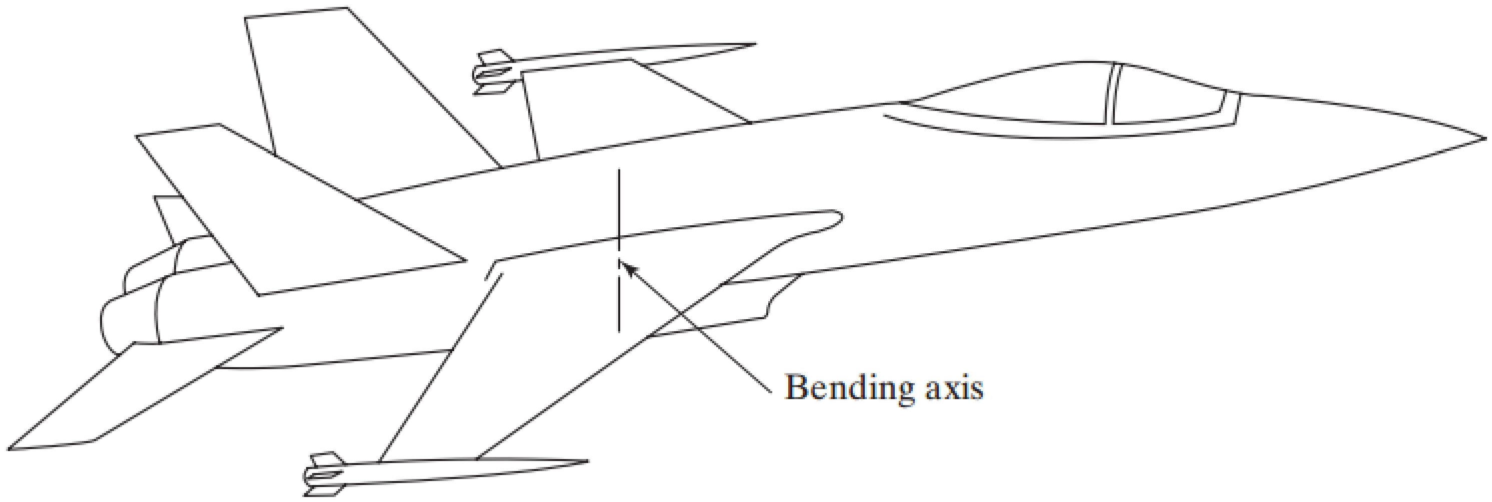


FIGURE 4.50

4.38 The wing of a fighter aircraft, carrying a missile at its tip, as shown in Fig. 4.57, can be approximated as an equivalent cantilever beam with $EI = 15 \times 10^9$ N-m^2 about the vertical axis and length $l = 10$ m. If the equivalent mass of the wing, including the mass of the missile and its carriage system, at the tip of the wing is $m = 2500$ kg, determine the vibration response of the wing (of m) due to the release of the missile. Assume that the force on m due to the release of the missile can be approximated as an impulse function of magnitude $\boldsymbol{F} = 50$ N-s.



(a) Real system

FIGURE 4.57