Dinamica Non Lineare di Strutture e Sistemi Meccanici

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Lezione 4

Non-linear Normal Modes

Historical notes

- Liapunov (1892)?

- Rosenberg (1966): synchronous modes; fixed modal relationships; dependent of modal displacements alone!

- Vakakis (1990): PhD Thesis CalTech

- Shaw & Pierre (1993): synchronous and asynchronous modes; conservative and non-conservative systems; invariant manifolds; dependent of modal displacements and velocities!

- Mazzilli & Soares & Baracho Neto (1998-2002): analytical methods; invariant manifolds and multiple time scales;

  application to finite-element models of 2D reticulated structures
- Kerschen, Cyril Touzé: numerical methods

Non-linear Normal Modes

Invariant manifold approach

Non-linear normal mode is a motion that takes place on a bidimensional invariant manifold within the system phase space. Such a manifold contains an equilibrium point and, there, is tangent to the corresponding eigenplane of the linearized system.

Consider a non-linear autonomous dynamical system

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = f_i\left(x_1, x_2 \dots x_n, y_1, y_2 \dots y_n\right) \end{cases}$$

Choose modal variables $u = x_k$ and $v = y_k$; Search for functions (modal relationships) $X_i\left(u,v\right)$ and $Y_i\left(u,v\right)$ such that:

$$\begin{cases} x_i = X_i\left(u,v\right) \\ y_i = Y_i\left(u,v\right) \end{cases}$$

Non-linear Normal Modes

Invariant manifold approach

$$\dot{x}_i = \frac{\partial X_i(u,v)}{\partial u}\dot{u} + \frac{\partial X_i(u,v)}{\partial v}\dot{v} = Y_i(u,v)$$

$$\frac{\partial X_i(u,v)}{\partial u}v + \frac{\partial X_i(u,v)}{\partial v}\underbrace{f_k\left[X_1(u,v), X_2(u,v), \ldots Y_n(u,v)\right]}_{g_k(u,v)} = Y_i(u,v)$$

$$\frac{\partial X_i(u,v)}{\partial u}v + \frac{\partial X_i(u,v)}{\partial v}g_k(u,v) = Y_i(u,v) \qquad (*)$$

$$\dot{y}_i = \frac{\partial Y_i(u,v)}{\partial u}\dot{u} + \frac{\partial Y_i(u,v)}{\partial v}\dot{v} = \underbrace{f_i(x_1, x_2 \ldots y_n)}_{g_i(u,v)}$$

$$\frac{\partial Y_i(u,v)}{\partial u}v + \frac{\partial Y_i(u,v)}{\partial v}g_k(u,v) = g_i(u,v) \qquad (*)$$

Non-linear Normal Modes

Invariant manifold approach

$X_i(u,v)$ and $Y_i(u,v)$ are expressed in power series, with coefficients a_i and b_i to be determined:

$$X_i(u,v) = a_1 u + a_2 v + a_3 u^2 + a_4 uv + a_5 v^2 + a_6 u^3 + a_7 u^2 v + a_8 uv^2 + a_9 v^3$$

$$Y_i(u,v) = b_1 u + b_2 v + b_3 u^2 + b_4 uv + b_5 v^2 + b_6 u^3 + b_7 u^2 v + b_8 uv^2 + b_9 v^3$$

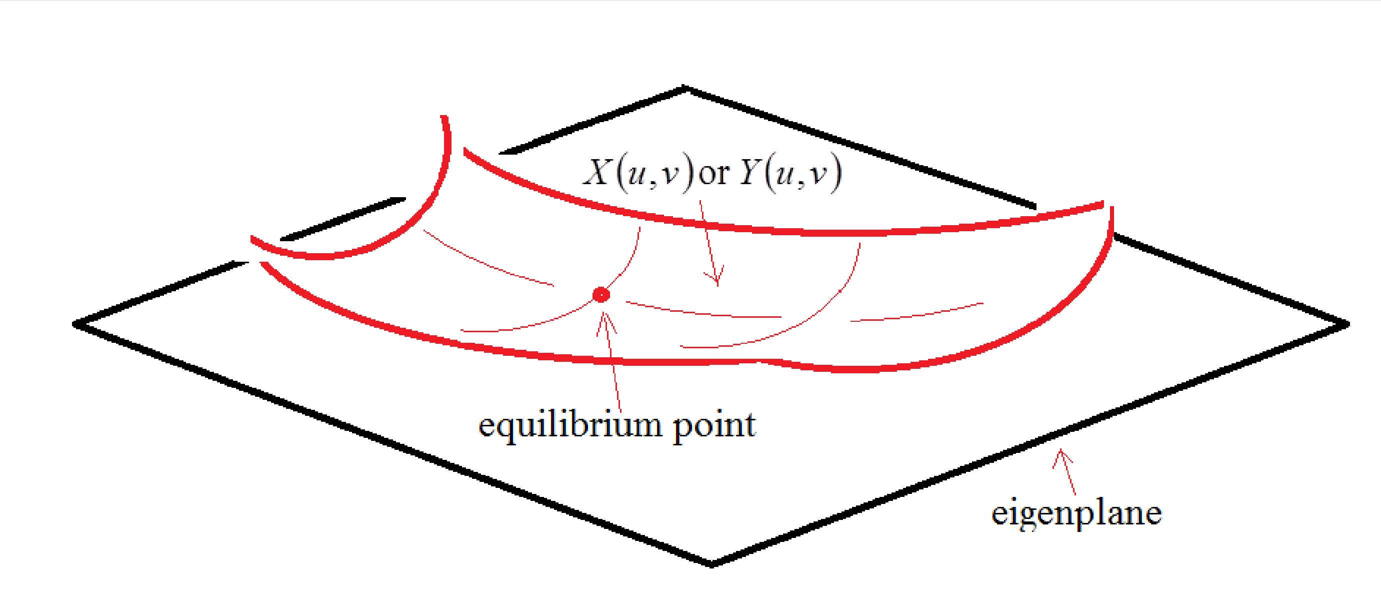
Known functions $g_i(u,v)$ are also expanded in power series.
Backsubstituting the expansions in equations (*)
allow for the determination of coefficients a_i and b_i and , hence,
the modal relationships $X_i(u,v)$ and $Y_i(u,v)$
that define the invariant manifold. On the manifold, the
(non-linear) modal oscillator is characterized by

$$\ddot{u} = Y_k(u,\dot{u})$$

Non-linear Normal Modes

Invariant manifold approach

Topological interpretation



Non-linear Normal Modes

Multiple time scales approach

- Asymptotic analytical solutions for the second-order equations of motion are searched, using the multiple time scales approach (Nayfeh & Mook, 1979)
- The time responses of all generalized coordinates are obtained explicitly.
- The topological structure of the embedded invariant manifold, if needed, has to be ‘extracted’ (post-processed)

Non-linear Normal Modes

Multiple time scales approach

- Asymptotic analytical solutions for the second-order equations of motion are searched, using the multiple time scales approach (Nayfeh & Mook, 1979)

$$\ddot{x}_i = h_i\left(x_1, x_2 \ldots x_n, y_1, y_2 \ldots y_n\right)$$

- The time responses of all generalized coordinates are obtained explicitly:

$$x_i(t) \text{ and } y_i(t) = \dot{x}_i(t)$$

- The topological structure of the embedded invariant manifold, if needed, has to be ‘extracted’ (post-processed): choose modal variables $u = x_k$ and $v = y_k$ and search coeficients a_i and b_i, $i = 1,2\ldots9$ that make true the equations

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = f_i\left(x_1, x_2 \ldots x_n, y_1, y_2 \ldots y_n\right) \end{cases}$$

Non-linear Normal Modes

Multiple time scales approach

Example of application of the multiple time scales approach to the unforced and undamped Duffing's equation (Nayfeh & Mook, 1979):

$$\ddot{x} + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 = 0 \qquad \text{with} \qquad \alpha_1 = \omega_0^2$$

Let $\varepsilon \in (0,1)$ be a small (dummy) parameter and $T_i = \varepsilon^i t$ be the time scales.

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots$$

$$\frac{d^2}{dt^2} = D_0^2 + \varepsilon 2 D_0 D_1 + \varepsilon^2 \left(2 D_0 D_2 + D_1^2 \right) + \ldots$$

$$D_i^j = \frac{\partial^j}{\partial T_i^j}$$

$$x(t,\varepsilon) = \varepsilon x_1 (T_0, T_1, T_2 \ldots) + \varepsilon^2 x_2 (T_0, T_1, T_2 \ldots) + \varepsilon^3 x_3 (T_0, T_1, T_2 \ldots) + \ldots$$

Functions $x_1(T_0, T_1, T_2 \ldots), x_2(T_0, T_1, T_2 \ldots), x_3(T_0, T_1, T_2 \ldots) \ldots$ to be determined!

Non-linear Normal Modes

Multiple time scales approach

Backsubstitution in the equation of motion and separation of terms of same order of ε leads to:

Order ε $\quad D_0^2 x_1 + \omega_0^2 x_1 = 0$

Order ε^2 $\quad D_0^2 x_2 + \omega_0^2 x_2 = -2D_0 D_1 x_1 - \alpha_2 x_1^2$

Order ε^3 $\quad D_0^2 x_3 + \omega_0^2 x_3 = -2D_0 D_1 x_2 - D_1^2 x_1 - 2D_0 D_2 x_1 - 2\alpha_2 x_1 x_2 - \alpha_3 x_1^3$

Non-linear Normal Modes

Multiple time scales approach

Order ε $\quad D_0^2 x_1 + \omega_0^2 x_1 = 0 \Rightarrow x_1 = A\left(T_1, T_2, \ldots\right) e^{i\omega_0 T_0} + cc$

Order ε^2 $\quad D_0^2 x_2 + \omega_0^2 x_2 = -2 D_0 D_1 x_1 - \alpha_2 x_1^2$

$$D_0^2 x_2 + \omega_0^2 x_2 = -2 i \omega_0 D_0 \left(D_1 A\right) e^{i\omega_0 T_0} - \alpha_2 A^2 e^{i2\omega_0 T_0} - \alpha_2 A\bar{A} + cc$$

Solvability condition: $D_1 A = 0 \Rightarrow x_2 = \frac{\alpha_2}{3\omega_0} A^2 e^{i2\omega_0 T_0} - \frac{\alpha_2}{\omega_0^2} A\bar{A} + cc$

Remark: only particular solution was considered!

Non-linear Normal Modes

Multiple time scales approach

Order ε^3 $\quad D_0^2 x_3 + \omega_0^2 x_3 = -2D_0 D_1 x_2 - D_1^2 x_1 - 2D_0 D_2 x_1 - 2\alpha_2 x_1 x_2 - \alpha_3 x_1^3$

$$\Downarrow$$

$$D_0^2 x_3 + \omega_0^2 x_3 = -\left[2i\omega_0 \left(D_2 A\right) - \frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{3\omega_0^2} A\bar{A}\right] e^{i\omega_0 T_0} - \frac{3\alpha_3\omega_0^2 + 2\alpha_2^2}{3\omega_0^2} A^3 e^{i3\omega_0 T_0} + cc$$

Sovability condition: $2i\omega_0 \left(D_2 A\right) - \frac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{3\omega_0^2} A\bar{A} = 0$

Non-linear Normal Modes

Multiple time scales approach

Polar notation: $A = \frac{1}{2} a\, e^{i\beta}, \quad a \in \mathbb{R}$ and $\beta \in \mathbb{R}$

Separating real and imaginary parts in the solvability condition:

$$\begin{cases} \omega_0 a' = 0 \Rightarrow a = a_0 = const. \\ \omega_0 a \beta' + \dfrac{10\alpha_2^2 - 9\alpha_3\omega_0^2}{24\omega_0^2} a^3 = 0 \Rightarrow \beta = \beta_0 + \dfrac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^2} a^2 T_2 \end{cases}$$

Hence: $x(t) = (\varepsilon a_0)\cos(\omega t + \beta_0) - \dfrac{\alpha_2}{2\omega_0^2}(\varepsilon a_0)^2 \left[1 - \dfrac{1}{3}\cos(2\omega t + 2\beta_0)\right] + O(\varepsilon^3)$

with $\omega = \omega_0 \left[1 + \dfrac{9\alpha_3\omega_0^2 - 10\alpha_2^2}{24\omega_0^2}(\varepsilon a_0)^2\right]$

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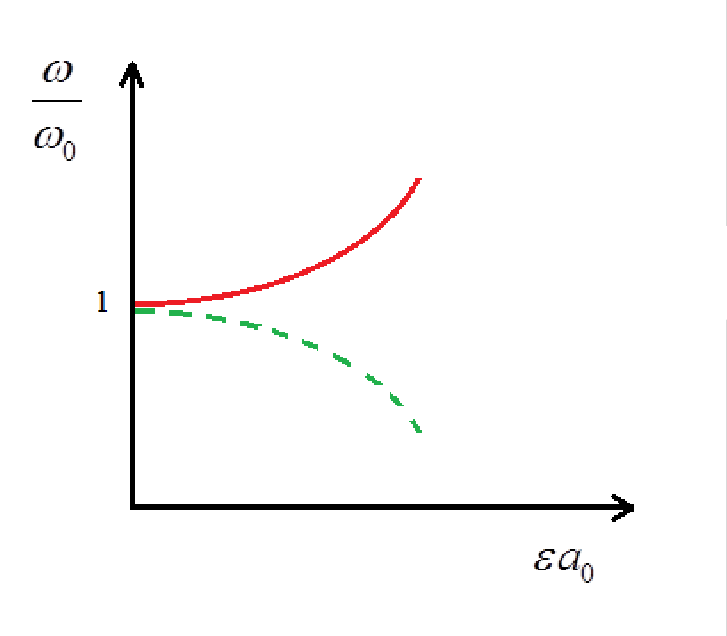
Multiple time scales approach



$\alpha_2 = 0$

Frequency-amplitude curve ('backbone' curve)

hardening for $\alpha_3 > 0$

softening for $\alpha_3 < 0$