

# Vibração Livre: 1 Grau de Liberdade

**Sistemas Dinâmicos II**

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Baseado no Livro:



# Visão geral

- Vibrações de sistemas de 1 GL
- Vibrações de sistemas com 2 GL
- Vibrações de sistemas com  $n$  GL
- Vibrações de sistemas contínuos

- Livres
- Forçadas harmônicas
- Forçadas gerais

- Soluções analíticas [integração e Laplace]
- Soluções numéricas
- Soluções numéricas por Elementos Finitos

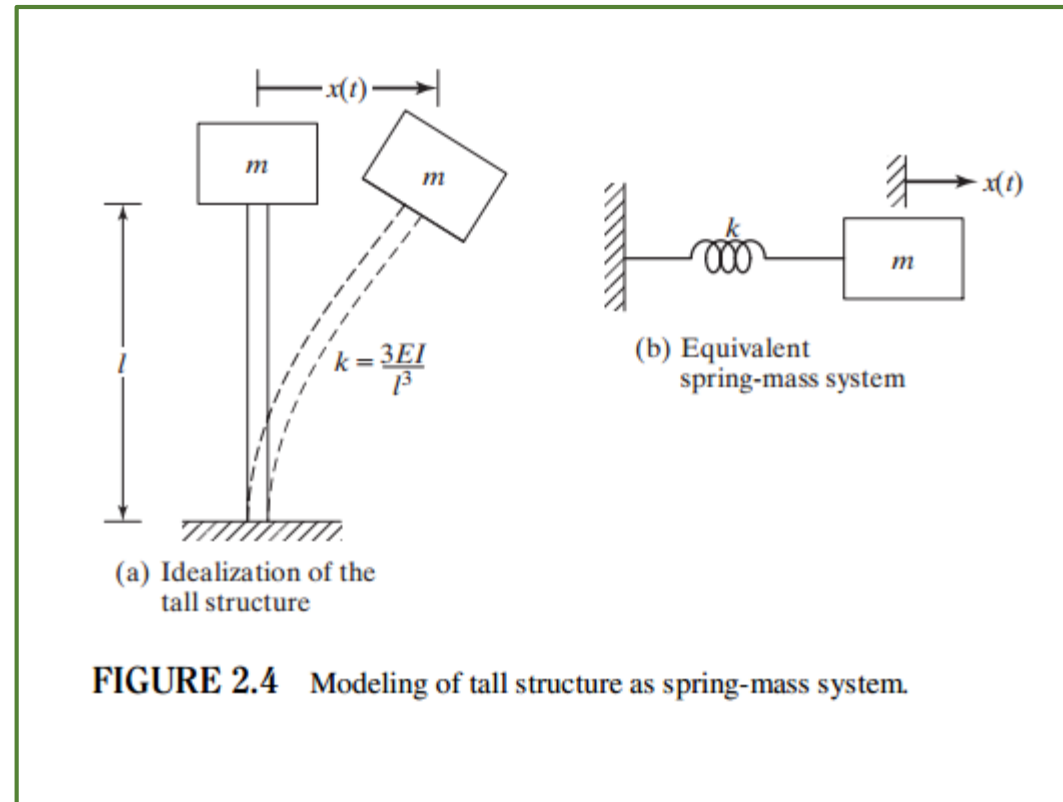
# Nesta fase do curso veremos

- Vibrações de 1 GL
- Vibrações de 2 GL
- Vibrações de hastes
- Vibrações de vigas
  
- Ilustrações experimentais

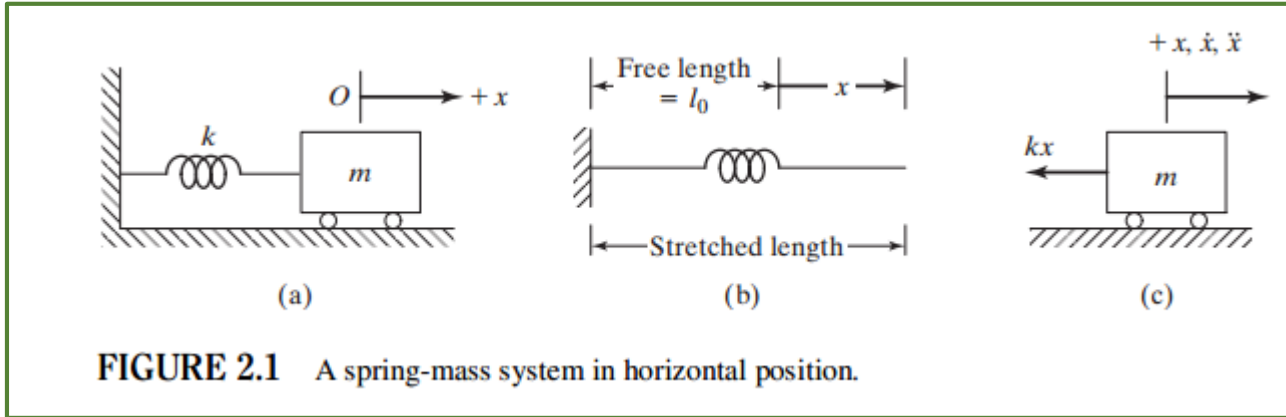
# Sistemas de 1 GL: comentários sobre idealização [ou modelagem]



FIGURE 2.3 The space needle (structure).



# Equação do movimento



$$F(t) = -kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0 \quad (2.3)$$

The solution of Eq. (2.3) can be found by assuming

$$x(t) = Ce^{st} \quad (2.11)$$

where  $C$  and  $s$  are constants to be determined. Substitution of Eq. (2.11) into Eq. (2.3) gives

$$C(ms^2 + k) = 0$$

Since  $C$  cannot be zero, we have

$$ms^2 + k = 0 \quad (2.12)$$

and hence

$$s = \pm \left( -\frac{k}{m} \right)^{1/2} = \pm i\omega_n \quad (2.13)$$

Equation (2.12) is called the *auxiliary* or the *characteristic* equation corresponding to the differential Eq. (2.3). The two values of  $s$  given by Eq. (2.13) are the roots of the characteristic equation, also known as the *eigenvalues* or the *characteristic values* of the problem. Since both values of  $s$  satisfy Eq. (2.12), the general solution of Eq. (2.3) can be expressed as

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \quad (2.15)$$

where  $C_1$  and  $C_2$  are constants. By using the identities

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

Eq. (2.15) can be rewritten as

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \quad (2.16)$$

where  $A_1$  and  $A_2$  are new constants. The constants  $C_1$  and  $C_2$  or  $A_1$  and  $A_2$  can be determined from the initial conditions of the system. Two conditions are to be specified to evaluate these constants uniquely. Note that the number of conditions to be specified is the same as the order of the governing differential equation. In the present case, if the values of displacement  $x(t)$  and velocity  $\dot{x}(t) = (dx/dt)(t)$  are specified as  $x_0$  and  $\dot{x}_0$  at  $t = 0$ , we have, from Eq. (2.16),

$$\begin{aligned} x(t = 0) &= A_1 = x_0 \\ \dot{x}(t = 0) &= \omega_n A_2 = \dot{x}_0 \end{aligned} \quad (2.17)$$

Hence  $A_1 = x_0$  and  $A_2 = \dot{x}_0/\omega_n$ . Thus the solution of Eq. (2.3) subject to the initial conditions of Eq. (2.17) is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \quad (2.18)$$

Equation (2.16) can be expressed in a different form by introducing the notation

$$\begin{aligned} A_1 &= A \cos \phi \\ A_2 &= A \sin \phi \end{aligned} \quad (2.19)$$

where  $A$  and  $\phi$  are the new constants, which can be expressed in terms of  $A_1$  and  $A_2$  as

$$\begin{aligned} A &= (A_1^2 + A_2^2)^{1/2} = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \text{amplitude} \\ \phi &= \tan^{-1} \left( \frac{A_2}{A_1} \right) = \tan^{-1} \left( \frac{\dot{x}_0}{x_0 \omega_n} \right) = \text{phase angle} \end{aligned} \quad (2.20)$$

Introducing Eq. (2.19) into Eq. (2.16), the solution can be written as

$$x(t) = A \cos (\omega_n t - \phi) \quad (2.21)$$

By using the relations

$$\begin{aligned} A_1 &= A_0 \sin \phi_0 \\ A_2 &= A_0 \cos \phi_0 \end{aligned} \quad (2.22)$$

Eq. (2.16) can also be expressed as

$$x(t) = A_0 \sin (\omega_n t + \phi_0) \quad (2.23)$$

where

$$A_0 = A = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} \quad (2.24)$$

and

$$\phi_0 = \tan^{-1} \left( \frac{x_0 \omega_n}{\dot{x}_0} \right) \quad (2.25)$$



The nature of harmonic oscillation can be represented graphically as in Fig. 2.8(a). If  $\vec{A}$  denotes a vector of magnitude  $A$ , which makes an angle  $\omega_n t - \phi$  with respect to the vertical ( $x$ ) axis, then the solution, Eq. (2.21), can be seen to be the projection of the vector  $\vec{A}$  on the  $x$ -axis. The constants  $A_1$  and  $A_2$  of Eq. (2.16), given by Eq. (2.19), are merely the rectangular components of  $\vec{A}$  along two orthogonal axes making angles  $\phi$  and  $-(\frac{\pi}{2} - \phi)$  with respect to the vector  $\vec{A}$ . Since the angle  $\omega_n t - \phi$  is a linear function of time, it increases linearly with time; the entire diagram thus rotates counterclockwise at an angular velocity  $\omega_n$ . As the diagram (Fig. 2.8a) rotates, the projection of  $\vec{A}$  onto the  $x$ -axis varies harmonically so that the motion repeats itself every time the vector  $\vec{A}$  sweeps an angle of  $2\pi$ . The projection of  $\vec{A}$ , namely  $x(t)$ , is shown plotted in Fig. 2.8(b) as a function of  $\omega_n t$ , and as a function of  $t$  in Fig. 2.8(c). The phase angle  $\phi$  can also be interpreted as the angle between the origin and the first peak.

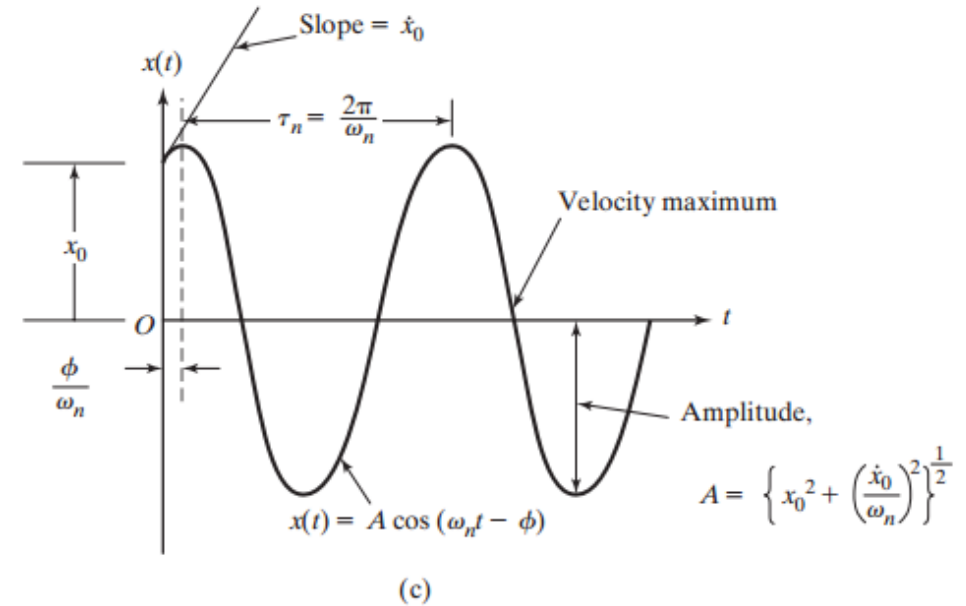
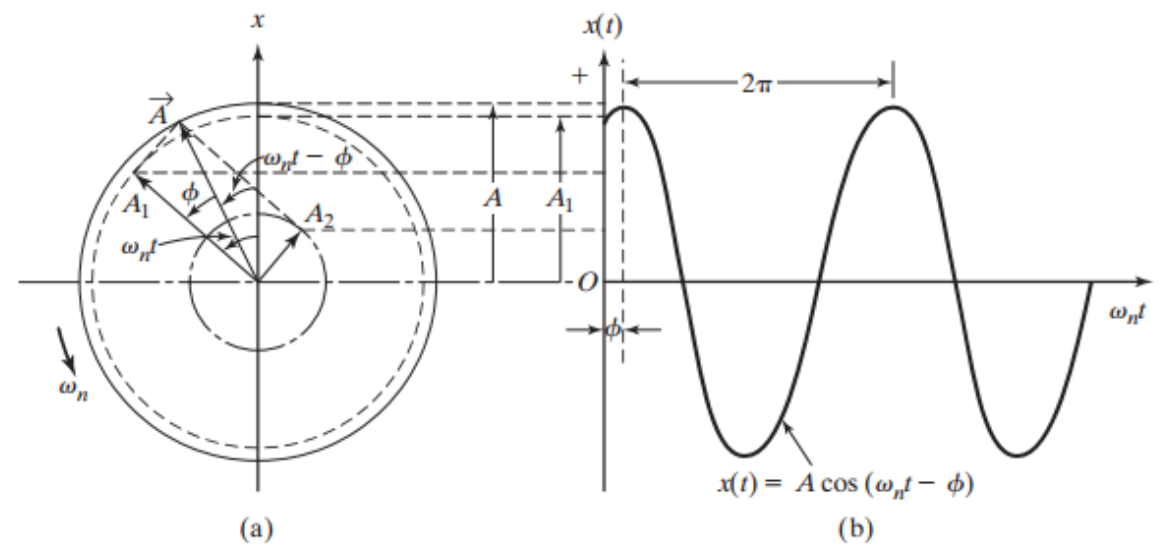


FIGURE 2.8 Graphical representation of the motion of a harmonic oscillator.

If the spring-mass system is in a vertical position, as shown in Fig. 2.7(a), the circular natural frequency can be expressed as

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \quad (2.26)$$

The spring constant  $k$  can be expressed in terms of the mass  $m$  from Eq. (2.9) as

$$k = \frac{W}{\delta_{st}} = \frac{mg}{\delta_{st}} \quad (2.27)$$

Substitution of Eq. (2.27) into Eq. (2.14) yields

$$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} \quad (2.28)$$

Hence the natural frequency in cycles per second and the natural period are given by

$$f_n = \frac{1}{2\pi} \left(\frac{g}{\delta_{st}}\right)^{1/2} \quad (2.29)$$

$$\tau_n = \frac{1}{f_n} = 2\pi \left(\frac{\delta_{st}}{g}\right)^{1/2} \quad (2.30)$$

Thus, when the mass vibrates in a vertical direction, we can compute the natural frequency and the period of vibration by simply measuring the static deflection  $\delta_{st}$ . We don't need to know the spring stiffness  $k$  and the mass  $m$ .

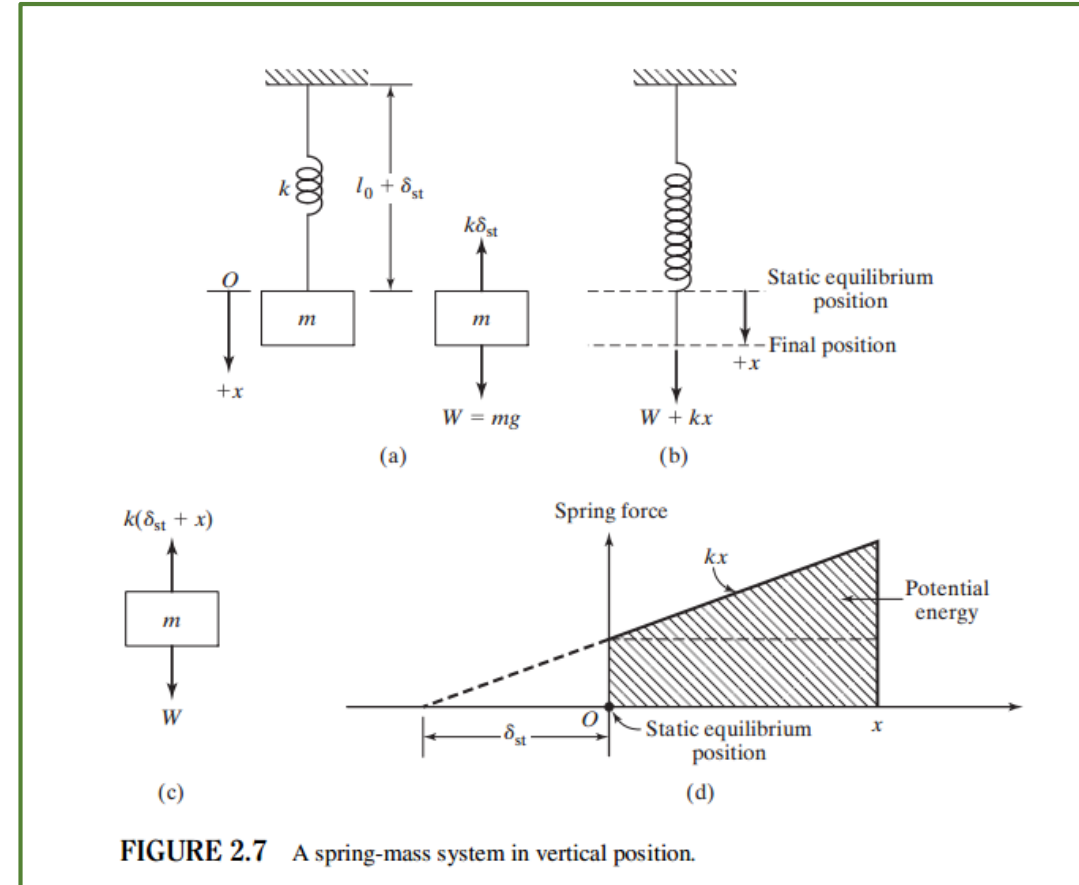


FIGURE 2.7 A spring-mass system in vertical position.

From Eq. (2.21), the velocity  $\dot{x}(t)$  and the acceleration  $\ddot{x}(t)$  of the mass  $m$  at time  $t$  can be obtained as

$$\begin{aligned}\dot{x}(t) &= \frac{dx}{dt}(t) = -\omega_n A \sin(\omega_n t - \phi) = \omega_n A \cos\left(\omega_n t - \phi + \frac{\pi}{2}\right) \\ \ddot{x}(t) &= \frac{d^2x}{dt^2}(t) = -\omega_n^2 A \cos(\omega_n t - \phi) = \omega_n^2 A \cos(\omega_n t - \phi + \pi)\end{aligned}\quad (2.31)$$

Equation (2.31) shows that the velocity leads the displacement by  $\pi/2$  and the acceleration leads the displacement by  $\pi$ .

If the initial displacement ( $x_0$ ) is zero, Eq. (2.21) becomes

$$x(t) = \frac{\dot{x}_0}{\omega_n} \cos\left(\omega_n t - \frac{\pi}{2}\right) = \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \quad (2.32)$$

If the initial velocity ( $\dot{x}_0$ ) is zero, however, the solution becomes

$$x(t) = x_0 \cos \omega_n t \quad (2.33)$$

The response of a single-degree-of-freedom system can be represented in the displacement ( $x$ )-velocity -plane, known as the state space or phase plane. For this we consider the displacement given by Eq. (2.21) and the corresponding velocity:

$$x(t) = A \cos(\omega_n t - \phi)$$

or

$$\begin{aligned} \cos(\omega_n t - \phi) &= \frac{x}{A} \\ \dot{x}(t) &= -A\omega_n \sin(\omega_n t - \phi) \end{aligned} \quad (2.34)$$

or

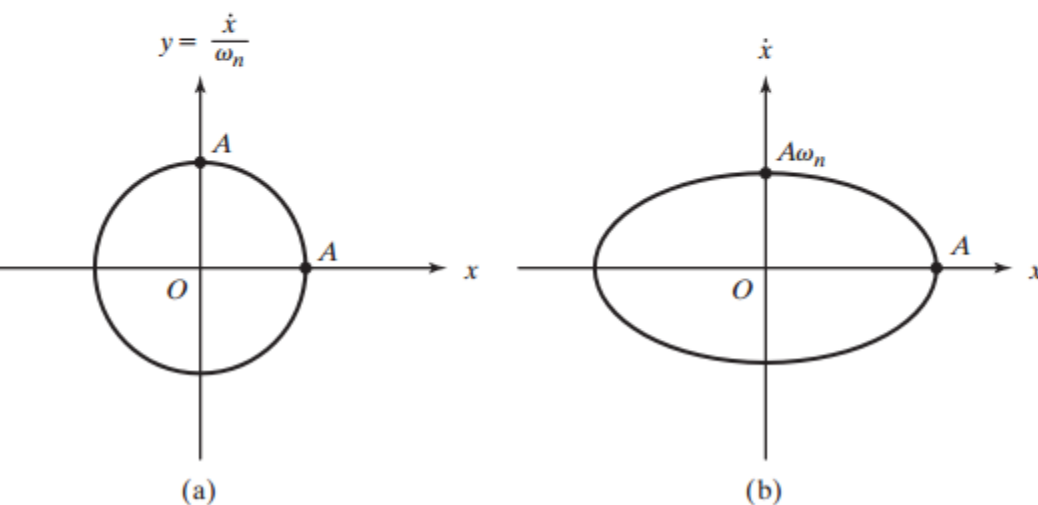
$$\sin(\omega_n t - \phi) = -\frac{\dot{x}}{A\omega_n} = -\frac{y}{A} \quad (2.35)$$

where  $y = \dot{x}/\omega_n$ . By squaring and adding Eqs. (2.34) and (2.35), we obtain

$$\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi) = 1$$

or

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1 \quad (2.36)$$



The graph of Eq. (2.36) in the ( $x, y$ )-plane is a circle, as shown in Fig. 2.9(a), and it constitutes the phase-plane or state-space representation of the undamped system. The radius of the circle,  $A$ , is determined by the initial conditions of motion. Note that the graph of Eq. (2.36) in the ( $x, \dot{x}$ )-plane will be an ellipse, as shown in Fig. 2.9(b).

# Exercícios

1. O extremo de um braço robótico de comprimento  $L$ , com propriedade de seção transversal  $EI$ , carrega uma massa  $G$  em seu extremo. O braço está girando com rotação  $\omega$  e repentinamente é travado pelo operador. Detalhe o movimento da massa em termos de deslocamento, velocidade e aceleração ao longo do tempo. Considere que o sistema possui amortecimento pequeno e que a massa do braço é pequena em relação a  $G$ .

## 2. Free-Vibration Response Due to Impact

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A cantilever beam carries a mass  $M$  at the free end as shown in Fig. 2.11(a). A mass  $m$  falls from a height  $h$  onto the mass  $M$  and adheres to it without rebounding. Determine the resulting transverse vibration of the beam.

**Solution:** When the mass  $m$  falls through a height  $h$ , it will strike the mass  $M$  with a velocity of  $v_m = \sqrt{2gh}$ , where  $g$  is the acceleration due to gravity. Since the mass  $m$  adheres to  $M$  without rebounding, the velocity of the combined mass ( $M + m$ ) immediately after the impact ( $\dot{x}_0$ ) can be found using the principle of conservation of momentum:

$$mv_m = (M + m)\dot{x}_0$$

$$\dot{x}_0 = \left(\frac{m}{M + m}\right)v_m = \left(\frac{m}{M + m}\right)\sqrt{2gh} \quad (\text{E.1})$$

The static equilibrium position of the beam with the new mass ( $M + m$ ) is located at a distance of  $\frac{mg}{k}$  below the static equilibrium position of the original mass ( $M$ ) as shown in Fig. 2.11(c). Here  $k$  denotes the stiffness of the cantilever beam, given by

$$k = \frac{3EI}{l^3}$$

Since free vibration of the beam with the new mass ( $M + m$ ) occurs about its own static equilibrium position, the initial conditions of the problem can be stated as

$$x_0 = -\frac{mg}{k}, \quad \dot{x}_0 = \left(\frac{m}{M + m}\right)\sqrt{2gh} \quad (\text{E.2})$$

Thus the resulting free transverse vibration of the beam can be expressed as (see Eq. (2.21)):

$$x(t) = A \cos(\omega_n t - \phi)$$

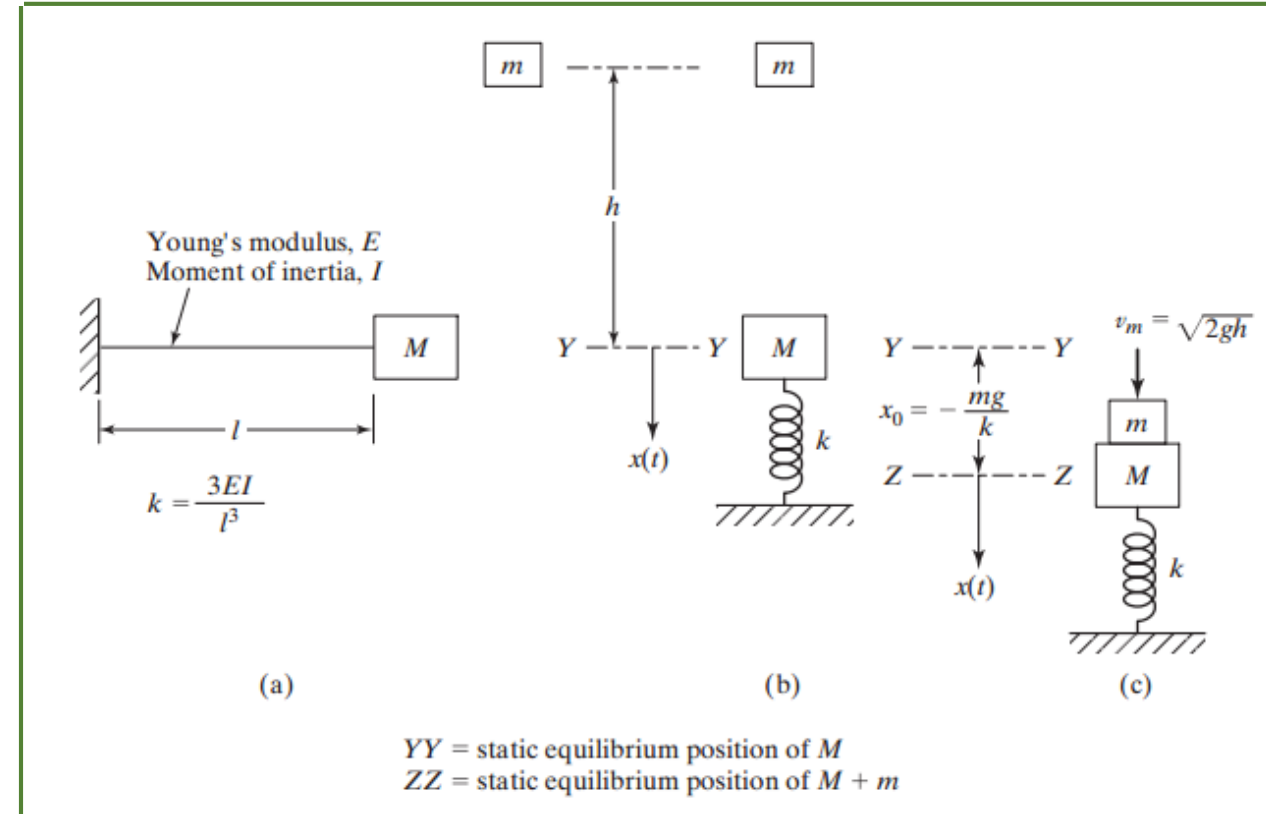
where

$$A = \left[ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right]^{1/2}$$

$$\phi = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right)$$

$$\omega_n = \sqrt{\frac{k}{M + m}} = \sqrt{\frac{3EI}{l^3(M + m)}}$$

with  $x_0$  and  $\dot{x}_0$  given by Eq. (E.2).

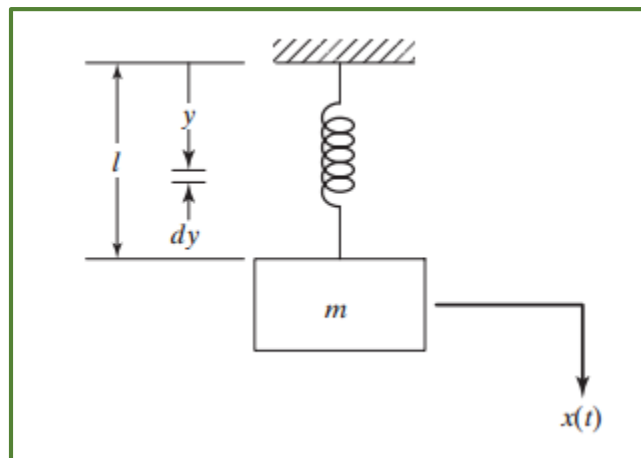


### 3. Effect of Mass on $\omega_n$ of a Spring

Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown in Fig. 2.19.

**Solution:** To find the effect of the mass of the spring on the natural frequency of the spring-mass system, we add the kinetic energy of the system to that of the attached mass and use the energy method to determine the natural frequency. Let  $l$  be the total length of the spring. If  $x$  denotes the displacement of the lower end of the spring (or mass  $m$ ), the displacement at distance  $y$  from the support is given by  $y(x/l)$ . Similarly, if  $\dot{x}$  denotes the velocity of the mass  $m$ , the velocity of a spring element located at distance  $y$  from the support is given by  $y(\dot{x}/l)$ . The kinetic energy of the spring element of length  $dy$  is

$$dT_s = \frac{1}{2} \left( \frac{m_s}{l} dy \right) \left( \frac{y\dot{x}}{l} \right)^2 \quad (\text{E.1})$$



where  $m_s$  is the mass of the spring. The total kinetic energy of the system can be expressed as

$$\begin{aligned} T &= \text{kinetic energy of mass } (T_m) + \text{kinetic energy of spring } (T_s) \\ &= \frac{1}{2}m\dot{x}^2 + \int_{y=0}^l \frac{1}{2} \left( \frac{m_s}{l} dy \right) \left( \frac{y^2 \dot{x}^2}{l^2} \right) \\ &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2} \frac{m_s}{3} \dot{x}^2 \end{aligned} \quad (\text{E.2})$$

The total potential energy of the system is given by

$$U = \frac{1}{2}kx^2 \quad (\text{E.3})$$

By assuming a harmonic motion

$$x(t) = X \cos \omega_n t \quad (\text{E.4})$$

where  $X$  is the maximum displacement of the mass and  $\omega_n$  is the natural frequency, the maximum kinetic and potential energies can be expressed as

$$T_{\max} = \frac{1}{2} \left( m + \frac{m_s}{3} \right) X^2 \omega_n^2 \quad (\text{E.5})$$

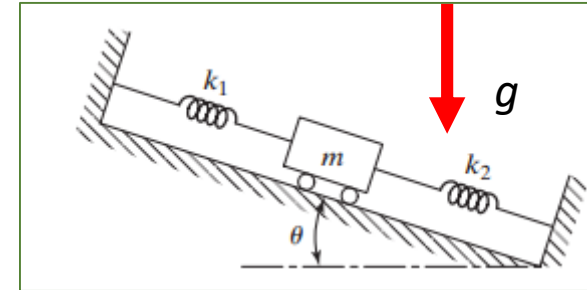
$$U_{\max} = \frac{1}{2} k X^2 \quad (\text{E.6})$$

By equating  $T_{\max}$  and  $U_{\max}$ , we obtain the expression for the natural frequency:

$$\omega_n = \left( \frac{k}{m + \frac{m_s}{3}} \right)^{1/2} \quad (\text{E.7})$$

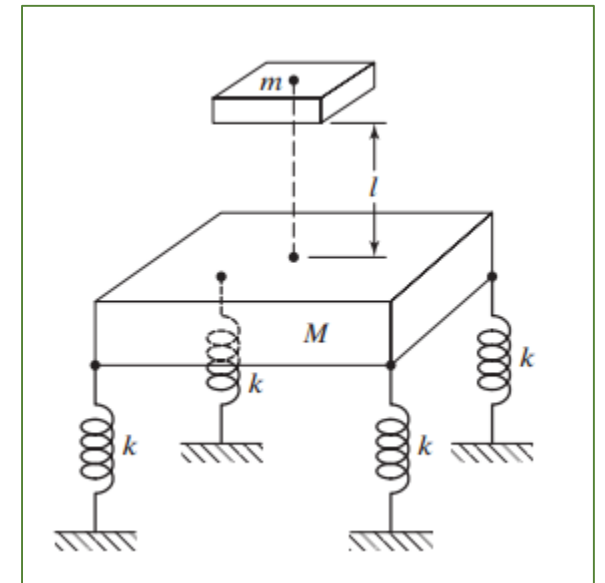


4. No braço robótico analisado no problema 1, como poderia ser considerado a influência da massa do braço,  $m$ , no movimento pós travamento?

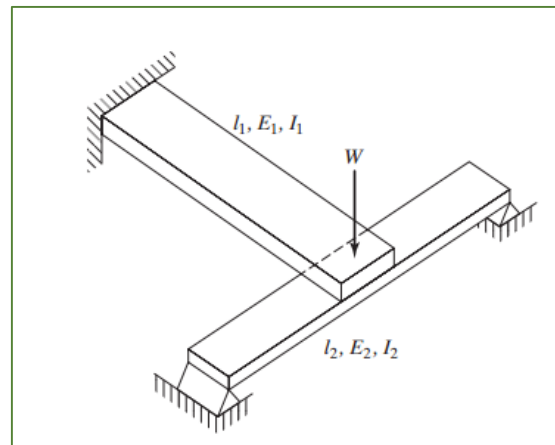


5. Qual a frequência natural do sistema da figura?

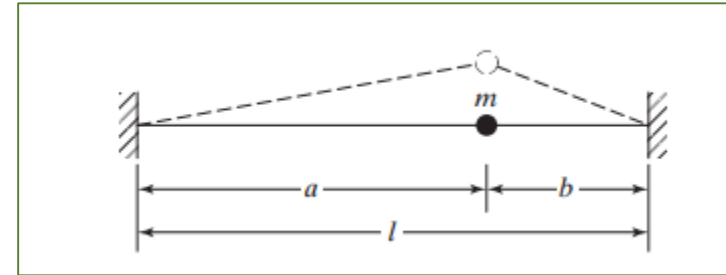
6. Qual a frequência natural do sistema da figura ao se considerar que a massa  $m$  cai de uma altura  $l$  e fica fixa ao conjunto? Descreva o movimento resultante.



7. Qual a frequência natural do sistema da figura, sendo  $W$  uma força peso?

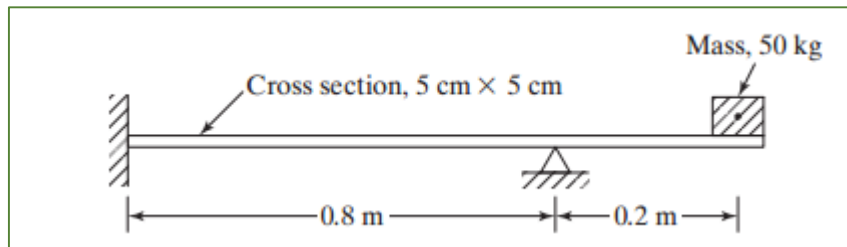


8. Obtenha a equação do movimento para a massa da figura e a frequência natural do sistema? Considere pequenos deslocamentos.



9. Descreva o movimento de um bungee jumper que se atira de uma altura  $H$ ?

10. Qual a frequência natural desta viga, quando modelada com um sistema massa-mola?



# Tracker = vídeo analysis

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