Programa de Pós-Graduação em Engenharia Civil
Escola Politécnica da Universidade de São Paulo

USP

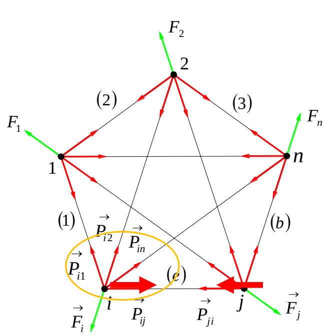
PEF-5750
Estruturas Leves
Ruy Marcelo de Oliveira Pauletti

Discrete Systems – Equilibrium of Systems of Central Forces

(Truss/cable element / Force Density Element / Sliding Cable Elements)

17/10/2017

A System of Central Forces :



Interaction forces between nodes i and j:

$$\mathbf{p}_{ij} = N_{ij}\mathbf{v}_{ij}$$

$$\mathbf{v}_{ij} = \mathbf{l}_{ij} / \left\| \mathbf{l}_{ij} \right\|$$

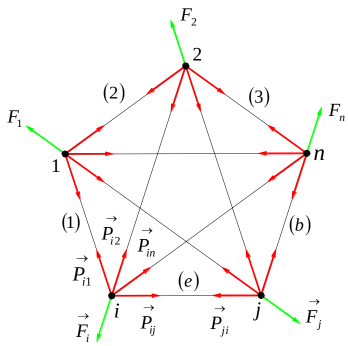
$$\mathbf{p}_{ij} = N_{ij} \frac{\mathbf{x}_j - \mathbf{x}_i}{\left\| \mathbf{x}_j - \mathbf{x}_i \right\|}$$

$$\mathbf{p}_{ij} = -\mathbf{p}_{ij} \quad ; \quad \mathbf{p}_{ii} = \mathbf{0}$$

Internal force at node i:

$$\mathbf{p}_i = \sum_{j=1}^{n} \mathbf{p}_{ij} = \sum_{j=1}^{n} N_{ij} \frac{\mathbf{x}_j - \mathbf{x}_i}{\left\| \mathbf{x}_j - \mathbf{x}_i \right\|}$$

A System of Central Forces :



Internal Loads Vector

$$\mathbf{P} = \begin{bmatrix} -\mathbf{p}_1 \\ -\mathbf{p}_2 \\ \vdots \\ -\mathbf{p}_n \end{bmatrix}$$

External Loads Vector

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{bmatrix}$$

Unbalanced load vector

$$\boxed{\mathbf{g} = \mathbf{P} - \mathbf{F}}$$

Solve for Equilibrium: Find $\mathbf{x}^* \;\big|\; \mathbf{g}\left(\mathbf{x}^*\right) = \mathbf{0}$

Defining a Displacement Vector:

$$\mathbf{x} = \mathbf{x}_r + \mathbf{u}$$

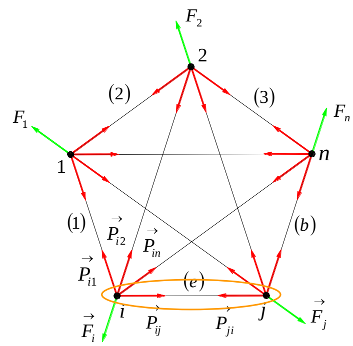
We can alterantively solve: Find $\mathbf{u}^* \;\big|\; \mathbf{g}\left(\mathbf{u}^*\right) = \mathbf{0}$

Newton's Method:

$$\mathbf{u}_{i+1} = \mathbf{u}_i - \left(\left. \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \right|_{\mathbf{u}_i} \right)^{-1} \mathbf{g}(\mathbf{u}_i) = \mathbf{u}_i - \left(\mathbf{K}_t^{\,i}\right)^{-1} \mathbf{g}(\mathbf{u}_i)$$

$$\mathbf{K}_t^{\,i} = \left. \frac{\partial \mathbf{g}}{\partial \mathbf{u}} \right|_{\mathbf{u}_i}$$ Tangent stiffness matrix

A System of Central Forces :



Element (e), conecting nodes i to j, contributes to P with a load $-P_{ij}$ at node i and a node $+P_{ij}$ at node j.

Considering the existence of *b* elements, we define a <u>vector of element loads</u>:

$$\mathbf{N} = \mathbf{N}(\mathbf{u}) = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_b \end{bmatrix}_{b\times 1}$$

$$\begin{bmatrix} 0 \\ [-P_{ij}] \\ \vdots \\ [-P_{ji}] \\ \vdots \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & -\mathbf{v}^{(e)} & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & 0 \\ 0 & 0 & +\mathbf{v}^{(e)} & 0 & 0 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}}_{\mathbf{C}^{(e)}} \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_e \\ \vdots \\ 0 \end{bmatrix}$$

"Geometric operator" for element (e): $\mathbf{C}^{(e)}$

Gobal geometric operator: $\mathbf{C} = \sum_{(e)=1}^{b} \mathbf{C}^{(e)}$

Global Internal load vector: $\boxed{\mathbf{P} = \mathbf{C}\,\mathbf{N}}$

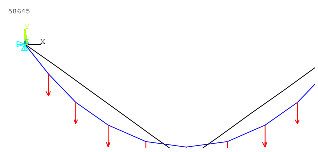
The tangent stiffness then becomes:

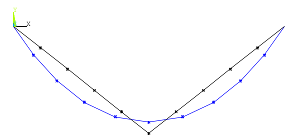
$$\mathbf{K}_t = \frac{\partial}{\partial \mathbf{u}}(\mathbf{C}\mathbf{N} - \mathbf{F}) = \mathbf{C}\frac{\partial \mathbf{N}}{\partial \mathbf{u}} + \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} - \frac{\partial \mathbf{F}}{\partial \mathbf{u}}$$

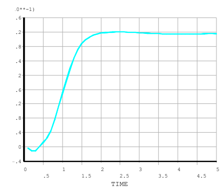
$$\mathbf{K}_t = \mathbf{K}_c + \mathbf{K}_g + \mathbf{K}_{ext}$$

Alternative Method: Dynamic Relaxation

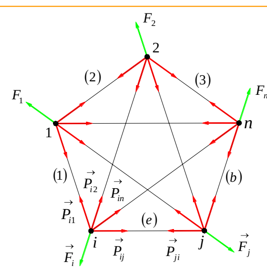
$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{P}(\mathbf{u}(t)) = \mathbf{F}_0$$







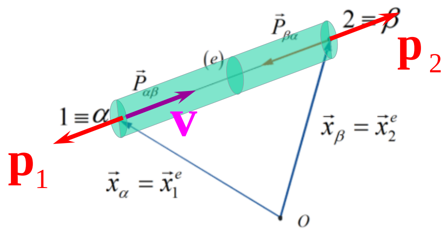
Force Density Method (shape finding):



$$\mathbf{p}_i = \sum_{j=1}^{n} \mathbf{p}_{ij} = \sum_{j=1}^{n} N_{ij} \mathbf{v}_{ij} = \sum_{j=1}^{n} N_{ij} \frac{\mathbf{x}_j - \mathbf{x}_i}{\left\| \mathbf{x}_j - \mathbf{x}_i \right\|} \quad , \quad i = 1, \ldots, n$$

$$\boxed{\sum_{j=1}^{m} n_{ij} \left(\mathbf{x}_j - \mathbf{x}_i \right) = \mathbf{f}_i \quad , \quad i = 1, \ldots, n}$$

Geometrically Exact Truss/Cable Element

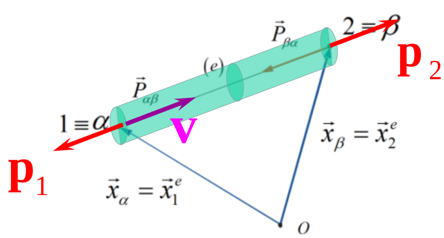


$$\mathbf{u}^{(e)} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}_{6x1} \qquad \mathbf{p}^{(e)} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix}_{6x1} = \begin{bmatrix} -\mathbf{v} \\ \mathbf{v} \end{bmatrix} N^{(e)} = \mathbf{C}^{(e)} N^{(e)}$$

$$\mathbf{p}^{(e)} = \mathbf{0}, \quad \text{if } \ell \le \ell^r$$

$$\mathbf{v}^{(e)} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{\left\| \mathbf{x}_2 - \mathbf{x}_1 \right\|} = \frac{\mathbf{l}}{\ell} \qquad N = \begin{cases} EA\left(\ell - \ell^r\right)/\,\ell^r \text{ , if } \ell > \ell^r \\ 0 \text{ , if } \ell \le \ell^r \end{cases} \qquad \ell^r = \frac{EA}{EA + N^0}\,\ell^0$$

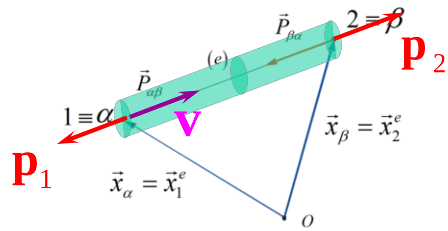
Geometrically Exact Truss Element



$$\begin{cases} \mathbf{u}^{(e)} = \mathbf{A}^{(e)} \mathbf{u} \\ \mathbf{P} = \sum_{(e)=1}^{b} \mathbf{A}^{(e)T} \mathbf{p}^{(e)} \end{cases}$$

$$\begin{bmatrix} \mathbf{u}_1^{(e)} \\ \mathbf{u}_2^{(e)} \end{bmatrix}_{6x1} = \begin{array}{c|ccccccc} & 1 & \cdots & i & \cdots & j & \cdots & n \\ \hline 1 & \mathbf{0} & \cdots & \mathbf{I}_3 & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ 2 & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{I}_3 & \cdots & \mathbf{0} \end{array}_{6x3n} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_j \\ \vdots \\ \mathbf{u}_n \end{bmatrix}_{3nx1}$$

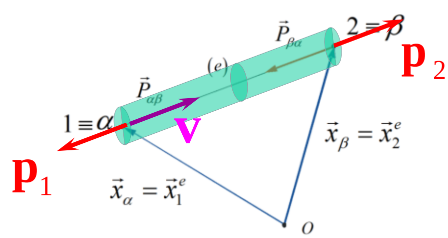
Geometrically Exact Truss Element



$$\mathbf{K}_t = \frac{\partial \mathbf{P}}{\partial \mathbf{u}} = \sum_{(e)=1}^{b} \mathbf{A}^{(e)T} \frac{\partial \mathbf{p}^{(e)}}{\partial \mathbf{u}} = \sum_{(e)=1}^{b} \mathbf{A}^{(e)T} \frac{\partial \mathbf{p}^{(e)}}{\partial \mathbf{u}^{(e)}} \frac{\partial \mathbf{u}^{(e)}}{\partial \mathbf{u}^{(e)}}$$

$$\mathbf{K}_t = \sum_{(e)=1}^{b} \mathbf{A}^{(e)T} \mathbf{k}_t^{(e)} \mathbf{A}^{(e)} \qquad \boxed{\mathbf{k}_t^{(e)} = \frac{\partial \mathbf{p}^{(e)}}{\partial \mathbf{u}^{(e)}}}$$

Geometrically Exact Truss Element



$$\mathbf{k}_t^{(e)} = \frac{\partial \mathbf{p}^{(e)}}{\partial \mathbf{u}^{(e)}} = \frac{\partial}{\partial \mathbf{u}^{(e)}} \left(\mathbf{C}^{(e)} N^{(e)} \right) = \mathbf{C} \left(\frac{\partial N}{\partial \mathbf{u}} \right)^T + N \frac{\partial \mathbf{C}}{\partial \mathbf{u}} = \mathbf{k}_e + \mathbf{k}_g$$

$$\boxed{\mathbf{k}_e = \mathbf{C}\left(\frac{\partial N}{\partial \mathbf{u}}\right)^T}$$

$$\frac{\partial N}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}}\left(\frac{EA}{\ell^r}\left(\ell - \ell^r\right)\right) = \frac{EA}{\ell^r}\frac{\partial \ell}{\partial \mathbf{u}}$$

$$\frac{\partial \ell}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}}\left(\mathbf{l}^T\mathbf{l}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\mathbf{l}^T\mathbf{l}\right)^{-\frac{1}{2}}\frac{\partial}{\partial \mathbf{u}}\left(\mathbf{l}^T\mathbf{l}\right) = \frac{1}{\ell}\left(\frac{\partial \mathbf{l}}{\partial \mathbf{u}}\right)^T \mathbf{l}$$

$$\frac{\partial \mathbf{l}}{\partial \mathbf{u}} = \begin{bmatrix} \left[\frac{\partial (\mathbf{l})_x}{\partial \mathbf{u}}\right]^T \\ \left[\frac{\partial (\mathbf{l})_y}{\partial \mathbf{u}}\right]^T \\ \left[\frac{\partial (\mathbf{l})_z}{\partial \mathbf{u}}\right]^T \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} = \left[-\mathbf{I}_3 \quad \mathbf{I}_3\right]$$

$$\frac{\partial \ell}{\partial \mathbf{u}} = \frac{1}{\ell}\begin{bmatrix} -\mathbf{I} \\ \mathbf{I} \end{bmatrix}\mathbf{l} = \begin{bmatrix} -\mathbf{v} \\ \mathbf{v} \end{bmatrix} = \mathbf{C}$$

$$\boxed{\mathbf{k}_e = \mathbf{C}\frac{EA}{\ell^r}\mathbf{C}^T = \frac{EA}{\ell^r}\begin{bmatrix} \mathbf{v}\mathbf{v}^T & -\mathbf{v}\mathbf{v}^T \\ -\mathbf{v}\mathbf{v}^T & \mathbf{v}\mathbf{v}^T \end{bmatrix}}$$

$$\boxed{\mathbf{k}_g = N\frac{\partial \mathbf{C}}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}}\left(\begin{bmatrix} -\mathbf{v}^{(e)} \\ \mathbf{v}^{(e)} \end{bmatrix}\right)}$$

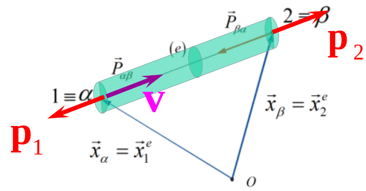
$$\frac{\partial \mathbf{v}}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}}\left(\frac{\mathbf{l}}{\ell}\right) = \frac{1}{\ell^2}\left(\ell\frac{\partial \mathbf{l}}{\partial \mathbf{u}} - \mathbf{l}\left(\frac{\partial \ell}{\partial \mathbf{u}}\right)^T\right)$$

$$\frac{\partial \mathbf{l}}{\partial \mathbf{u}} = \left[-\mathbf{I}_3 \quad \mathbf{I}_3\right]$$

$$\frac{\partial \ell}{\partial \mathbf{u}} = \begin{bmatrix} -\mathbf{v} \\ \mathbf{v} \end{bmatrix}$$

$$\boxed{\mathbf{k}_g = \frac{N}{\ell}\begin{bmatrix} \left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) & -\left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) \\ -\left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) & \left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) \end{bmatrix}}$$

Summary: Geometrically Exact Truss Element



$$\mathbf{p}^{(e)} = \begin{cases} N\begin{bmatrix} -\mathbf{v} \\ \mathbf{v} \end{bmatrix}, \quad N = \dfrac{EA\left(\ell - \ell^r\right)}{\ell^r}, \text{ if } \ell > \ell \\ \mathbf{0}, \qquad \text{if } \ell \le \ell^r \end{cases}$$

$$\mathbf{k}_t^{(e)} = \frac{EA}{\ell^r}\begin{bmatrix} \mathbf{v}\mathbf{v}^T & -\mathbf{v}\mathbf{v}^T \\ -\mathbf{v}\mathbf{v}^T & \mathbf{v}\mathbf{v}^T \end{bmatrix} + \frac{N}{\ell}\begin{bmatrix} \left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) & -\left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) \\ -\left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) & \left(\mathbf{I}_3 - \mathbf{v}\mathbf{v}^T\right) \end{bmatrix}$$

$$\mathbf{k}_t^{(e)} = \mathbf{0}, \quad \text{if } \ell \le \ell^r$$

Finite-difference approximations to k_t

For each element, we can partition the stiffness matrix into columns:

$$\mathbf{k}_t^e = \begin{bmatrix} \mathbf{k}_1^e & \mathbf{k}_2^e & \cdots & \mathbf{k}_{n_{dof}^e}^e \end{bmatrix}$$

where $\mathbf{k}_j^e$ are the <u>directional derivatives</u> of g with respect to each component (or "*degree of freedom*") of $\mathbf{u}^e$, approximated by a central finite-difference schemes:

$$\mathbf{k}_j^e = \frac{1}{2h}\left[\mathbf{g}^e\left(\mathbf{u}^e + h\boldsymbol{\delta}_j^e\right) - \mathbf{g}\left(\mathbf{u}^e - h\boldsymbol{\delta}_j^e\right)\right], \; j = 1, \ldots, n_{dof}^e$$

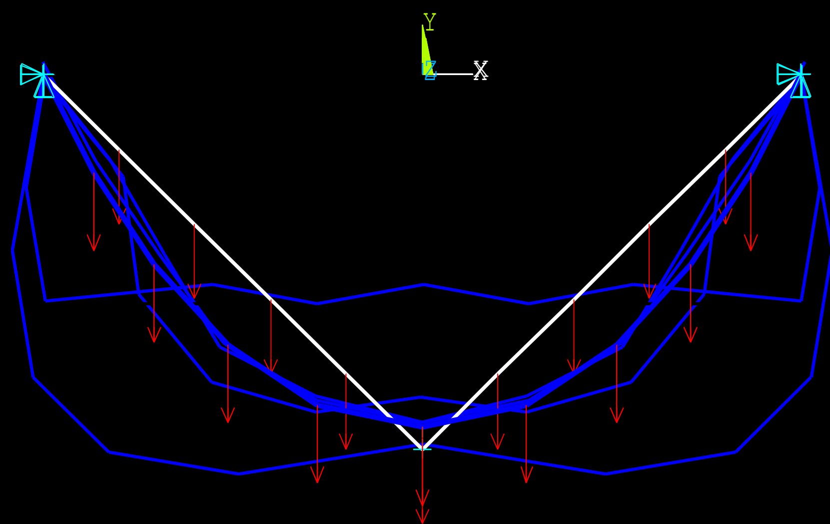
and where h is a scalar parameter and $h\boldsymbol{\delta}_j$ is a perturbation of the j^{th} degree of freedom of the system, such that

$$\boldsymbol{\delta}_j = \left[\delta_i\right], \quad \begin{cases} \delta_j = 1, & i = j \\ \delta_i = 0, & i \ne j \end{cases}, \quad i = 1, \ldots, n_{dof}$$

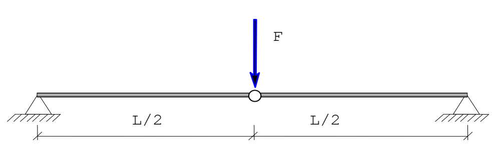
In MATLAB environment, in a desktop computer with Intel's i7-3770s CPU, we found a good performance with

$$h = \sqrt[4]{\varepsilon} \sim 10^{-10}$$

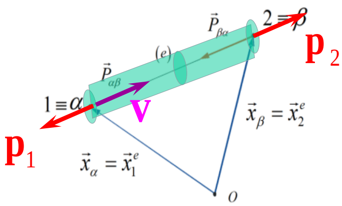
A hanging cable modeled with truss elements,
solved through Newton's Method. Several iterations are shown:



Exercise 6: Consider an assembling of two elements of length L/2, as shown in figure below.
Assemble the stiffness matrix of the system, according to the formalism shown previously, and show that the stiffness of the intermediate node, in the vertical direction, is $k_0 = 4N_0 / L$



Force densities for truss / cable elements



Global internal forces vector:

$$\mathbf{P} = \sum_{e=1}^{b} \mathbf{A}^{eT} \mathbf{p}^e$$

$$\mathbf{P} = \sum_{e=1}^{b} \mathbf{A}^{eT} N_e \begin{bmatrix} -\mathbf{v}^e \\ \mathbf{v}^e \end{bmatrix} = \sum_{e=1}^{b} \mathbf{A}^{eT} \begin{bmatrix} -\mathbf{x}_j^e - \mathbf{x}_i^e \\ \mathbf{x}_j^e - \mathbf{x}_i^e \end{bmatrix} \frac{N_e}{\|\mathbf{x}_j^e - \mathbf{x}_i^e\|}$$

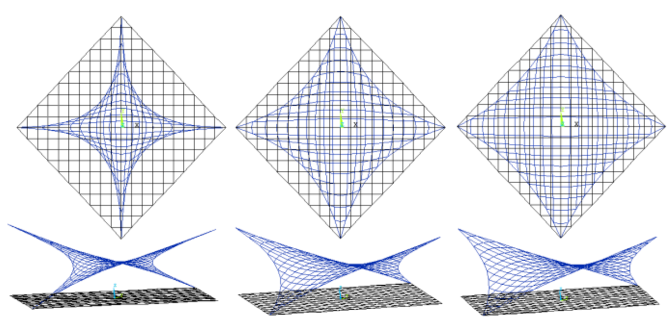
Force Densities: $\boxed{n_e = \frac{N_e}{\|\mathbf{x}_j^e - \mathbf{x}_i^e\|} = \frac{N_e}{\ell^e}}$

$$\mathbf{P} = \sum_{e=1}^{b} \mathbf{A}^{eT} n_e \begin{bmatrix} \mathbf{x}_i^e - \mathbf{x}_j^e \\ \mathbf{x}_j^e - \mathbf{x}_i^e \end{bmatrix} = \left(\sum_{e=1}^{b} \mathbf{A}^{eT} n_e \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix} \mathbf{A}^e \right) \mathbf{x}$$

Constant element stiffness: $\boxed{\mathbf{k}_d^e = n_e \begin{bmatrix} \mathbf{I}_3 & -\mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{I}_3 \end{bmatrix}}$

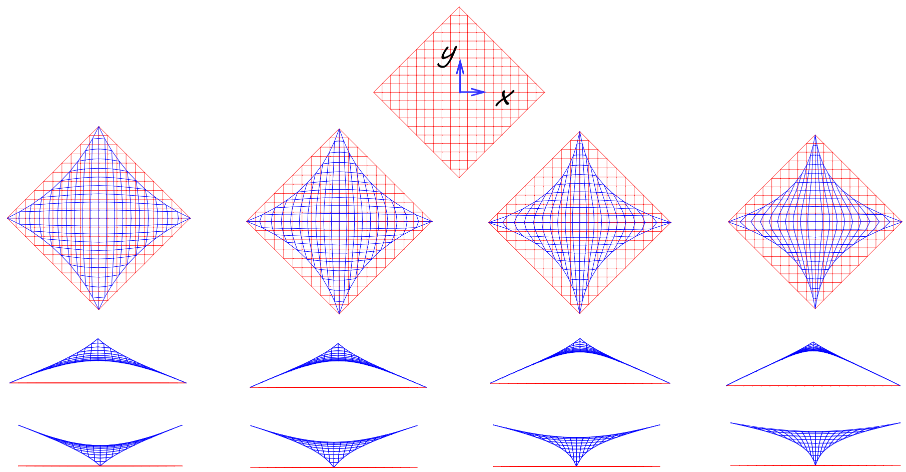
Global Equilibrium $\boxed{\mathbf{P} = \mathbf{F}}$ *A linear system:* $\boxed{\left(\sum_{e=1}^{b} \mathbf{A}^{eT} n_e \mathbf{k}_d^e \mathbf{A}^e \right) \mathbf{x} = \mathbf{K}_d \mathbf{x} = \mathbf{F}}$

Some isotropic solutions:



$n_b = n_i$ $n_b = 5n_i$ $n_b = 10n_i$

Some non-isotropic solutions:



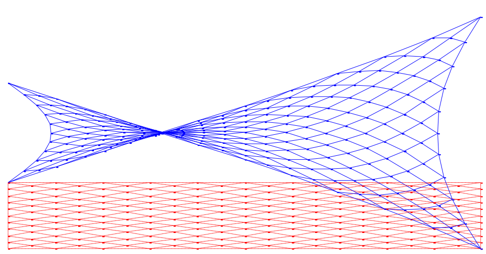
$n_x = 1; n_y = 1$ $n_b = 10$

$n_x = 2; n_y = 1$ $n_b = 10$

$n_x = 4; n_y = 1$ $n_b = 10$

$n_x = 10; n_y = 1$ $n_b = 20$

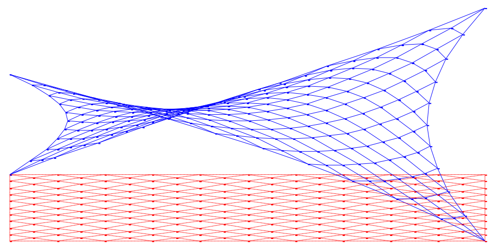
Some non-isotropic solutions:



$n_x = 1 \quad ; \quad n_y = 1 \quad ; \quad n_b = 10$

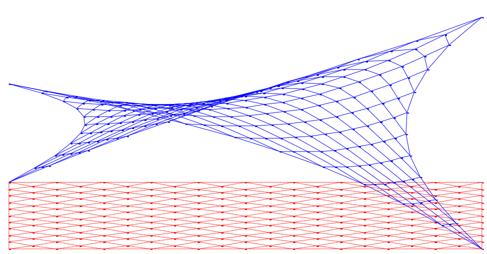
(Isotropic case)

Some non-isotropic solutions:



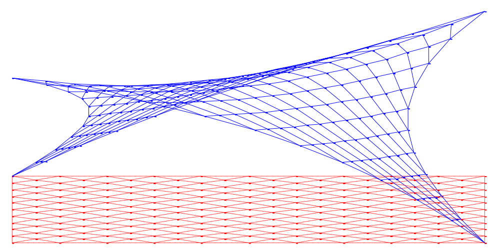
$n_x = 2 \quad ; \quad n_y = 1 \quad ; \quad n_b = 10$

Some non-isotropic solutions:



$n_x = 4 \quad ; \quad n_y = 1 \quad ; \quad n_b = 10$

Some non-isotropic solutions:



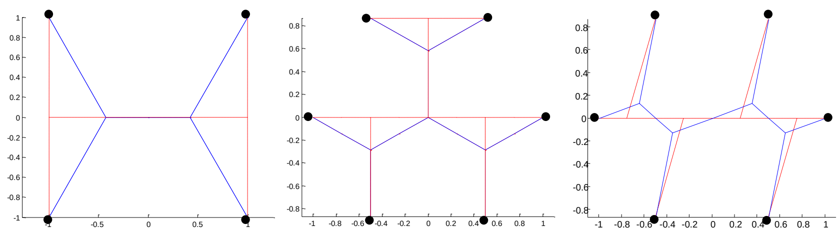
$n_x = 10 \quad ; \quad n_y = 1 \quad ; \quad n_b = 20$

Exercise 7: The Motorway problem.

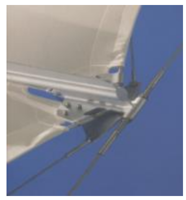
Finding the minimum path joining a number of points, iterating the FDM. Solutions consist of straight lines, connecting at 120º (Isenberg, 1992). For four points located at the vertices of a square of side *L*, the minimum configuration is the 'hourglass' shape shown in blue in the first figure. The length of the horizontal segment is equal to $b = \left(\sqrt{3}-1\right)L/\sqrt{3}$

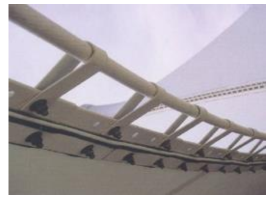
For *L*=2, one gets *b*=0.84530

Furthermore, find the two minimal configurations (in blue) which are possible for six points located at the vertices of a regular hexagon. They depend n the different topologies assumed for the reference configuration (in red).

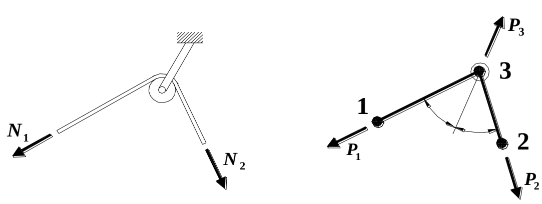


Sliding cables





Aufare's sliding-cable element



Nodal displacements: $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}$ **Internal loads vector:** $\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ -(\mathbf{v}_1 + \mathbf{v}_2) \end{bmatrix} N = \mathbf{C}N$

$$N = \frac{EA}{\ell^r}\left(\ell - \ell^r\right)$$

Aufare's sliding-cable element

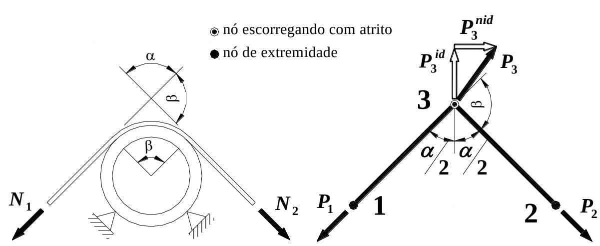
Tangent stiffness matrix: $\mathbf{k}_t = \mathbf{k}_e + \mathbf{k}_g$

$$\mathbf{k}_e = \frac{EA}{\ell_r}\mathbf{C}\mathbf{C}^T = \frac{EA}{\ell_r}\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & -(\mathbf{M}_{11}+\mathbf{M}_{12}) \\ \mathbf{M}_{21} & \mathbf{M}_{22} & -(\mathbf{M}_{21}+\mathbf{M}_{22}) \\ -(\mathbf{M}_{11}+\mathbf{M}_{21}) & -(\mathbf{M}_{12}+\mathbf{M}_{22}) & (\mathbf{M}_{11}+\mathbf{M}_{12}+\mathbf{M}_{21}+\mathbf{M}_{22}) \end{bmatrix}$$

$$\mathbf{k}_g = \begin{bmatrix} \frac{N}{\ell_1}\mathbf{M}_1 & \mathbf{0} & -\frac{N}{\ell_1}\mathbf{M}_1 \\ \mathbf{0} & \frac{N}{\ell_2}\mathbf{M}_2 & -\frac{N}{\ell_2}\mathbf{M}_2 \\ -\frac{N}{\ell_1}\mathbf{M}_1 & -\frac{N}{\ell_2}\mathbf{M}_2 & \left(\frac{N}{\ell_1}\mathbf{M}_1 + \frac{N}{\ell_2}\mathbf{M}_2\right) \end{bmatrix}$$

$$\mathbf{M}_{ij} = \mathbf{v}_i\mathbf{v}_j^T \qquad \mathbf{M}_i = \mathbf{I}_3 - \mathbf{v}_i\mathbf{v}_i^T \qquad i = 1,2$$

Sliding cable with Coulomb Friction



$$\eta = \frac{N_1}{N_2} = e^{\mu\beta}$$

$$N_1 = \frac{EA}{\ell_1^r}\left(\ell_1 - \ell_1^r\right) = \frac{2\eta}{\eta+1}N^{id}$$

$$N_2 = \frac{EA}{\ell_2^r}\left(\ell_2 - \ell_2^r\right) = \frac{2}{\eta+1}N^{id}$$

$$\mathbf{N}^{(e)} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \dfrac{2\eta}{\eta+1} N^{id} \\ \dfrac{2}{\eta+1} N^{id} \end{bmatrix} \qquad \mathbf{C}^{(e)} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_2 \\ -\mathbf{v}_1 & -\mathbf{v}_2 \end{bmatrix}$$

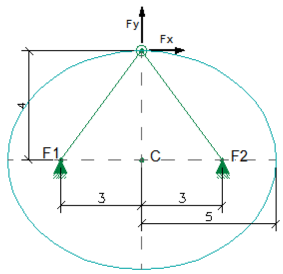
$$\mathbf{p}^{(e)} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} N_1 \mathbf{v}_1 \\ N_2 \mathbf{v}_2 \\ -\left(N_1 \mathbf{v}_1 + N_2 \mathbf{v}_2 \right) \end{bmatrix} = \mathbf{C}^{(e)} \mathbf{N}^{(e)}$$

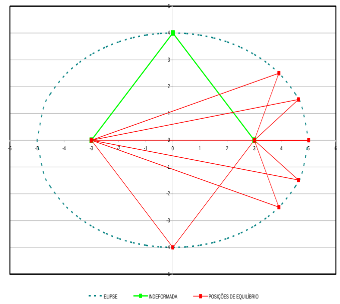
$$\mathbf{k}_t = \frac{\partial \mathbf{p}}{\partial \mathbf{u}} = \mathbf{C} \frac{\partial \mathbf{N}}{\partial \mathbf{u}} + \mathbf{N}^T \frac{\partial \mathbf{C}}{\partial \mathbf{u}} = \mathbf{k}_c + \mathbf{k}_g$$

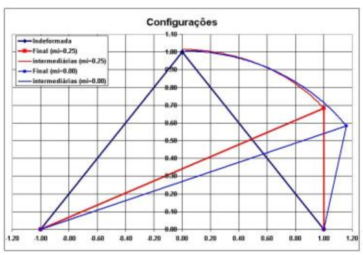
$$\mathbf{k}_t = \frac{2}{\eta+1} \begin{bmatrix} \eta \mathbf{k}_{11}^{id} & \eta \mathbf{k}_{12}^{id} & \eta \mathbf{k}_{13}^{id} \\ \mathbf{k}_{21}^{id} & \mathbf{k}_{22}^{id} & \mathbf{k}_{23}^{id} \\ -\left(\eta \mathbf{k}_{11}^{id} + \mathbf{k}_{21}^{id} \right) & -\left(\eta \mathbf{k}_{12}^{id} + \mathbf{k}_{22}^{id} \right) & -\left(\eta \mathbf{k}_{13}^{id} + \mathbf{k}_{23}^{id} \right) \end{bmatrix} +$$

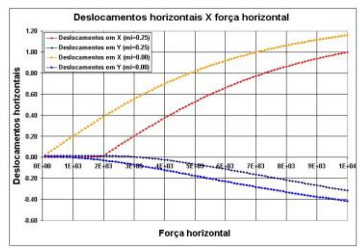
$$+ \frac{2\mu\eta N^{id}}{(\eta+1)^2 \sin\beta} \begin{bmatrix} \mathbf{M}_{12}\mathbf{M}_1 / \ell_1 & \mathbf{M}_{11}\mathbf{M}_2 / \ell_2 & -\left(\mathbf{M}_{12}\mathbf{M}_1 / \ell_1 + \mathbf{M}_{11}\mathbf{M}_2 / \ell_2 \right) \\ \mathbf{M}_{22}\mathbf{M}_1 / \ell_1 & -\mathbf{M}_{21}\mathbf{M}_2 / \ell_2 & -\left(\mathbf{M}_{22}\mathbf{M}_1 / \ell_1 - \mathbf{M}_{21}\mathbf{M}_2 / \ell_2 \right) \\ -\begin{pmatrix} \mathbf{M}_{12}\mathbf{M}_1 / \ell_1 + \\ \mathbf{M}_{22}\mathbf{M}_1 / \ell_2 \end{pmatrix} & -\xi N^{id} \begin{pmatrix} \mathbf{M}_{11}\mathbf{M}_2 / \ell_2 + \\ -\mathbf{M}_{21}\mathbf{M}_2 / \ell_2 \end{pmatrix} & \begin{pmatrix} \left(\mathbf{M}_{12}\mathbf{M}_1 + \mathbf{M}_{22}\mathbf{M}_1 \right) / \ell_1 + \\ \left(\mathbf{M}_{11}\mathbf{M}_2 - \mathbf{M}_{21}\mathbf{M}_2 \right) / \ell_2 \end{pmatrix} \end{bmatrix}$$

Aufare's sliding-cable element

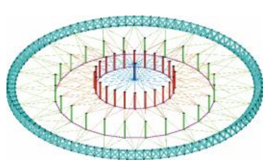


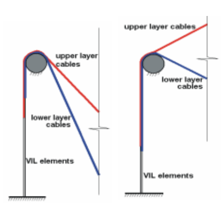


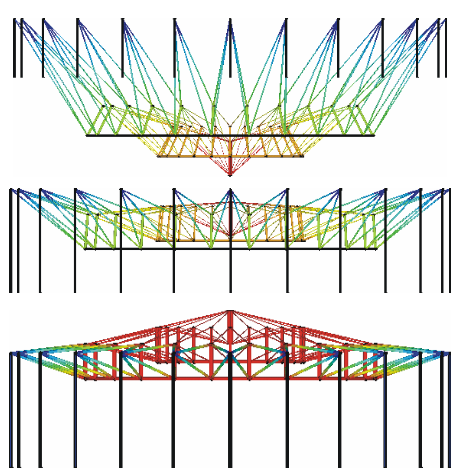




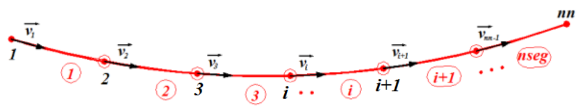
Aufare's sliding-cable element







An ideal sliding-cable super-element



A super-element with n nodes and $n_{seg}=n-1$ segments, sliding over $n_p=n-2$ pulleys

$$\mathbf{l}_k = \mathbf{x}^0_{k+1} + \mathbf{u}_{k+1} - \mathbf{x}^0_k - \mathbf{u}_k \qquad \mathbf{l}^0_k = \mathbf{x}^0_{k+1} - \mathbf{x}^0_k$$

$$\ell = \sum_{k=1}^{n_{seg}} \ell_k = \sum_{k=1}^{n_{seg}} \|\mathbf{l}_k\| \qquad \ell_0 = \sum_{k=1}^{n_{seg}} \|\mathbf{l}_0\| \qquad \ell_r = \frac{EA\ell_0}{EA+N_0} \qquad N = EA\left(\frac{\ell-\ell_r}{\ell_r}\right)$$

Nodal displacements:

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \qquad \mathbf{v}_i = \mathbf{l}_i / \ell_i \qquad \mathbf{v}_0 = \mathbf{v}_n = \mathbf{0} \qquad \mathbf{p}_i = N\left(\mathbf{v}_{i-1} - \mathbf{v}_i\right)$$

Internal loads vector:

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_i \\ \vdots \\ \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0 - \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{i-1} - \mathbf{v}_i \\ \vdots \\ \mathbf{v}_{n-1} - \mathbf{v}_n \end{bmatrix} N = \mathbf{C}N$$

An ideal sliding-cable super-element

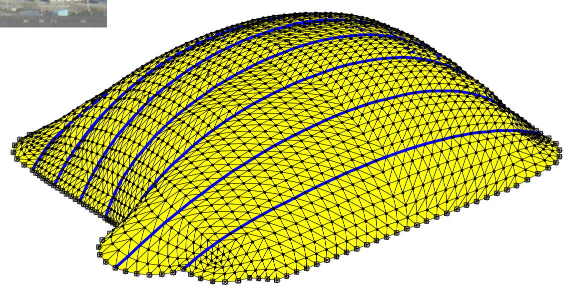
Tangent stiffness matrix: $\mathbf{k}_t = \mathbf{k}_e + \mathbf{k}_g$

$$\mathbf{k}_e = \frac{EA}{\ell_r}\mathbf{C}\mathbf{C}^T \qquad \mathbf{k}_g = \left[\mathbf{k}_{ij}\right] \qquad i,j = 1,\ldots,n$$

$$\mathbf{k}_{i,i-1} = -N\frac{\mathbf{M}_{i-1}}{\ell_{i-1}} \quad ; \quad \mathbf{k}_{ii} = N\left(\frac{\mathbf{M}_{i-1}}{\ell_{i-1}} + \frac{\mathbf{M}_i}{\ell_i}\right) \quad ; \quad \mathbf{k}_{i,i+1} = -N\frac{\mathbf{M}_i}{\ell_i}$$

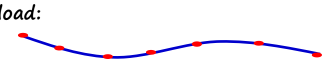
$$\mathbf{k}_g = N\begin{bmatrix}
\left(\frac{\mathbf{M}_0}{\ell_0}+\frac{\mathbf{M}_1}{\ell_1}\right) & -\frac{\mathbf{M}_1}{\ell_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-\frac{\mathbf{M}_1}{\ell_1} & \left(\frac{\mathbf{M}_1}{\ell_1}+\frac{\mathbf{M}_2}{\ell_2}\right) & -\frac{\mathbf{M}_2}{\ell_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -\frac{\mathbf{M}_2}{\ell_2} & \left(\frac{\mathbf{M}_2}{\ell_2}+\frac{\mathbf{M}_3}{\ell_3}\right) & -\frac{\mathbf{M}_3}{\ell_3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{\mathbf{M}_{i-1}}{\ell_{i-1}} & \left(\frac{\mathbf{M}_{i-1}}{\ell_{i-1}}+\frac{\mathbf{M}_i}{\ell_i}\right) & -\frac{\mathbf{M}_i}{\ell_i} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{\mathbf{M}_i}{\ell_i} & \left(\frac{\mathbf{M}_i}{\ell_i}+\frac{\mathbf{M}_{i+1}}{\ell_{i+1}}\right) & -\frac{\mathbf{M}_{i+1}}{\ell_{i+1}} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \vdots & \mathbf{0} & -\frac{\mathbf{M}_{i+1}}{\ell_{i+1}} & \ddots & & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & -\frac{\mathbf{M}_{n-2}}{\ell_{n-2}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\frac{\mathbf{M}_{n-2}}{\ell_{n-2}} & \left(\frac{\mathbf{M}_{n-2}}{\ell_{n-2}}+\frac{\mathbf{M}_{n-1}}{\ell_{n-1}}\right) & -\frac{\mathbf{M}_{n-1}}{\ell_{n-1}} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\frac{\mathbf{M}_{n-1}}{\ell_{n-1}} & \left(\frac{\mathbf{M}_{n-1}}{\ell_{n-1}}+\frac{\mathbf{M}_n}{\ell_n}\right)
\end{bmatrix}$$

Application of the sliding-cable super-element



A Geodesic String super-element

Consider a a chain of n-1 cable elements, connecting n nodes, under a constant, uniform normal load:

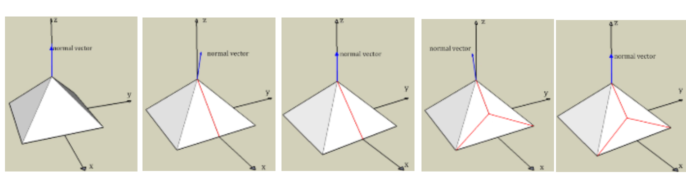


Internal loads vector:

$$\mathbf{P}^{gs} = \begin{bmatrix} \mathbf{p}_1^{gs} \\ \vdots \\ \mathbf{p}_i^{gs} \\ \vdots \\ \mathbf{p}_n^{gs} \end{bmatrix} \qquad \mathbf{p}_1^{gs} = \mathbf{p}_n^{gs} = \mathbf{0}$$

For every node, calculate an average normal unit vector:

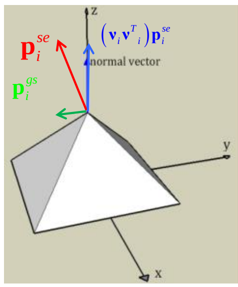
$$\mathbf{v}_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|} \qquad \mathbf{w}_i = \sum_{k=1}^{n_{ei}(i)} \mathbf{n}_k \theta_k \qquad \mathbf{n}_k,\ k = 1, \ldots, n_{ei}(i)$$



Average normal vectors at the apex of the same pyramid, due to different average criteria
(a) simple average; (b) simple average; (c) area-weighted; (d) area-weighted; (e) angle-weighted

A Geodesic String super-element

at each node, remove the normal components of the load vector:



$$\mathbf{p}_i^{gs} = \mathbf{p}_i^{se} - \left(\mathbf{v}_i\mathbf{v}^T{}_i\right)\mathbf{p}_i^{se} = \mathbf{p}_i^{gs} = \left(\mathbf{I} - \mathbf{v}_i\mathbf{v}^T{}_i\right)\mathbf{p}_i^{se}$$

$$\mathbf{M}_i^{gs} = \left(\mathbf{I} - \mathbf{v}_i\mathbf{v}^T{}_i\right), \quad \mathbf{M}_1^{gs} = \mathbf{M}_{n_{gs}}^{gs} = \mathbf{0}$$

$$\mathbf{p}_i^{gs} = \mathbf{M}_i^{gs}\mathbf{p}_i^{se}$$

$$\mathbf{M}^{gs} = \begin{bmatrix} \mathbf{M}_1^{gs} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_2^{gs} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{n_{gs}}^{gs} \end{bmatrix}$$

Internal loads vector:

$$\boxed{\mathbf{p}^{gs} = \mathbf{M}^{gs}\mathbf{p}^{se}}$$

A Geodesic String super-element

Tangent / secant stiffness matrix:

$$\mathbf{k}_t^{gs} = \frac{\partial \mathbf{p}^{gs}}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}}\left(\mathbf{M}^{gs}\mathbf{p}^{se}\right) = \mathbf{M}^{gs}\frac{\partial \mathbf{p}^{se}}{\partial \mathbf{u}} + \left(\mathbf{p}^{se}\right)^T\frac{\partial \mathbf{M}^{gs}}{\partial \mathbf{u}} = \mathbf{M}^{gs}\mathbf{k}_t^{se} + \left(\mathbf{p}^{se}\right)^T\frac{\partial \mathbf{M}^{gs}}{\partial \mathbf{u}}$$

$$\frac{\partial \mathbf{M}^{gs}}{\partial \mathbf{u}} \sim 0$$

$$\boxed{\mathbf{k}_t^{gs} \simeq \mathbf{M}^{gs}\mathbf{k}_g^{se}}$$

$$\tilde{\mathbf{k}}_{ii}^{gs} = N^{gs}\mathbf{M}_i^{gs}\left(\frac{\mathbf{M}_{i-1}}{\ell_{i-1}} + \frac{\mathbf{M}_i}{\ell_i}\right)$$

$$\tilde{\mathbf{k}}_{i,i+1}^{gs} = -\frac{N^{gs}}{\ell_i}\mathbf{M}_i^{gs}\mathbf{M}_i$$

$$\tilde{\mathbf{k}}_{i,i-1}^{gs} = -\frac{N^{gs}}{\ell_{i-1}}\mathbf{M}_i^{gs}\mathbf{M}_{i-1}$$

$$\tilde{\mathbf{k}}_{ij}^{gs} = \mathbf{0} \quad , j < i-1, j > i+1$$

A Geodesic String super-element