Vibração Livre com Amortecimento: 1 Grau de Liberdade

Sistemas Dinâmicos II

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Baseado no Livro:

2.6 Free Vibration with Viscous Damping

2.6.1 Equation of Motion

As stated in Section 1.9, the viscous damping force F is proportional to the velocity $\dot{x}$ or v and can be expressed as

$$F = -c\dot{x} \tag{2.58}$$

where c is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity. A single-degree-of-freedom system with a viscous damper is shown in Fig. 2.21. If x is measured from the equilibrium position of the mass m, the application of Newton's law yields the equation of motion:

$$m\ddot{x} = -c\dot{x} - kx$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.59}$$

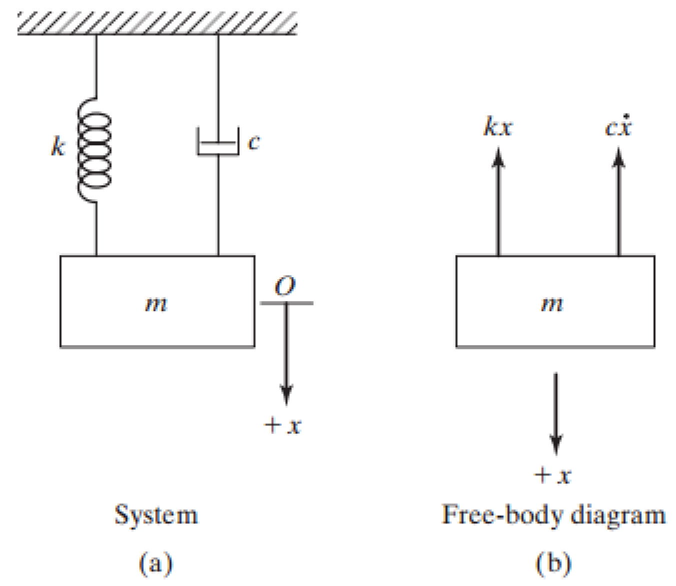


FIGURE 2.21 Single-degree-of-freedom system with viscous damper.

To solve Eq. (2.59), we assume a solution in the form

$$x(t) = Ce^{st} \tag{2.60}$$

where C and s are undetermined constants. Inserting this function into Eq. (2.59) leads to the characteristic equation

$$ms^2 + cs + k = 0 \tag{2.61}$$

the roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{2.62}$$

These roots give two solutions to Eq. (2.59):

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t} \tag{2.63}$$

Thus the general solution of Eq. (2.59) is given by a combination of the two solutions $x_1(t)$ and $x_2(t)$:

$$\begin{aligned} x(t) &= C_1 e^{s_1 t} + C_2 e^{s_2 t} \\ &= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} \end{aligned} \tag{2.64}$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions of the system.

Critical Damping Constant and the Damping Ratio. The critical damping c_c is defined as the value of the damping constant c for which the radical in Eq. (2.62) becomes zero:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n \qquad (2.65)$$

For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = c/c_c \qquad (2.66)$$

Using Eqs. (2.66) and (2.65), we can write

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n \qquad (2.67)$$

and hence

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \qquad (2.68)$$

Thus the solution, Eq. (2.64), can be written as

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t} \qquad (2.69)$$

The nature of the roots s_1 and s_2 and hence the behavior of the solution, Eq. (2.69), depends upon the magnitude of damping. It can be seen that the case $\zeta = 0$ leads to the undamped vibrations discussed in Section 2.2. Hence we assume that $\zeta \neq 0$ and consider the following three cases.

Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$s_1 = (-\zeta + i\sqrt{1-\zeta^2})\omega_n$$
$$s_2 = (-\zeta - i\sqrt{1-\zeta^2})\omega_n$$

and the solution, Eq. (2.69), can be written in different forms:

$$\begin{aligned} x(t) &= C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t} \\ &= e^{-\zeta\omega_n t}\left\{C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}\right\} \\ &= e^{-\zeta\omega_n t}\left\{(C_1 + C_2)\cos\sqrt{1-\zeta^2}\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\zeta^2}\omega_n t\right\} \\ &= e^{-\zeta\omega_n t}\left\{C_1'\cos\sqrt{1-\zeta^2}\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\omega_n t\right\} \\ &= X_0 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\omega_n t + \phi_0\right) \\ &= X e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\omega_n t - \phi\right) \end{aligned} \tag{2.70}$$

where (C_1', C_2'), (X, ϕ), and (X_0, ϕ_0) are arbitrary constants to be determined from the initial conditions.

For the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$, C_1' and C_2' can be found:

$$C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \tag{2.71}$$

and hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t\right\} \tag{2.72}$$

The constants (X, ϕ) and (X_0, ϕ_0) can be expressed as

$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} = \frac{\sqrt{x_0^2\omega_n^2 + \dot{x}_0^2 + 2x_0\dot{x}_0\zeta\omega_n}}{\sqrt{1-\zeta^2}\,\omega_n} \tag{2.73}$$

$$\phi_0 = \tan^{-1}\left(\frac{C_1'}{C_2'}\right) = \tan^{-1}\left(\frac{x_0\omega_n\sqrt{1-\zeta^2}}{\dot{x}_0 + \zeta\omega_n x_0}\right) \tag{2.74}$$

$$\phi = \tan^{-1}\left(\frac{C_2'}{C_1'}\right) = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{x_0\omega_n\sqrt{1-\zeta^2}}\right) \tag{2.75}$$

The motion described by Eq. (2.72) is a damped harmonic motion of angular frequency $\sqrt{1-\zeta^2}\,\omega_n$, but because of the factor $e^{-\zeta\omega_n t}$, the amplitude decreases exponentially with time, as shown in Fig. 2.22. The quantity

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{2.76}$$

is called the *frequency of damped vibration*. It can be seen that the frequency of damped vibration ω_d is always less than the undamped natural frequency ω_n. The decrease in the frequency of damped vibration with increasing amount of damping, given by Eq. (2.76), is shown graphically in Fig. 2.23. The underdamped case is very important in the study of mechanical vibrations, as it is the only case that leads to an oscillatory motion [2.10].

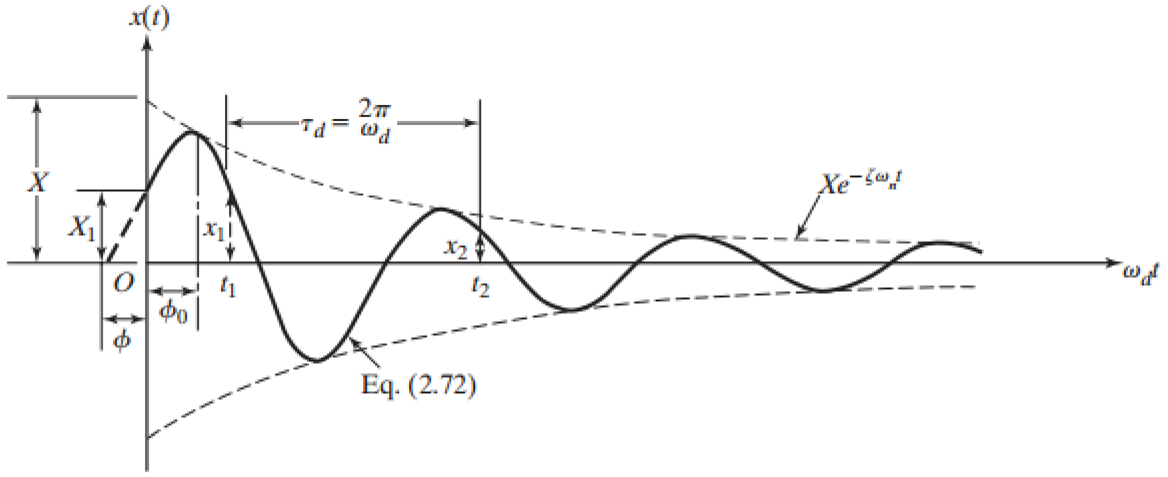


FIGURE 2.22 Underdamped solution.

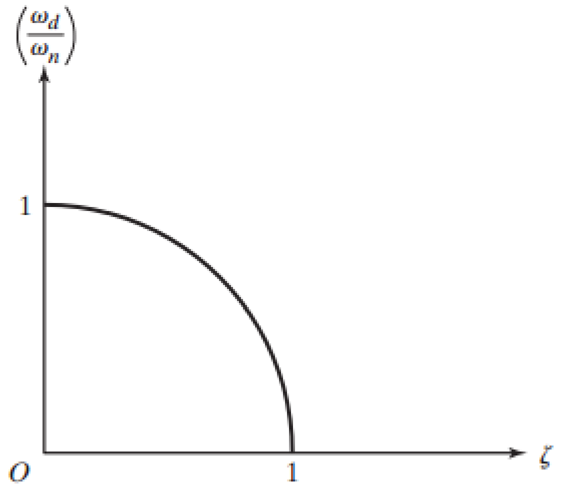


FIGURE 2.23 Variation of ω_d with damping.

Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case the two roots s_1 and s_2 in Eq. (2.68) are equal:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n \tag{2.77}$$

Because of the repeated roots, the solution of Eq. (2.59) is given by [2.6][1]

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{2.78}$$

The application of the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$ for this case gives

$$C_1 = x_0$$
$$C_2 = \dot{x}_0 + \omega_n x_0 \tag{2.79}$$

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{2.80}$$

It can be seen that the motion represented by Eq. (2.80) is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \to 0$ as $t \to \infty$, the motion will eventually diminish to zero, as indicated in Fig. 2.24.

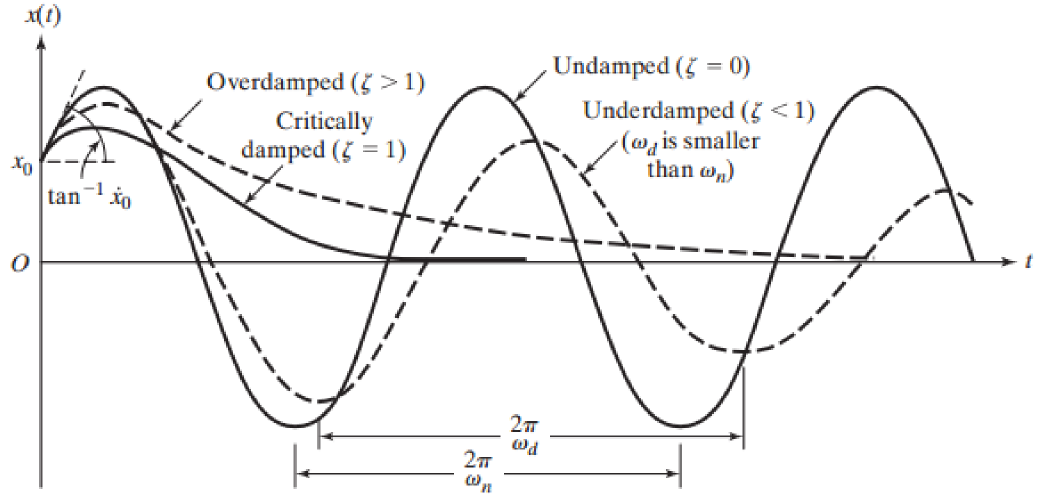


FIGURE 2.24 Comparison of motions with different types of damping.

Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, Eq. (2.68) shows that the roots s_1 and s_2 are real and distinct and are given by

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$$

with $s_2 \ll s_1$. In this case, the solution, Eq. (2.69), can be expressed as

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \tag{2.81}$$

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, the constants C_1 and C_2 can be obtained:

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0\omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}} \tag{2.82}$$

Equation (2.81) shows that the motion is aperiodic regardless of the initial conditions imposed on the system. Since roots s_1 and s_2 are both negative, the motion diminishes exponentially with time, as shown in Fig. 2.24.

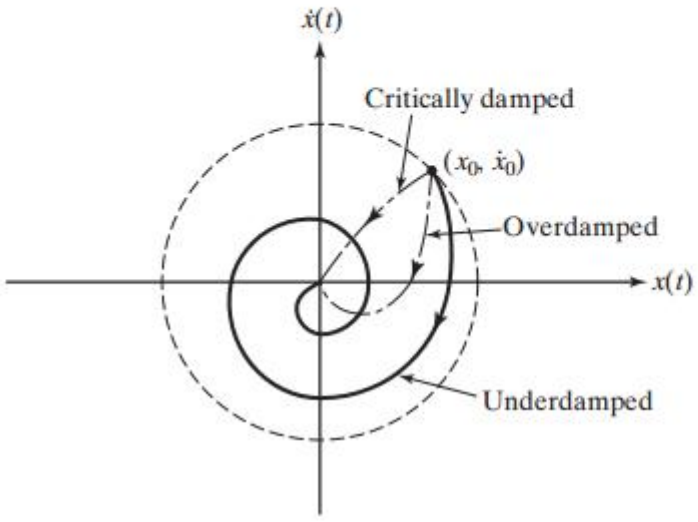


FIGURE 2.25 Phase plane of a damped system.

2.6.3 Logarithmic Decrement

The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes. Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes (displacements), measured one cycle apart for an underdamped system, as in Fig. 2.22. Using Eq. (2.70), we can form the ratio

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)} \tag{2.83}$$

But $t_2 = t_1 + \tau_d$, where $\tau_d = 2\pi/\omega_d$ is the period of damped vibration. Hence $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$, and Eq. (2.83) can be written as

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = e^{\zeta\omega_n\tau_d} \tag{2.84}$$

The logarithmic decrement δ can be obtained from Eq. (2.84):

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m} \tag{2.85}$$

For small damping, Eq. (2.85) can be approximated:

$$\delta \simeq 2\pi\zeta \quad \text{if} \quad \zeta \ll 1 \tag{2.86}$$

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ. Once δ is known, ζ can be found by solving Eq. (2.85):

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{2.87}$$

If we use Eq. (2.86) instead of Eq. (2.85), we have

$$\zeta \simeq \frac{\delta}{2\pi} \tag{2.88}$$

If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2. By taking the natural logarithm of the ratio of x_1 and x_2, we obtain δ. By using Eq. (2.87), we can compute the damping ratio ζ. In fact, the damping ratio ζ can also be found by measuring two displacements separated by any number of complete cycles. If x_1 and x_{m+1} denote the amplitudes corresponding to times t_1 and $t_{m+1} = t_1 + m\tau_d$, where m is an integer, we obtain

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}} \tag{2.89}$$

Since any two successive displacements separated by one cycle satisfy the equation

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d} \tag{2.90}$$

Eq. (2.89) becomes

$$\frac{x_1}{x_{m+1}} = (e^{\zeta\omega_n\tau_d})^m = e^{m\zeta\omega_n\tau_d} \tag{2.91}$$

Equations (2.91) and (2.85) yield

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right) \tag{2.92}$$

which can be substituted into Eq. (2.87) or Eq. (2.88) to obtain the viscous damping ratio ζ.

The free vibration of a single-degree-of-freedom spring-mass-viscous-damper system shown in Fig. 2.21 is governed by Eq. (2.59):

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.106}$$

whose characteristic equation can be expressed as (Eq. (2.61)):

$$ms^2 + cs + k = 0 \tag{2.107}$$

or

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{2.108}$$

The roots of this characteristic equation, called the *characteristic roots* or, simply, *roots*, help us in understanding the behavior of the system. The roots of Eq. (2.107) or (2.108) are given by (see Eqs. (2.62) and (2.68)):

$$s_1, s_2 = \frac{-c \pm \sqrt{c^2 - 4mk}}{2\,m} \tag{2.109}$$

or

$$s_1, s_2 = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} \tag{2.110}$$

The roots given by Eq. (2.110) can be plotted in a complex plane, also known as the s-plane, by denoting the real part along the horizontal axis and the imaginary part along the vertical axis. Noting that the response of the system is given by

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{2.111}$$

where C_1 and C_2 are constants, the following observations can be made by examining Eqs. (2.110) and (2.111):

1. Because the exponent of a larger real negative number (such as e^{-2t}) decays faster than the exponent of a smaller real negative number (such as e^{-t}), the roots lying farther to the left in the s-plane indicate that the corresponding responses decay faster than those associated with roots closer to the imaginary axis.
2. If the roots have positive real values of s—that is, the roots lie in the right half of the s-plane—the corresponding response grows exponentially and hence will be unstable.
3. If the roots lie on the imaginary axis (with zero real value), the corresponding response will be naturally stable.
4. If the roots have a zero imaginary part, the corresponding response will not oscillate.
5. The response of the system will exhibit an oscillatory behavior only when the roots have nonzero imaginary parts.
6. The farther the roots lie to the left of the s-plane, the faster the corresponding response decreases.
7. The larger the imaginary part of the roots, the higher the frequency of oscillation of the corresponding response of the system.

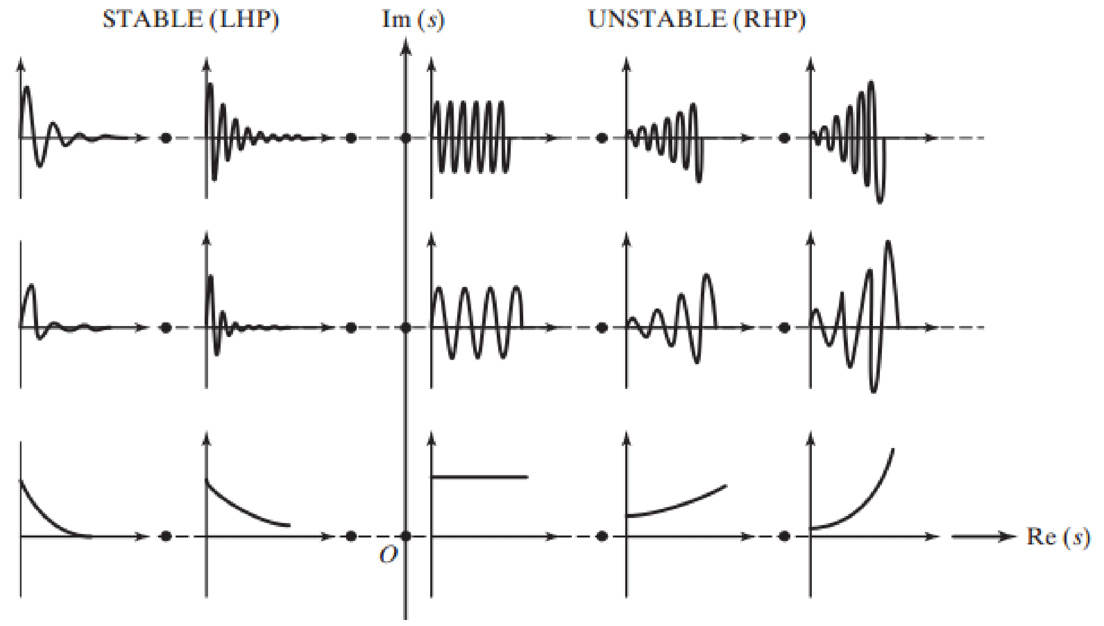


FIGURE 2.32 Locations of characteristic roots (•) and the corresponding responses of the system.

Although the roots s_1 and s_2 appear as complex conjugates, we consider only the roots in the upper half of the s-plane. The root s_1 is plotted as point A with the real value as $\zeta\omega_n$ and the complex value as $\omega_n\sqrt{1-\zeta^2}$, so that the length of OA is ω_n (Fig. 2.33). Thus the roots lying on the circle of radius ω_n correspond to the same natural frequency (ω_n) of the system (PAQ denotes a quarter of the circle). Thus different concentric circles represent systems with different natural frequencies as shown in Fig. 2.34. The horizontal line passing through point A corresponds to the damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Thus, lines parallel to the real axis denote systems having different damped natural frequencies, as shown in Fig. 2.35.

It can be seen, from Fig. 2.33, that the angle made by the line OA with the imaginary axis is given by

$$\sin\theta = \frac{\zeta\omega_n}{\omega_n} = \zeta \tag{2.112}$$

or

$$\theta = \sin^{-1}\zeta \tag{2.113}$$

Thus, radial lines passing through the origin correspond to different damping ratios, as shown in Fig. 2.36. Therefore, when $\zeta = 0$, we have no damping ($\theta = 0$), and the damped natural frequency will reduce to the undamped natural frequency. Similarly, when $\zeta = 1$,

we have critical damping and the radical line lies along the negative real axis. The time constant of the system, τ, is defined as

$$\tau = \frac{1}{\zeta\omega_n} \tag{2.114}$$

and hence the distance DO or AB represents the reciprocal of the time constant, $\zeta\omega_n = \frac{1}{\tau}$. Hence different lines parallel to the imaginary axis denote reciprocals of different time constants (Fig. 2.37).

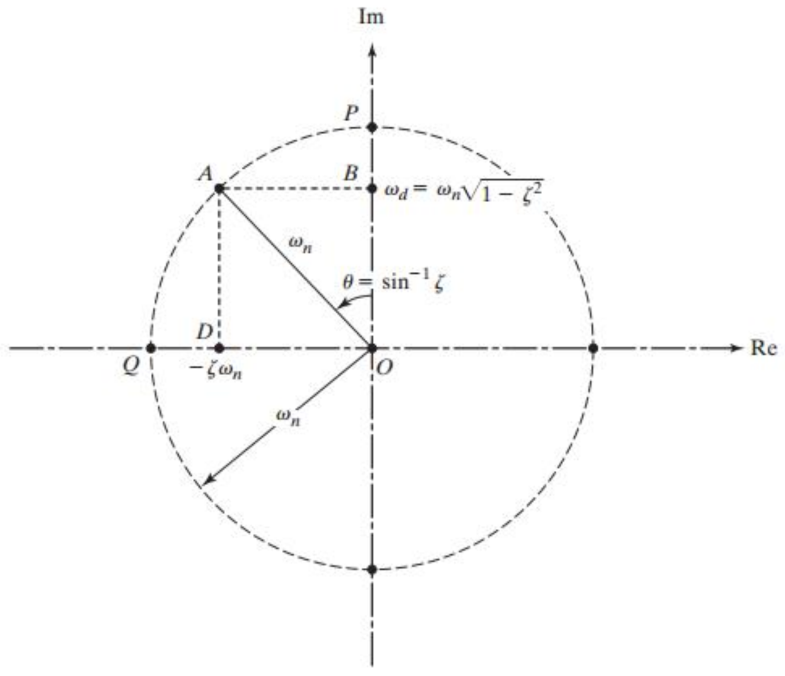


FIGURE 2.33 Interpretations of ω_n, ω_d, and ζ.

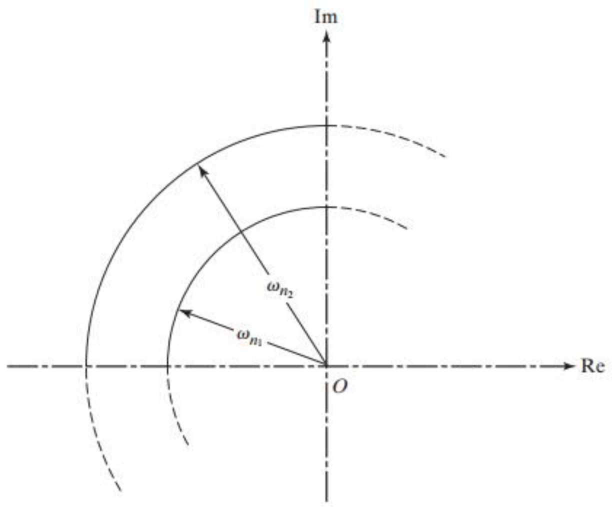


FIGURE 2.34 ω_n in s-plane.

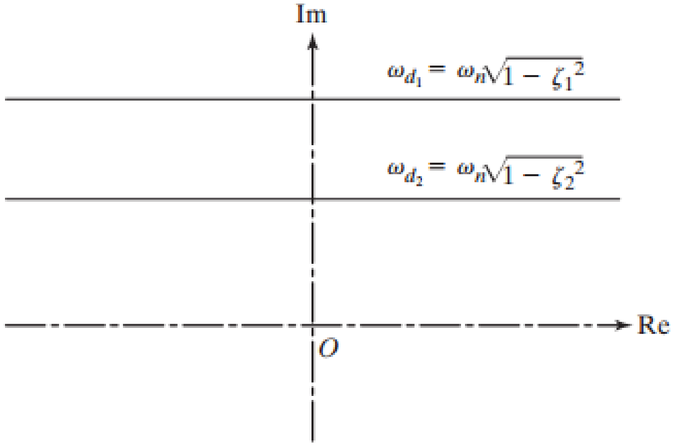


FIGURE 2.35 ω_d in s-plane.

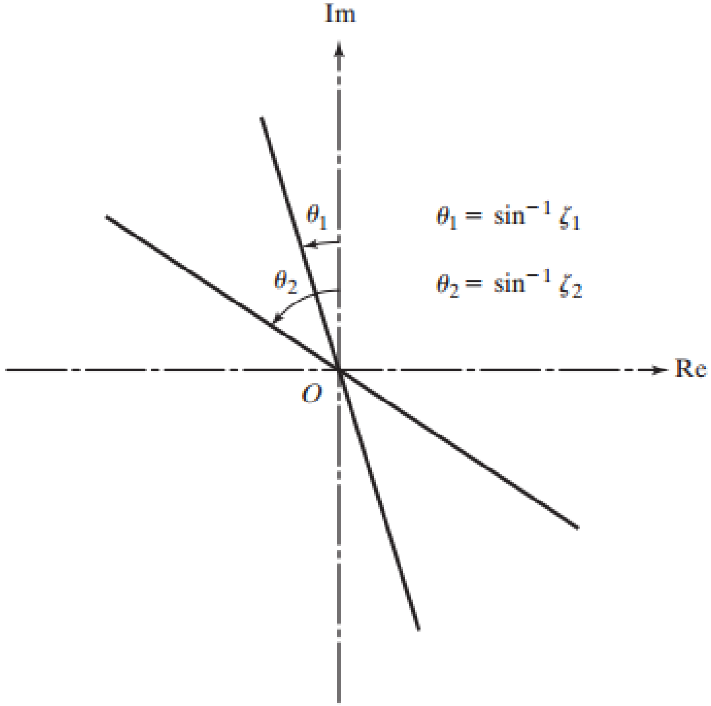


FIGURE 2.36 ζ in s-plane.

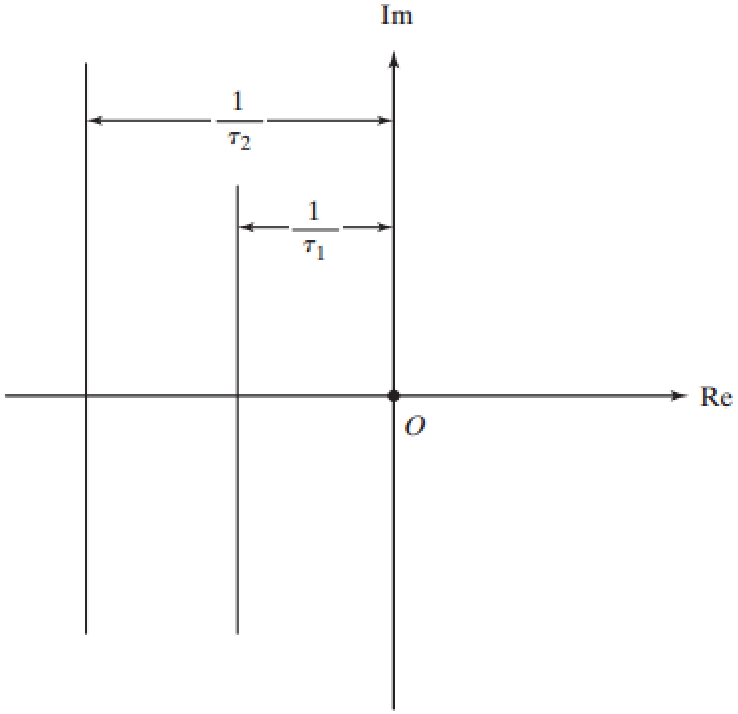


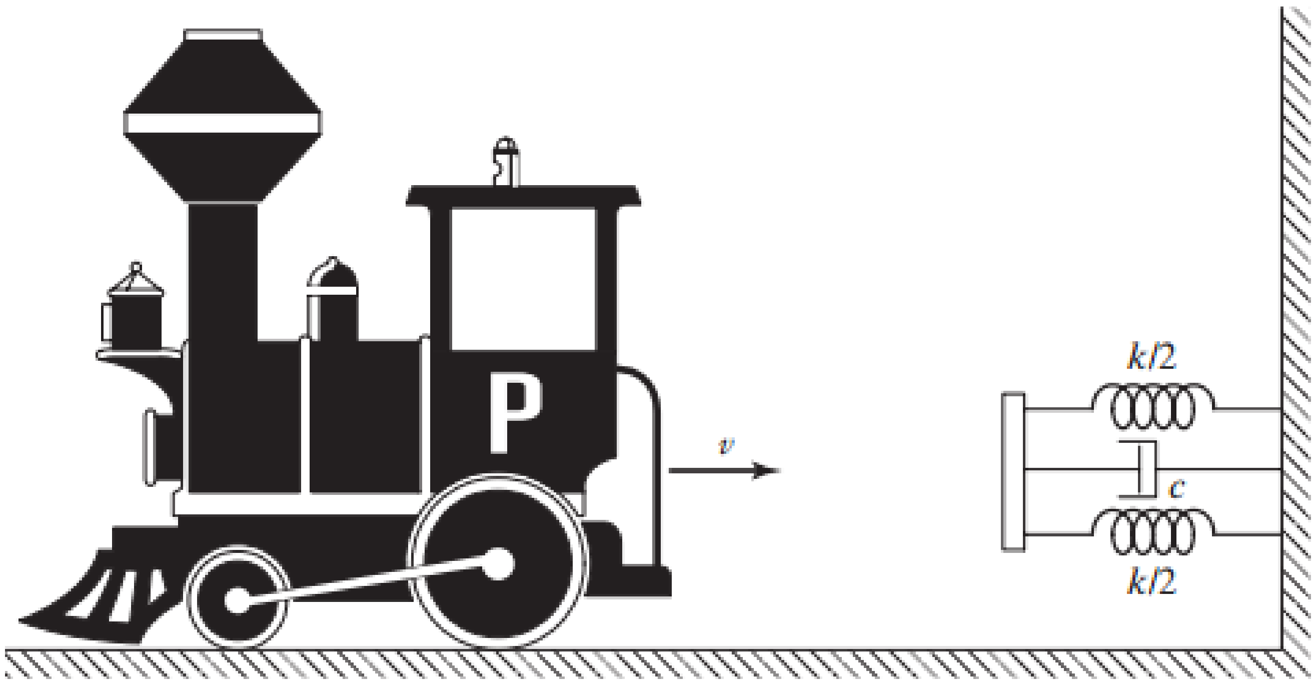
FIGURE 2.37 τ in s-plane.

Exercícios

1. A shock absorber is to be designed to limit its overshoot to 15 percent of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$?

2.

3. A railroad car of mass 2,000 kg traveling at a velocity $v = 10$ m/s is stopped at the end of the tracks by a spring-damper system, as shown in Fig. 2.108. If the stiffness of the spring is $k = 80$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

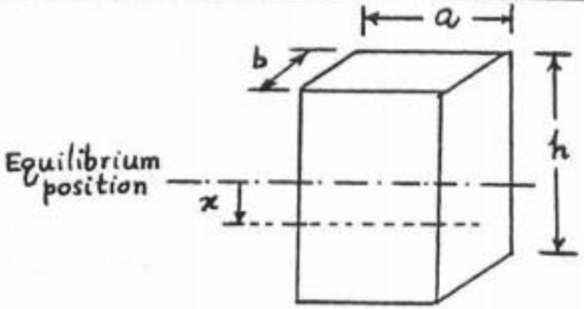


4. A wooden rectangular prism of cross section 40 cm × 60 cm, height 120 cm, and mass 40 kg floats in a fluid as shown in Fig. 2.105. When disturbed, it is observed to vibrate freely with a natural period of 0.5 s. Determine the density of the fluid.

When prism is displaced by x from equilibrium position, the weight of oil displaced

$= \rho_o\, g\, a b x$ = restoring force

Mass of prism = $m = \rho_w\, a b h$



Equation of motion:

$$m \ddot{x} + \text{restoring force} = 0$$

$$\rho_w\, a b h\, \ddot{x} + \rho_o\, g\, a b\, x = 0$$

$$\omega_n = \sqrt{\frac{\rho_o\, g\, a b}{\rho_w\, a b\, h}} = \sqrt{\frac{\rho_o\, g}{\rho_w\, h}} \qquad (E_1)$$

Since ω_n is independent of cross-section of the prism, ω_n remains same even for a circular wooden prism.