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Escola Politécnica da Universidade de São Paulo

PEF-5750
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ARGYRIS' NATURAL MEMBRANE ELEMENT
THE NATURAL FORCE DENSITY METHOD

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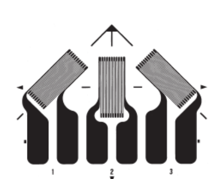
Argyris' Natural Membrane Element

Argyris ~1974
A membrane finite element based on natural deformations
(mesured along the sides of the element),
able to cope with large displacements and large deformations.

- Akin to a "strain rosette" plane stress finite element:



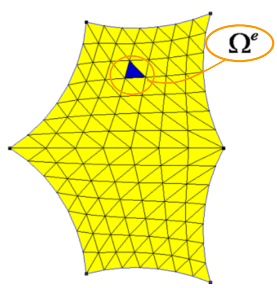
Meek ~1991
- A corrotational description;
- Small strains.

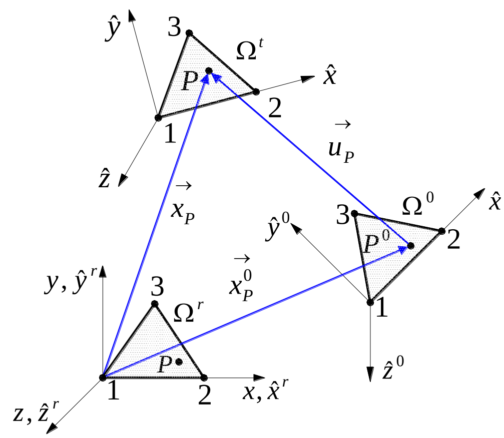
Pauletti ~2003
- a more concise notation;
- distinction between the constitutive and geometric parts of the element tangent stiffness;
- the "simplest possible membrane finite element":
- large displacements / small strains (a few percent...)

Pauletti (2006)
- first publication on the natural force density concept

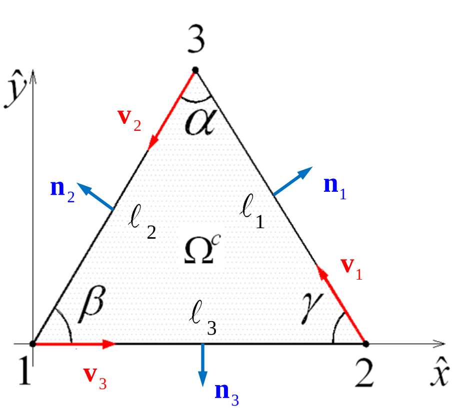
R.M.O. Pauletti, "An extension of the force density procedure to membrane structures"
IASS Symposium / APCS Conference – New Olympics, New Shell and Spacial Structures, Beijing, 2006

Reference, Initial and Current Configurations For Argyris Element





Element Description



$$\mathbf{x}_{0,c} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix}_{0,c} \qquad \mathbf{u} = \mathbf{x}_c - \mathbf{x}_0 = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}$$

Side lenghts:

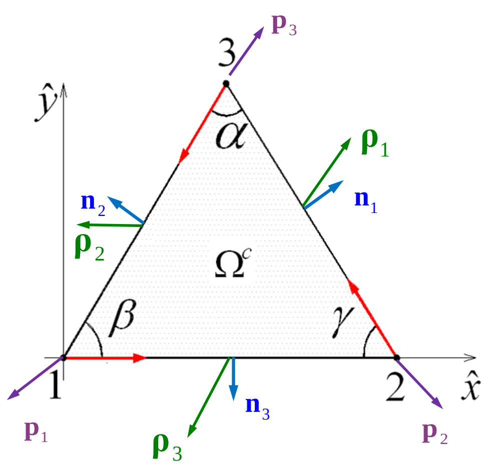
$$\ell_i = \left\| \mathbf{x}_k - \mathbf{x}_j \right\| \qquad \{i, j, k\} \equiv \{1, 2, 3\}$$

(in cyclic permutation)

Unit side vectors: $\mathbf{v}_i = \dfrac{\mathbf{x}_k^e - \mathbf{x}_j^e}{\ell_i}$

Unit normal vectors: $\mathbf{n}_i = -\hat{\mathbf{k}} \times \mathbf{v}_i$

Element Stress Field and Vector of Internal Nodal Forces



Cauchy Plane Stress Tensor:

$$\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\hat{x}} & \tau_{\hat{x}\hat{y}} & 0 \\ \tau_{\hat{x}\hat{y}} & \sigma_{\hat{y}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \sigma_{\hat{x}} \\ \sigma_{\hat{y}} \\ \tau_{\hat{x}\hat{y}} \end{bmatrix}$$

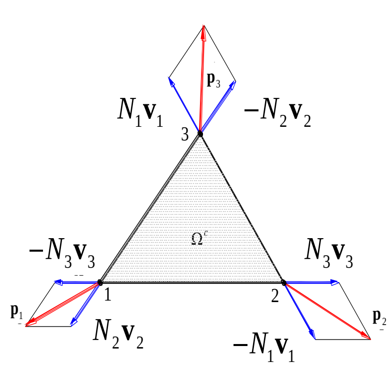
Side stress vectors:

$$\boldsymbol{\rho}_i = \hat{\boldsymbol{\sigma}} \mathbf{n}_i$$

Vector of internal nodal forces:

$$\mathbf{p}^e = \begin{bmatrix} \mathbf{p}_1^e \\ \mathbf{p}_2^e \\ \mathbf{p}_3^e \end{bmatrix} = \frac{t}{2} \begin{bmatrix} \ell_2 \boldsymbol{\rho}_2 + \ell_3 \boldsymbol{\rho}_3 \\ \ell_1 \boldsymbol{\rho}_1 + \ell_3 \boldsymbol{\rho}_3 \\ \ell_1 \boldsymbol{\rho}_1 + \ell_2 \boldsymbol{\rho}_2 \end{bmatrix}$$

Vector of Natural Forces



The vector of internal forces can be decomposed into components parallel to the element sides:

$$\mathbf{p}^e = \begin{bmatrix} N_2 \mathbf{v}_2^e - N_3 \mathbf{v}_3^e \\ N_3 \mathbf{v}_3^e - N_1 \mathbf{v}_1^e \\ N_1 \mathbf{v}_1^e - N_2 \mathbf{v}_2^e \end{bmatrix}$$

Vector of Natural Forces

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

Natural Stresses

Comparing both expression available for p^e:

$$\begin{bmatrix} N_2\mathbf{v}_2^e - N_3\mathbf{v}_3^e \\ N_3\mathbf{v}_3^e - N_1\mathbf{v}_1^e \\ N_1\mathbf{v}_1^e - N_2\mathbf{v}_2^e \end{bmatrix} = \frac{t}{2}\begin{bmatrix} \ell_2\boldsymbol{\rho}_2 + \ell_3\boldsymbol{\rho}_3 \\ \ell_1\boldsymbol{\rho}_1 + \ell_3\boldsymbol{\rho}_3 \\ \ell_1\boldsymbol{\rho}_1 + \ell_2\boldsymbol{\rho}_2 \end{bmatrix}$$

We obtain the Vector of Natural Forces, as function of Cauchy Stresses, and we identity some "Natural Stresses" ($\sigma_1, \sigma_2, \sigma_3$):

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{t}{2}\begin{bmatrix} h_1\left(\dfrac{\cos\beta}{\sin\gamma\sin\alpha}\sigma_{\hat{y}} - \dfrac{\sin\beta}{\sin\gamma\sin\alpha}\tau_{\hat{x}\hat{y}}\right) \\ h_2\left(\dfrac{\cos\gamma}{\sin\beta\sin\alpha}\sigma_{\hat{y}} + \dfrac{\sin\gamma}{\sin\beta\sin\alpha}\tau_{\hat{x}\hat{y}}\right) \\ h_3\left(\sigma_{\hat{x}} - \dfrac{\cos\beta\cos\gamma}{\sin\beta\sin\gamma}\sigma_{\hat{y}} + \dfrac{\cos(\beta-\gamma)}{\sin\beta\sin\gamma}\tau_{\hat{x}\hat{y}}\right) \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \frac{t}{2}\begin{bmatrix} h_1\sigma_1 \\ h_2\sigma_2 \\ h_3\sigma_3 \end{bmatrix}$$

Vector of Natural Stresses

We group the Natural Stresses" ($\sigma_1, \sigma_2, \sigma_3$) in a Vector of Natural Stresses:

$$\boldsymbol{\sigma}_n = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{\cos\beta}{\sin\gamma\sin\alpha} & -\dfrac{\sin\beta}{\sin\gamma\sin\alpha} \\ 0 & \dfrac{\cos\gamma}{\sin\beta\sin\alpha} & \dfrac{\sin\gamma}{\sin\beta\sin\alpha} \\ 1 & -\dfrac{\cos\beta\cos\gamma}{\sin\beta\sin\gamma} & \dfrac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma} \end{bmatrix}\begin{bmatrix} \sigma_{\hat{x}} \\ \sigma_{\hat{y}} \\ \tau_{\hat{x}\hat{y}} \end{bmatrix}$$

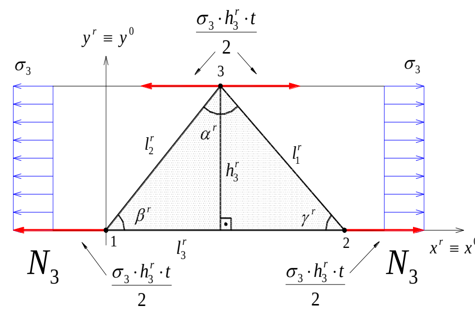
$$\boxed{\boldsymbol{\sigma}_n = \mathbf{T}^{-T}\hat{\boldsymbol{\sigma}}}$$

Exercise 11. Verify the above expression!

Vector of Natural Stresses

Each natural force N_i can be understood as the nodal resultant of each natural normal stress field σ_i

$$\boxed{N_i = \frac{t}{2} h_i \sigma_i}$$



And since $h_i = \dfrac{2A}{\ell_i}$ $\therefore$ $\boxed{N_i = V \dfrac{\sigma_i}{\ell_i}}$

Relationship between the Vectors of Natural Forces and Stresses

In matrix form:

$$\mathbf{N} = V \mathcal{L}^{-1} \boldsymbol{\sigma}_n$$

Length matrix: $\mathcal{L} = \begin{bmatrix} \ell_1 & 0 & 0 \\ 0 & \ell_2 & 0 \\ 0 & 0 & \ell_3 \end{bmatrix}$

$\boldsymbol{\sigma}_n = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$ *Vector of Natural Stresses*

Vector of Natural Deformations

The deformations along the sides of the element are collected in a '<u>Vector of Natural Deformations</u>':

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \cos^2\gamma & \sin^2\gamma & -\sin\gamma\cos\gamma \\ \cos^2\beta & \sin^2\beta & -\sin\beta\cos\beta \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{\hat{x}} \\ \varepsilon_{\hat{y}} \\ \gamma_{\hat{x}\hat{y}} \end{bmatrix}$$

$$\boxed{\boldsymbol{\varepsilon}_n = \mathbf{T}\hat{\boldsymbol{\varepsilon}}}$$

Linearized Green Strains

Exercise 12. Verify the above expression!

We remark that $\boldsymbol{\sigma}_n$ and $\boldsymbol{\varepsilon}_n$ are energetically conjugate.

Indeed, by the Principle of Virtual Work:

$$\delta\hat{\boldsymbol{\varepsilon}}^T\hat{\boldsymbol{\sigma}} = \delta\boldsymbol{\varepsilon}_n{}^T\boldsymbol{\sigma}_n \quad , \forall\,\delta\hat{\boldsymbol{\varepsilon}}$$

$$\delta\hat{\boldsymbol{\varepsilon}}^T\hat{\boldsymbol{\sigma}} = \left(\mathbf{T}\,\delta\hat{\boldsymbol{\varepsilon}}\right)^T\boldsymbol{\sigma}_n = \delta\hat{\boldsymbol{\varepsilon}}^T\mathbf{T}^T\boldsymbol{\sigma}_n \quad , \forall\,\delta\hat{\boldsymbol{\varepsilon}}$$

Thus: $\boxed{\boldsymbol{\sigma}_n = \mathbf{T}^{-T}\hat{\boldsymbol{\sigma}}}$, as deduced before.

Tangent Stiffness Matrix for Argyris' Element

$$\mathbf{k}_t = \frac{\partial\mathbf{g}}{\partial\mathbf{u}} = \mathbf{N}^T\frac{\partial\mathbf{C}}{\partial\mathbf{u}} + \mathbf{C}\frac{\partial\mathbf{N}}{\partial\mathbf{u}} - \frac{\partial\mathbf{f}}{\partial\mathbf{u}}$$

<u>Geometric Stiffness Matrix</u>

$$\mathbf{k}_g = \mathbf{N}^T\frac{\partial\mathbf{C}}{\partial\mathbf{u}} = \begin{bmatrix} N_2\dfrac{\partial\mathbf{v}_2}{\partial\mathbf{u}} - N_3\dfrac{\partial\mathbf{v}_3}{\partial\mathbf{u}} \\ -N_1\dfrac{\partial\mathbf{v}_1}{\partial\mathbf{u}} + N_3\dfrac{\partial\mathbf{v}_3}{\partial\mathbf{u}} \\ N_1\dfrac{\partial\mathbf{v}_1}{\partial\mathbf{u}} - N_2\dfrac{\partial\mathbf{v}_2}{\partial\mathbf{u}} \end{bmatrix}$$

$$\boxed{\mathbf{k}_g = \begin{bmatrix} \frac{N_2}{\ell_2}\left(\mathbf{I}_3-\mathbf{v}_2\mathbf{v}_2^T\right)+\frac{N_3}{\ell_3}\left(\mathbf{I}_3-\mathbf{v}_3\mathbf{v}_3^T\right) & -\frac{N_3}{\ell_3}\left(\mathbf{I}_3-\mathbf{v}_3\mathbf{v}_3^T\right) & -\frac{N_2}{\ell_2}\left(\mathbf{I}_3-\mathbf{v}_2\mathbf{v}_2^T\right) \\ -\frac{N_3}{\ell_3}\left(\mathbf{I}_3-\mathbf{v}_3\mathbf{v}_3^T\right) & \frac{N_1}{\ell_1}\left(\mathbf{I}_3-\mathbf{v}_1\mathbf{v}_1^T\right)+\frac{N_3}{\ell 3}\left(\mathbf{I}_3-\mathbf{v}_3\mathbf{v}_3^T\right) & -\frac{N_1}{\ell_1}\left(\mathbf{I}_3-\mathbf{v}_1\mathbf{v}_1^T\right) \\ -\frac{N_2}{\ell_2}\left(\mathbf{I}_3-\mathbf{v}_2\mathbf{v}_2^T\right) & -\frac{N_1}{\ell_1}\left(\mathbf{I}_3-\mathbf{v}_1\mathbf{v}_1^T\right) & \frac{N_1}{\ell_1}\left(\mathbf{I}_3-\mathbf{v}_1\mathbf{v}_1^T\right)+\frac{N_2}{\ell_2}\left(\mathbf{I}_3-\mathbf{v}_2\mathbf{v}_2^T\right) \end{bmatrix}}$$

Exact!

Tangent Stiffness Matrix for Argyris' Element

$$\mathbf{k}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} + \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

External Stiffness Matrix $\quad \mathbf{k}_{\text{ext}} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$

External force vector: $\quad \mathbf{f} = \mathbf{f}_{\text{weight}} - \mathbf{f}_{\text{wind}} = \frac{V\rho}{3}\begin{bmatrix}\mathbf{I}_3\\ \mathbf{I}_3\\ \mathbf{I}_3\end{bmatrix}\mathbf{g} - \frac{pA}{3}\begin{bmatrix}\mathbf{I}_3\\ \mathbf{I}_3\\ \mathbf{I}_3\end{bmatrix}\mathbf{n}$

$$\boxed{\mathbf{k}_{\text{ext}} = \frac{\partial \mathbf{f}_{\text{wind}}}{\partial \mathbf{u}} = \ldots = \frac{p}{6}\begin{bmatrix}\mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3\\ \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3\\ \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 & \mathbf{\Lambda}_3\end{bmatrix}}$$

Exact!

$$\mathbf{\Lambda}_i = \text{Skew}(\mathbf{l}_i) = \ell_i \begin{bmatrix} 0 & -\mathrm{v}_i^z & \mathrm{v}_i^y \\ \mathrm{v}_i^z & 0 & -\mathrm{v}_i^x \\ -\mathrm{v}_i^y & \mathrm{v}_i^x & 0 \end{bmatrix}, \; i = 1,2,3$$

Exercise 13. Verify the above expression for k_{ext}!

Tangent Stiffness Matrix for Argyris' Element

$$\mathbf{k}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{u}} = \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} + \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}} - \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$$

Constitutive Stiffness Matrix $\quad \mathbf{k}_c = \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}} \quad$ *Exact!*

Defining the vector of Natural Displacements $\quad \mathbf{a} = \begin{bmatrix} \ell_1 - \ell_1^0 \\ \ell_2 - \ell_2^0 \\ \ell_3 - \ell_3^0 \end{bmatrix}$

There exist some kind of relationship $\mathbf{N} = \mathbf{N}(\mathbf{a})$ *so that*

$$\mathbf{k}_c = \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{a}}\frac{\partial \mathbf{a}}{\partial \mathbf{u}} = \mathbf{C}\mathbf{k}_n\mathbf{C}^T$$

$$\mathbf{k}_n = \frac{\partial \mathbf{N}}{\partial \mathbf{a}}$$ *'is the Natural Stiffness Matrix'*

Tangent Stiffness Matrix for Argyris' Element

A simplification: *Linear elastic material behavior*

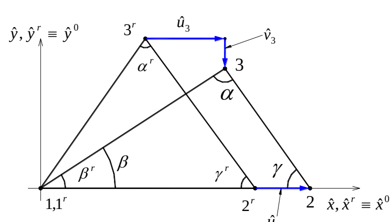
Thus, a linear relationship $\mathbf{N} = \mathbf{k}_n^r \mathbf{a}$ *exists*

Where $\mathbf{k}_n^r = \dfrac{\partial \mathbf{N}}{\partial \mathbf{a}}$ *is a 3x3 constant natural stiffness matrix*

And therefore $$\boxed{\mathbf{k}_c = \mathbf{C}\mathbf{k}_n^r\mathbf{C}^T}$$

?!

A linear elastic simplification for K_c



$$\hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_{\hat{x}} \\ \varepsilon_{\hat{y}} \\ \gamma_{\hat{x}\hat{y}} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \hat{u}}{\partial \hat{x}} \\ \dfrac{\partial \hat{v}}{\partial \hat{y}} \\ \dfrac{\partial \hat{v}}{\partial \hat{x}} + \dfrac{\partial \hat{u}}{\partial \hat{y}} \end{bmatrix} = \begin{bmatrix} \dfrac{u_2}{x_2^r} \\ \dfrac{v_3}{y_3^r} \\ \dfrac{x_2^r u_3 - x_3^r u_2}{x_2^r y_3^r} \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_n = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\Delta\ell_1}{\ell_1^r} \\ \dfrac{\Delta\ell_2}{\ell_2^r} \\ \dfrac{\Delta\ell_3}{\ell_3^r} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\ell_1^r} & 0 & 0 \\ 0 & \dfrac{1}{\ell_2^r} & 0 \\ 0 & 0 & \dfrac{1}{\ell_3^r} \end{bmatrix} \begin{bmatrix} \Delta\ell_1 \\ \Delta\ell_2 \\ \Delta\ell_3 \end{bmatrix} = \mathcal{L}_r^{-1}\mathbf{a}$$

$$\boxed{\boldsymbol{\varepsilon}_n = \mathbf{T}_r \hat{\boldsymbol{\varepsilon}}}$$

$$\mathbf{T}_r = \begin{bmatrix} \cos^2\gamma_r & \sin^2\gamma_r & -\sin\gamma_r\cos\gamma_r \\ \cos^2\beta_r & \sin^2\beta_r & -\sin\beta_r\cos\beta_r \\ 1 & 0 & 0 \end{bmatrix}$$

A linear elastic simplification for K_c

Hooke's Law: $\hat{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_{\hat{x}} \\ \sigma_{\hat{y}} \\ \tau_{\hat{x}\hat{y}} \end{bmatrix} = \hat{\mathbf{D}}\hat{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\sigma}}_0$ $\qquad \hat{\mathbf{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

But, now: $\boldsymbol{\sigma}_n = \mathbf{T}_r^{-T}\hat{\boldsymbol{\sigma}} = \mathbf{T}_r^{-T}\left(\hat{\mathbf{D}}\hat{\boldsymbol{\varepsilon}} + \hat{\boldsymbol{\sigma}}_0\right) = \mathbf{T}_r^{-T}\hat{\mathbf{D}}\hat{\boldsymbol{\varepsilon}} + \mathbf{T}_r^{-T}\hat{\boldsymbol{\sigma}}_0$

$$\boldsymbol{\sigma}_n = \mathbf{T}_r^{-T}\hat{\mathbf{D}}\mathbf{T}_r^{-1}\boldsymbol{\varepsilon}_n + \mathbf{T}_r^{-T}\hat{\boldsymbol{\sigma}}_0$$

That is $\quad \boldsymbol{\sigma}_n = \mathbf{D}_n\boldsymbol{\varepsilon}_n + \boldsymbol{\sigma}_{n0}$

Where $\quad \mathbf{D}_n = \mathbf{T}_r^{-T}\hat{\mathbf{D}}\mathbf{T}_r^{-1}$

A linear elastic simplification for K_c

Recaling the Natural Forces: $\mathbf{N} = V^r \mathcal{L}_r^{-1}\boldsymbol{\sigma}_n$

$$\mathbf{N} = V^r \mathcal{L}_r^{-1}\mathbf{D}_n\boldsymbol{\varepsilon}_n = V^r \mathcal{L}_r^{-1}\left(\mathbf{T}_r^{-T}\hat{\mathbf{D}}\mathbf{T}_r^{-1}\right)\mathcal{L}_r^{-1}\mathbf{a}$$

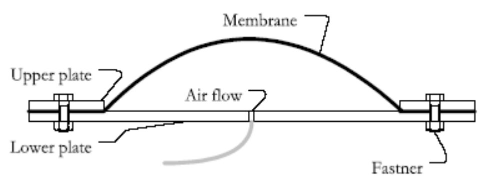
And we arrive to the Natural Stiffness Matrix, (considering small deformations):

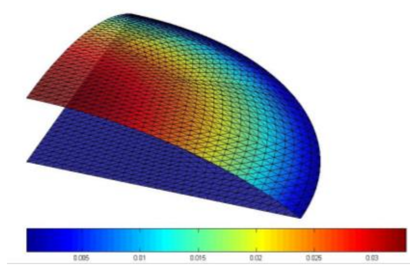
$$\boxed{\mathbf{k}_n^r = \frac{\partial \mathbf{N}}{\partial \mathbf{a}} = V^r \mathcal{L}_r^{-1}\left(\mathbf{T}_r^{-T}\hat{\mathbf{D}}\mathbf{T}_r^{-1}\right)\mathcal{L}_r^{-1}}$$

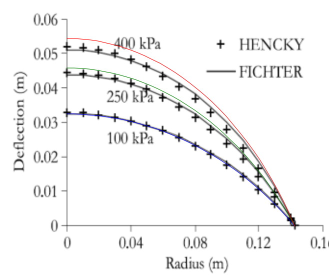
An order 3, symmetric matrix, that can be calculated and stored at the start, and rotated at each Newton's iteration, according to the co-rotational element coordinate system:

$$\boxed{\mathbf{k}_c = \mathbf{C}\mathbf{k}_n^r\mathbf{C}^T}$$

A benchmark:
an axisymmetric pressurized membrane





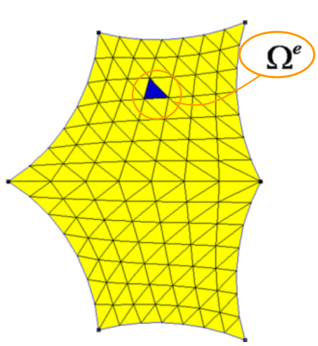


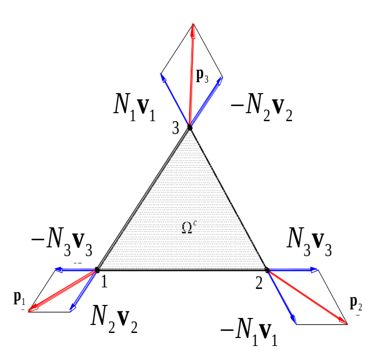
deformed shape,
as calculated by SATS

comparison between SATS
and theoretical results

THE NATURAL FORCE DENSITY METHOD

Vector of Natural Forces for Argyris' Natural Membrane Element:





The vector of internal forces can be decomposed into components parallel to the element sides:

$$\mathbf{p}^e = \begin{bmatrix} N_2\mathbf{v}_2^e - N_3\mathbf{v}_3^e \\ N_3\mathbf{v}_3^e - N_1\mathbf{v}_1^e \\ N_1\mathbf{v}_1^e - N_2\mathbf{v}_2^e \end{bmatrix}$$

$$\mathbf{p}^e = \begin{bmatrix} N_2\mathbf{v}_2^e - N_3\mathbf{v}_3^e \\ N_3\mathbf{v}_3^e - N_1\mathbf{v}_1^e \\ N_1\mathbf{v}_1^e - N_2\mathbf{v}_2^e \end{bmatrix} = \begin{bmatrix} \frac{N_2}{\ell_2}\left(\mathbf{x}_1^e - \mathbf{x}_3^e\right) - \frac{N_3}{\ell_3}\left(\mathbf{x}_2^e - \mathbf{x}_1^e\right) \\ \frac{N_3}{\ell_3}\left(\mathbf{x}_2^e - \mathbf{x}_1^e\right) - \frac{N_1}{\ell_1}\left(\mathbf{x}_3^e - \mathbf{x}_2^e\right) \\ \frac{N_1}{\ell_1}\left(\mathbf{x}_3^e - \mathbf{x}_2^e\right) - \frac{N_2}{\ell_2}\left(\mathbf{x}_1^e - \mathbf{x}_3^e\right) \end{bmatrix}$$

$$\mathbf{p}^e = \begin{bmatrix} n_2\left(\mathbf{x}_1^e - \mathbf{x}_3^e\right) - n_3\left(\mathbf{x}_2^e - \mathbf{x}_1^e\right) \\ n_3\left(\mathbf{x}_2^e - \mathbf{x}_1^e\right) - n_1\left(\mathbf{x}_3^e - \mathbf{x}_2^e\right) \\ n_1\left(\mathbf{x}_3^e - \mathbf{x}_2^e\right) - n_2\left(\mathbf{x}_1^e - \mathbf{x}_3^e\right) \end{bmatrix} = \underbrace{\begin{bmatrix} (n_2+n_3)\mathbf{I}_3 & -n_3\mathbf{I}_3 & -n_2\mathbf{I}_3 \\ -n_3\mathbf{I}_3 & (n_1+n_3)\mathbf{I}_3 & -n_1\mathbf{I}_3 \\ -n_2\mathbf{I}_3 & -n_1\mathbf{I}_3 & (n_1+n_2)\mathbf{I}_3 \end{bmatrix}}_{\boxed{\mathbf{k}_d^e}} \begin{bmatrix} \mathbf{x}_1^e \\ \mathbf{x}_2^e \\ \mathbf{x}_3^e \end{bmatrix}$$

Natural Force Density Element Stiffness:

$$\boxed{\mathbf{k}_d^e = \begin{bmatrix} (n_2+n_3)\mathbf{I}_3 & -n_3\mathbf{I}_3 & -n_2\mathbf{I}_3 \\ -n_3\mathbf{I}_3 & (n_1+n_3)\mathbf{I}_3 & -n_1\mathbf{I}_3 \\ -n_2\mathbf{I}_3 & -n_1\mathbf{I}_3 & (n_1+n_2)\mathbf{I}_3 \end{bmatrix}}$$

Vector of element internal forces: $\boxed{\mathbf{p}^e = \mathbf{k}_d^e \mathbf{x}^e}$

Relationships between element and global vectors: $\mathbf{x}^e = \mathbf{A}^e\mathbf{x} \quad ; \quad \mathbf{P} = \sum_{e=1}^{b} \mathbf{A}^{eT}\mathbf{p}^e$

Natural Force Densities
Global Stiffness:

$$\boxed{\mathbf{K}_d = \sum_{e=1}^{b} \mathbf{A}^{eT}\mathbf{k}_d^e\mathbf{A}^e}$$

Equilibrium: $\mathbf{P} = \mathbf{F}$

A system of linear equations: $\boxed{\mathbf{K}_d\mathbf{x} = \mathbf{F}}$

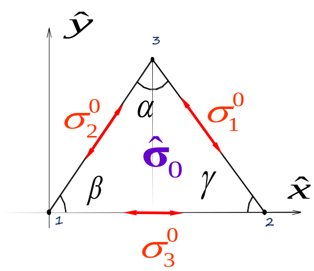
The Natural Force Densities $\{n_1, n_2, n_3\}$ *can be collected into a*
Vector of Natural Force Densities:

$$\boxed{\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \dfrac{N_1}{\ell_1} \\ \dfrac{N_2}{\ell_2} \\ \dfrac{N_3}{\ell_3} \end{bmatrix} = \mathcal{L}^{-1}\mathbf{N}}$$

Remembering the relationship between Natural Forces and Stresses:

$$\mathbf{N} = V\boldsymbol{\mathcal{L}}^{-1}\boldsymbol{\sigma}_n$$

Natural force densities can be calculated according to a given geometry and an initial stress field:

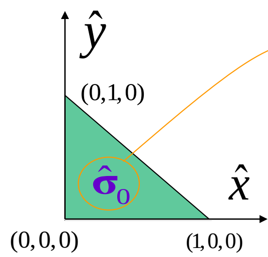


$$\mathbf{n}_0 = \boldsymbol{\mathcal{L}}_0^{-1}\mathbf{N}_0 = V_0\boldsymbol{\mathcal{L}}_0^{-2}\boldsymbol{\sigma}_n^0$$

But $\boldsymbol{\sigma}_n^0 = \mathbf{T}_0^{-T}\hat{\boldsymbol{\sigma}}_0$

Therefore $\boxed{\mathbf{n}_0 = V_0\boldsymbol{\mathcal{L}}_0^{-2}\mathbf{T}_0^{-T}\hat{\boldsymbol{\sigma}}_0}$

For instance:



Isotropic stress field

$$\hat{\boldsymbol{\sigma}}_0 = \begin{bmatrix}\sigma_{\hat{x}}^0\\ \sigma_{\hat{y}}^0\\ \tau_{\hat{x}\hat{y}}^0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} \Rightarrow \mathbf{n}_0 = \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

Once the solution for $\mathbf{x}$ is obtained, Cauchy stresses at the final configuration can be computed according to:

$$\boxed{\hat{\boldsymbol{\sigma}} = \left(V^{-1}\mathcal{L}^2\mathbf{T}\right)\mathbf{n}_0}$$

$$\boxed{\hat{\boldsymbol{\sigma}} = \left(V^{-1}\mathcal{L}^2\mathbf{T}\right)\left(V_0\mathcal{L}_0^{-2}\mathbf{T}_0^{-T}\right)\hat{\boldsymbol{\sigma}}_0}$$

In general, even for uniform stresses at the reference configuration, non-uniform stresses result at the equilibrium configuration!

This is fully coherent with the original force density method, for which normal loads in the equilibrium configuration also vary, even thou initial normal loads are uniform!

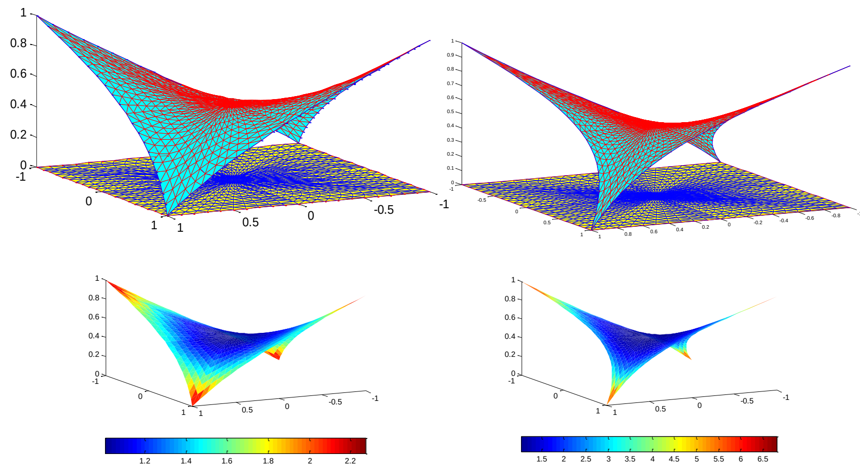
It can be shown that imposition of $\hat{\boldsymbol{\sigma}}_0$ at a reference configuration corresponds to imposition of the 2nd Piola-Kirchhoff stresses, associated to the Cauchy stresses $\hat{\boldsymbol{\sigma}}$ at the equilibrium configuration!

$\Rightarrow$ R.M.O. Pauletti & P.M. Pimenta,
"The natural force density method for the shape finding of taut structures"
Computer Methods in Applied Mechanics and Engineering
Volume 197, Issues 49-50, 15 September 2008, Pages 4419-4428

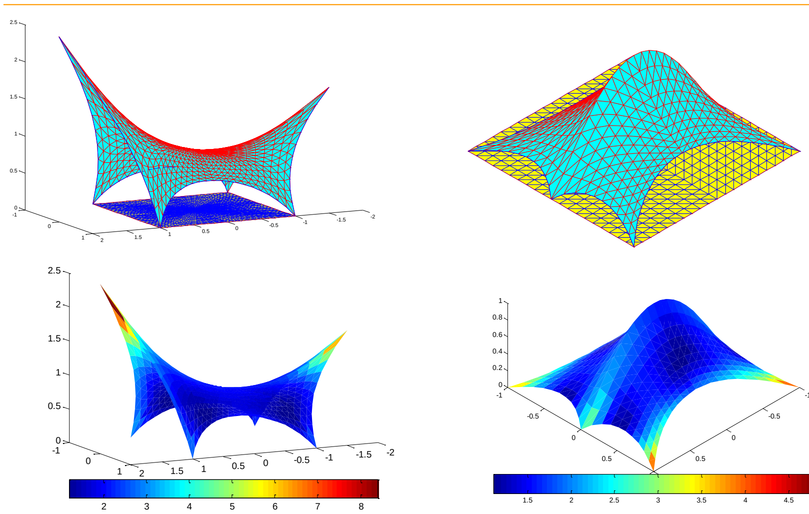
This result extends to membranes a conclusion already stated by Bletzinger & Ramm, for the original force density concept (i.e., for cables).

$\Rightarrow$ K.-U. Bletzinger & E. Ramm,
"A General Finite element Approach to the Form Finding of Tensile Structures by the Updated Reference Strategy'
Int. J. Space Struct. 14 (2) (1999) 131–145

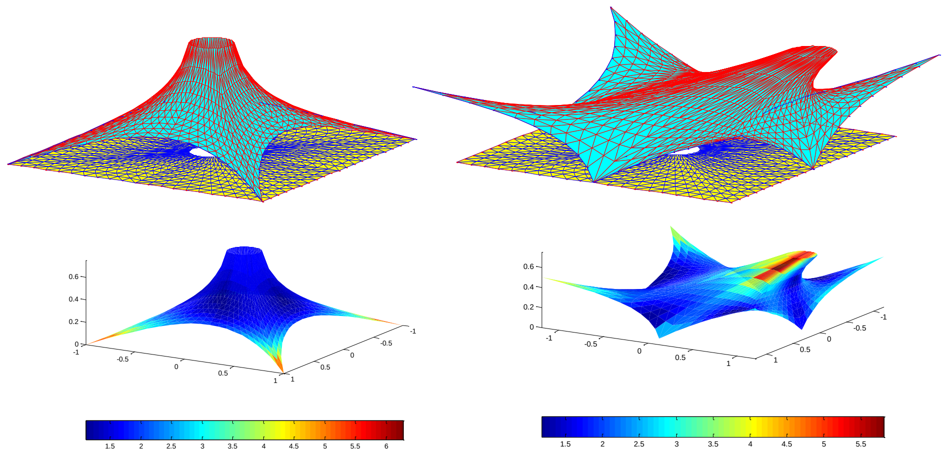
Some solutions:



Some solutions:



Some solutions:



Iterative Natural Force Density Method:

Although Cauchy stresses at the final configuration cannot be imposed in a single force density step, 2nd P-K stresses can be imposed recursively

$$\boxed{\hat{\boldsymbol{\sigma}}_i = \prod_{k=0}^{i}\left(V_k^{-1}\mathcal{L}_k^2\mathbf{T}_k\right) = \left(V_i^{-1}\mathcal{L}_i^2\mathbf{T}_i\right)\dots\left(V_1^{-1}\mathcal{L}_1^2\mathbf{T}_1\right)\left(V_0\mathcal{L}_0^{-2}\mathbf{T}_0^{-T}\right)\hat{\boldsymbol{\sigma}}_0}$$

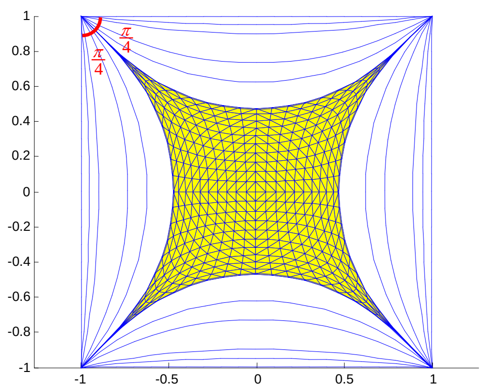
If an uniform isotropic 2nd P-K stress field is recursively imposed, the geometry converges (through a succession of viable shapes) to a minimal surface, with an uniform isotropic Cauchy stress field!

$$\boxed{\hat{\boldsymbol{\sigma}}_i \to \hat{\boldsymbol{\sigma}}_0}$$

We note that a sequence of nonlinear structural analyses can also converge to a minimal surface, but through a succession of non-equilibrium, unviable shapes! This is a clear advantage of the iterative NFDM, which can be stopped at any iteration, always giving a viable shape!

Minimal surfaces:

Consider the minimal flat square membrane fixed at the corner and bounded by border cables:



The following relationship holds:

$$\boxed{T = \frac{\sigma t L}{2\sin\alpha}}$$

Upper limit condition:

$$\alpha = 0 \Rightarrow T = \infty$$

Lower limit condition:

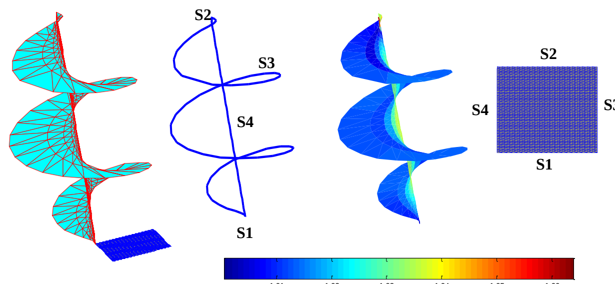
$$\left.\begin{array}{l}\sigma t L = 2 \\ \alpha = \frac{\pi}{4}\end{array}\right\} \quad T = \sqrt{2} \simeq 1.41$$

Exercise 14. Deduce the relationship between the membrane stresses and the normal force on border cables, and numerically verify the upper and lower limit conditions stated above.

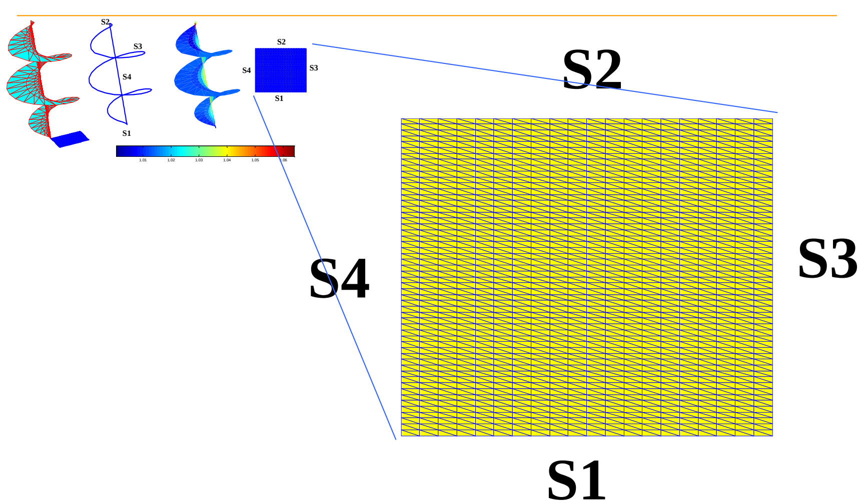
Minimal surfaces:

C. Isenberg, The science of soap films and soap bubbles, Dover Pub. Inc., New York, 1992.

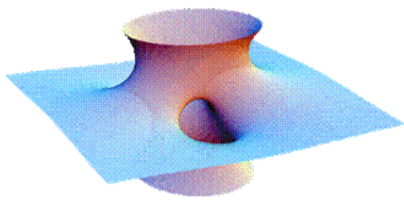
Minimal surfaces:



Minimal surfaces:



Costa's Surface:



The Costa surface is a complete minimal embedded surface of finite topology (i.e., it has no boundary and does not intersect itself). It has genus 1 with three punctures (Schwalbe and Wagon 1999). Until this surface was discovered by Costa (1984), the only other known complete minimal embeddable surfaces in R^3 with no self-intersections were the plane (genus 0), catenoid (genus 0 with two punctures), and helicoid (genus 0 with two punctures), and it was conjectured that these were the only such surfaces. Rather amazingly, the Costa surface belongs to the dihedral group of symmetries.

Helaman Ferguson, 1999 / 2008

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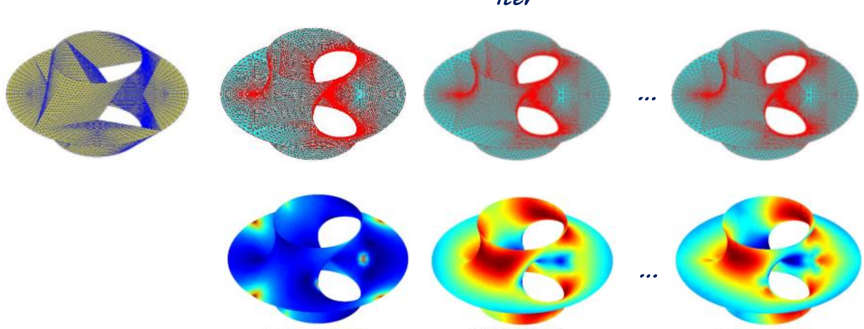
Costa's Surface:

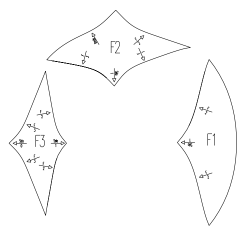
1st iter | 2nd iter | … | 5th iter

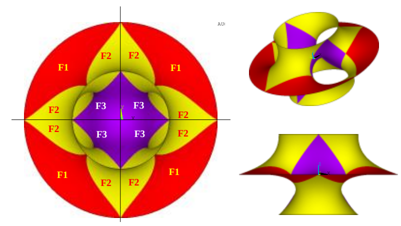


σ_I 1.0288 – 1.8086 | σ_I 1.0015 – 1.0597 | … | σ_I 1.0002 – 1.0124

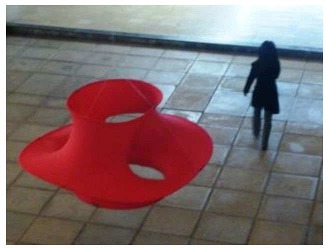
Costa's Surface:

Symmetry & Patterns:

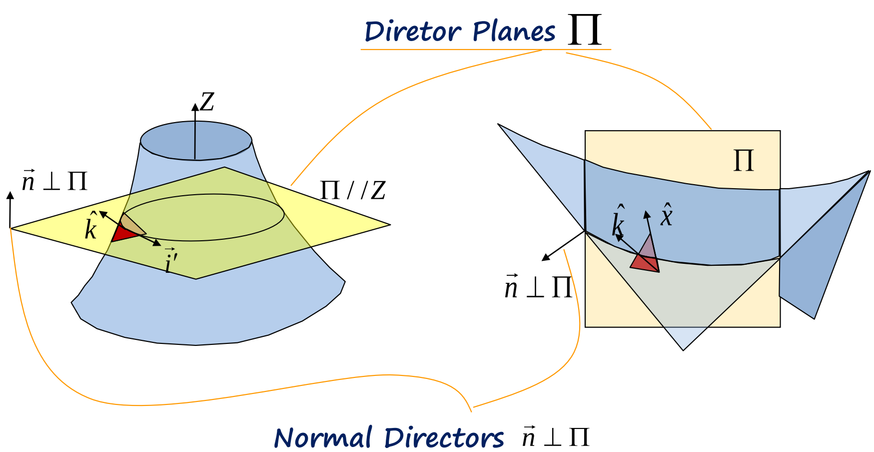




A physical model:



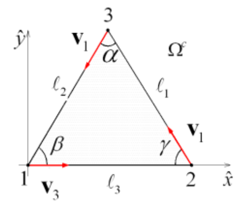
Non-minimal surfaces

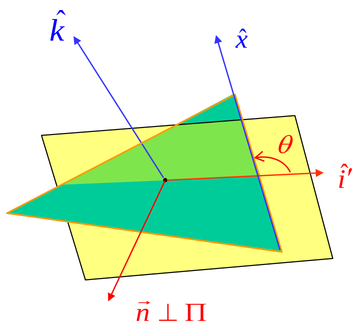


Non-minimal surfaces

Local base vectors in global coordinates:

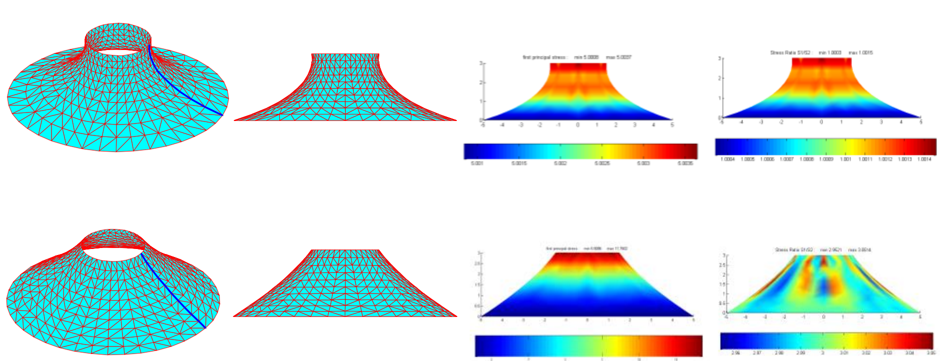
$$\hat{i} = \vec{v}_3 \qquad \hat{k} = \frac{\vec{v}_3 \times \vec{v}_1}{\left\| \vec{v}_3 \times \vec{v}_1 \right\|}; \quad \hat{j} = \hat{k} \times \hat{i}$$





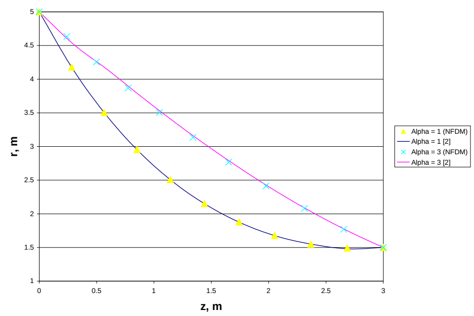
$$\begin{cases} \hat{i}' = \hat{k} \times \vec{n} \\ \theta = \arcsin\left(\left(\hat{i}' \times \hat{i}\right) \cdot \hat{k}\right) \end{cases}$$

Minimal and non-minimal conoids

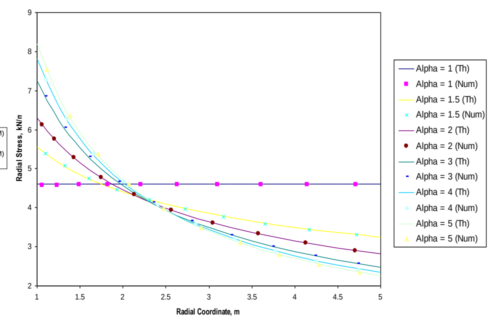


Comparison with an analytical solution

SLADE GELLIN[1] & RUY M.O. PAULETTI[2] – FORM FINDING OF TENSIONED FABRIC CONE STRUCTURES USING THE NATURAL FORCE DENSITY METHOD **(in IASS 2010)**

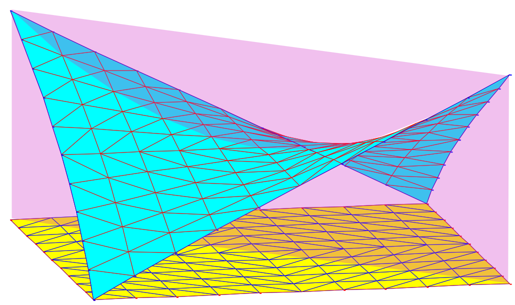


Generatrix Profile

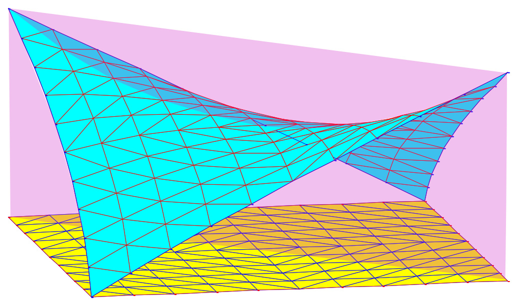


Stress Profile

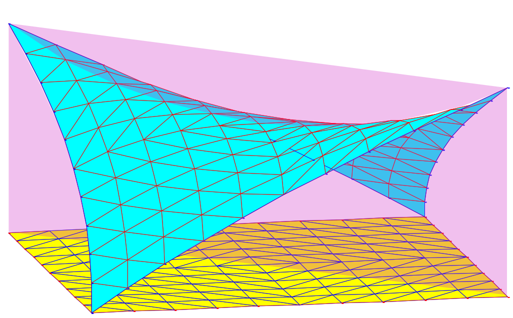
Minimal saddle surface:



Non-minimal saddle surfaces:



Non-minimal saddle surfaces:



Non-minimal saddle surfaces: