Programa de Pós-Graduação em Engenharia Civil
Escola Politécnica da Universidade de São Paulo

PEF-5750
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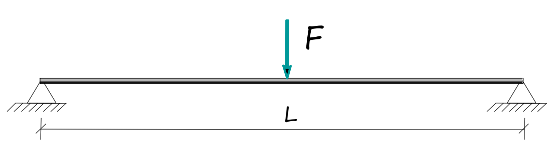
Some basic concepts
19/09/2017

Geometric nonlinearity

Consider a string of undeformed lenght ℓ_r *, stretched to a length* $\ell_0 = L > \ell_r$ *under an initial normal load* N_0 *and under a concentrated load at midspan:*

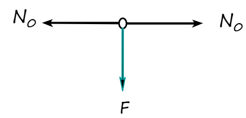


Consider also a linear elastic material: $N_0 = \dfrac{EA}{\ell_r}\left(\ell_0 - \ell_r\right) = k\left(\ell_0 - \ell_r\right)$

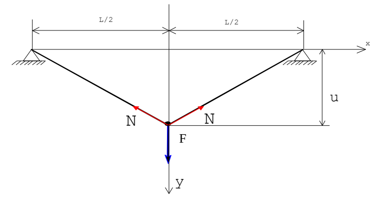
Where $k = \dfrac{EA}{\ell_r}$ *is a "spring constant"*

Geometric nonlinearity

•Equilibrium of node C is impossible at the undeformed configuration:



• Equilibrium is possible at a deformed configuration:



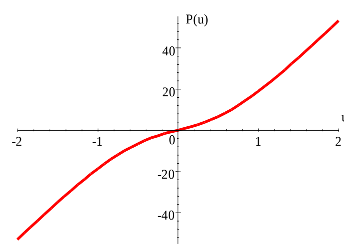
$$\ell(u) = \sqrt{\ell_0^2 + 4u^2}$$

$$N(u) = k(\ell - \ell_r)$$

• The resultant of the internal forces at node C is:

$$P(u) = 2N\sin\alpha = 2N\frac{u}{\left(\frac{\ell}{2}\right)} = \frac{4N}{\ell}u$$

$$P(u) = 4k\left(1 - \frac{\ell_r}{\sqrt{\ell_0^2 + 4u^2}}\right)u$$



For $u << \ell_0 \quad \therefore \quad \ell(u) \simeq \ell_0 \quad ; \quad N(u) \simeq N_0$

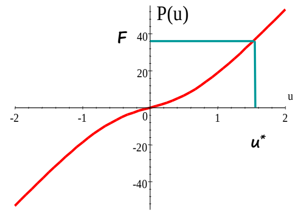
$$P(u) \simeq \frac{4N_0}{\ell_0}u = k_0 u$$

k_0 *is a initial stiffness, around the initial straight configuration*

A non-linear equilibrium problem:

We seek u^* cuch that

$$P\left(u^*\right) = F$$



Defining the unbalanced load function:

$$g\left(u\right) = P\left(u\right) - F$$

Equlibrium corresponds to $\quad g\left(u^*\right) = P\left(u^*\right) - F = 0$

Newton´s Method

Expanding g(u) in Taylor´s series, around a trial displacement u_i

$$g\left(u^*\right) = g\left(u_i\right) + \left.\frac{dg}{du}\right|_{u_i}\left(u^* - u_i\right) + \frac{1}{2}\left.\frac{d^2g}{du^2}\right|_{u_i}\left(u^* - u_i\right)^2 + \ldots = 0$$

Truncating this expansion at the linear term we obtain a non-zero umbalanced load, thus corresponding to a diaplacement

$$u' \neq u^*$$

We seek an approximation u_{i+1} such that

$$g\left(u_{i+1}\right) + \left.\frac{dg}{du}\right|_{u_i}\left(u_{i+1} - u_i\right) = 0$$

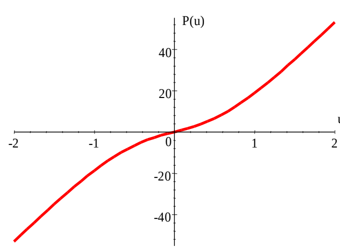
That is $$u_{i+1} = u_i - \left.\frac{dg}{du}\right|_{u_i} g(u_i)$$

Defining that Tangent Stiffness $$k_t^i = \left.\frac{dg}{du}\right|_{u_i}$$

$$\boxed{u_{i+1} = u_i - \left(k_t^i\right)^{-1} g(u_i)}$$

It can be shown that this recurrence, in a suficiently small vicinity of u* , converges to it with quadratic rates (provided u* is a stable point).

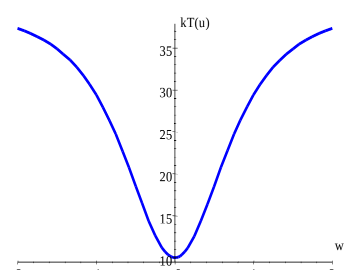
For the taut string:



$$k_t = 4k\left[\left(1-\frac{\ell_r}{\ell}\right)+4\frac{\ell_r}{\ell^3}u^2\right]$$

where

$$\ell = \ell(u) = \sqrt{\ell_0^2 + 4u^2}$$



Note that, as expected:

$$u \to 0 \therefore k_t \to 4k\left(1-\frac{\ell_r}{\ell_0}\right) = \frac{4N_0}{\ell_0} = k_0$$

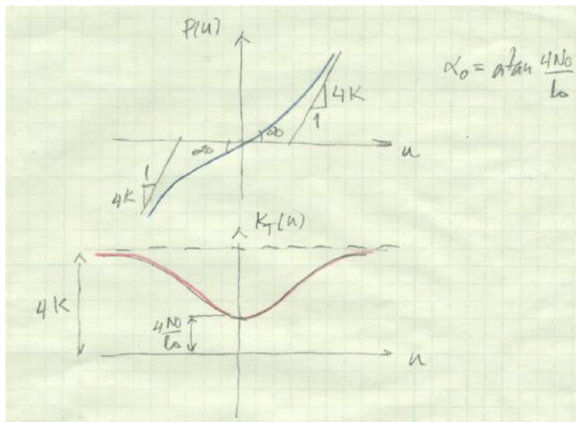
(initial, geometric stiffness)

$$u \to \infty \therefore\ k_t \to 4k$$

(equivalent to two rods of lenght $\ell_r/2$ acting in parallel!

Exercise 1 – Derive k_t for a string loaded at midspan

$$k_t = 4k\left[\left(1-\frac{\ell_r}{\sqrt{L^2+4u^2}}\right)+4\frac{\ell_r}{\sqrt{\left(L^2+4u^2\right)^3}}u^2\right]$$



Exercise 2 – Consider a string with k=10N/m, L=2m, l_r=1.5m.

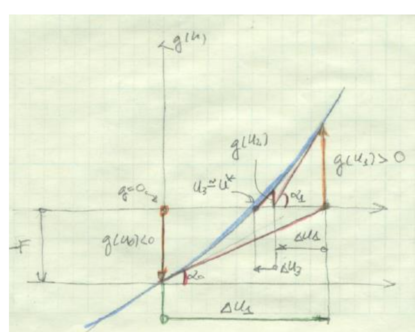
Plot P(u) and k_t(u).

Find u* for F=20N using Newton´s Method

NEWTON´S METHOD

$i = 0 \quad \therefore \quad \text{guess } u_0$

$$\left\|\begin{array}{l} u_{i+1} = u_i - \left(k_t^i\right)^{-1} g\left(u_i\right) \\ \text{if } \left\{ \left(\dfrac{\left|\Delta u_{i+1}\right|}{\left|\Delta u_i\right|} < \varepsilon_u \right) \wedge \left(\dfrac{\left|\Delta g_{i+1}\right|}{\left|\Delta g_i\right|} < \varepsilon_u \right) \right. \quad \text{stop} \\ \text{else} \quad i = i+1 \\ \text{continue} \end{array}\right.$$



The Dynamic Relaxation Method

DRM solves complicated nonlinear equilibrium problems,

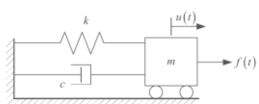
$$\mathbf{g}(\mathbf{u}^*) = \mathbf{p}(\mathbf{u}^*) - \mathbf{f} = 0$$

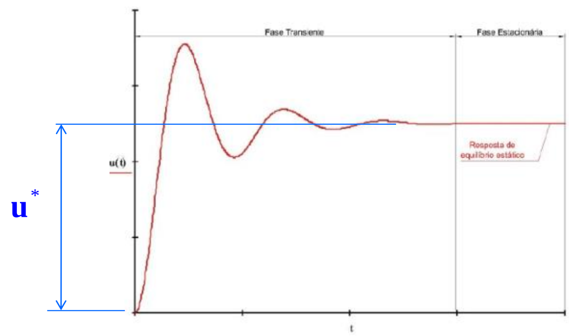
replacing the static problem by a pseudo-dynamic analysis, with fictitious masses and damping matrices

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{p}\big(\mathbf{u}(t)\big) = \mathbf{f}$$

For a single DOF, apply:

$$\mathbf{f}(t) = \mathbf{f}_0$$



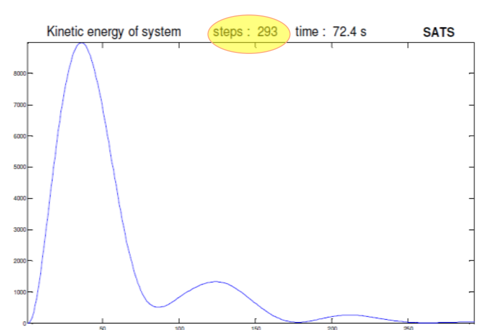


DRM with Viscous Damping

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{p}(\mathbf{u}) = \mathbf{0}$$

Apply $\mathbf{f}(\mathbf{u}) = \mathbf{f}_0$

Kinetic Energy

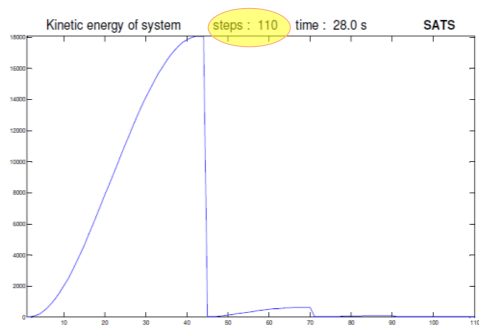


DRM with Kinetic Damping

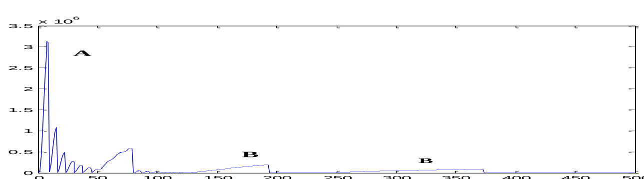
$$\mathbf{C} = \mathbf{0} \quad \Rightarrow \quad \mathbf{M}\ddot{\mathbf{u}} + \mathbf{p}(\mathbf{u}) = \mathbf{0}$$

If the systems kinetic energy is arbitrary zeroed whenever it reaches a maximum, the system will eventually come to a rest, usually faster than with viscous damping:

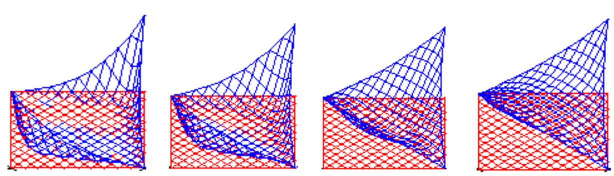
Kinetic Energy



DRM with Kinetic Damping



Transient of kinetic energy during the shape finding of a cable network via DR, with kinetic damping



Several instants of the DRM applied to a cable network

DRM shows no advantage to solve small to medium sized problems, whenever Newton's Method shows good, 2nd order convergence;

DRM is a robust technique, much useful in cases where Newton's Method fails to converge;

DRM may brings economy for solution of very large problems, since the computational costs for Newton's method grows with the square of the number of DOF, whilst the cost of DRM grows linearly;

However, when the discretization is refined, the critical time-step is also reduced, and more steps are required for the system to come to a rest.

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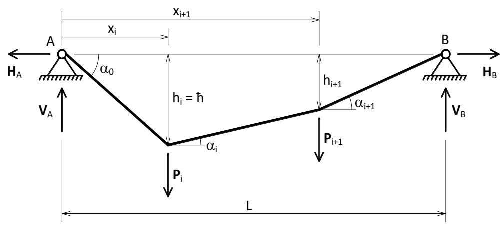
Static Equilibrium of cables

19/09/2017

Polygonal cable



$$\sum M_{(A)}=0 \quad \therefore \quad V_B=\left(\sum P_i\, x_i\right)/\,L$$

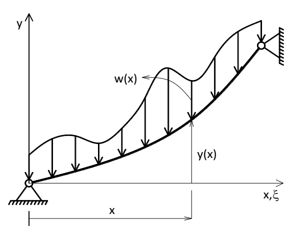
$$\sum M_{(B)}=0 \quad \therefore \quad V_A=\left(\sum P_i\,(L-x_i)\right)L$$

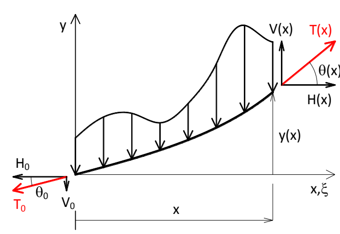
$$\left.\begin{array}{l} h_i = h^* \\ \sum M_{(i)}^{left}=0 \\ \sum F_x = 0 \end{array}\right\} \Rightarrow \quad H_A = H_B = \frac{V_A\, x_i}{h^*} = H$$ 'Thrust'

$$\sum M_{(i+1)}^{right}=0 \quad \Rightarrow \quad h_{i+1}=\frac{V_B}{H}\left(L-x_{i+1}\right)$$

$$\left(x_i, h_i\right) \quad \Rightarrow \quad \left(\ell_i, \alpha_i\right) \quad \Rightarrow \quad N_i = \frac{H}{\cos\alpha_i}$$

Plane cable under distributed vertical loads

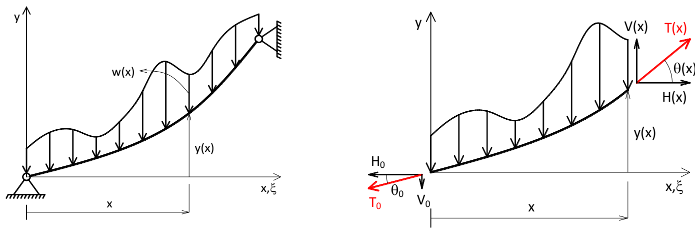




$$\sum F_x = 0 \quad \Rightarrow \quad -T_0\cos\theta_0 + T(x)\cos\theta(x) = 0 \quad \forall x$$

$$H = T(x)\cos\theta(x) = T_0\cos\theta_0$$ constant! ('Thrust')

Plane cable under distributed vertical loads

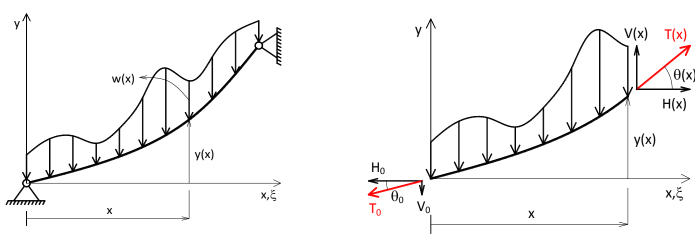


$$\sum F_V = \quad -V_0 - \int_0^x w(\xi)\,d\xi + V(x) = 0, \qquad \forall x$$

$$V(x) - V_0 = \int_0^x w(\xi)\,d\xi = 0, \qquad \forall x \qquad\qquad \therefore \quad V(x) \text{ is a primitive of } w(x)$$

$$\frac{dV}{dx} = \frac{d}{dx}\left(\int_0^x w(\xi)\,d\xi\right) = w(x)$$

Plane cable under distributed vertical loads

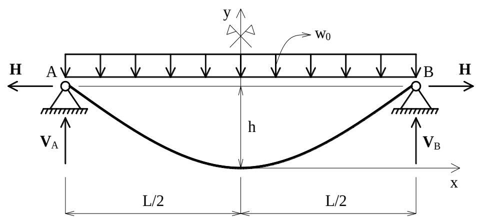


$$\sum M_{(0)} = \quad Vx - H\,y - \int_0^x \left(\xi w(\xi)\right) d\xi = 0, \qquad \forall x$$

Deriving with respect to x: $\quad \cancel{\frac{dV}{dx}}x + V - H\,\frac{d\,y}{dx} - \cancel{xw(x)} = 0$

$$\frac{dy}{dx} = \frac{V}{H} = \frac{1}{H}\int w dx \qquad \therefore \qquad \boxed{y = \frac{1}{H}\int\left(\int w dx\right)dx}$$

Cable under uniform vertical distributed load



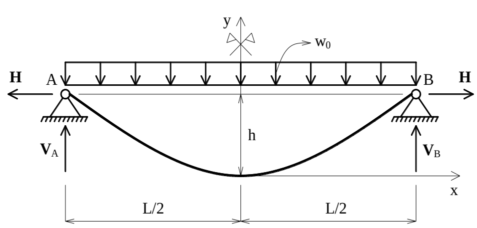
$$w(x) = w_0 \qquad \therefore \qquad y = \frac{1}{H}\int\left(\int w_0 dx\right)dx = \frac{w_0}{2H}x^2 + Cx + D$$

$$\left.\begin{array}{l} y(0) = 0 \\ \left.\dfrac{dy}{dx}\right|_{x=0} = 0 \end{array}\right\} \Rightarrow \quad C = D = 0 \quad \therefore \quad y = \frac{w_0}{2H}x^2$$, a parabola!

$$y\left(\frac{L}{2}\right) = h \quad \Rightarrow \quad \boxed{H = \frac{w_0 L^2}{8h}}$$ *"Thrust formula"*

$$\boxed{y = \left(\frac{4h}{L^2}\right)x^2}$$

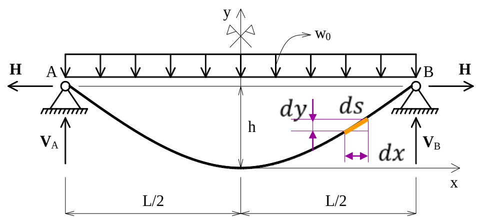
Cable under uniform vertical distributed load



$T = \dfrac{H}{\cos\theta}$ is maximum at the supports, where $\cos\theta$ is minimum!

$$T_{\max} = \sqrt{H^2 + V_A^2} = \frac{w_0 L}{2}\sqrt{1 + \left(\frac{L}{4h}\right)^2}$$

Cable under uniform vertical distributed load



The lenght of a differential cable element is: $ds = \sqrt{dx^2 + dy^2} = dx\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}$

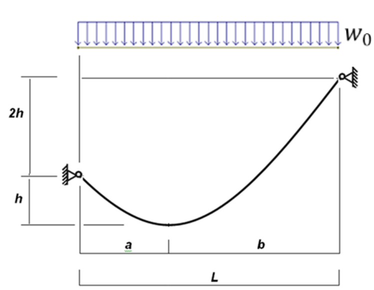
The total lenght of the cable is:

$$\ell = \int_{-L/2}^{L/2} ds = \int_{-L/2}^{L/2}\left(1 + \left(\frac{8hx}{L^2}\right)^2\right)^{\frac{1}{2}} dx = \frac{L}{2}\left(\left(1+\lambda^2\right)^{\frac{1}{2}} + \frac{L}{4h}\arcsin\lambda\right), \qquad \text{where: } \lambda = \frac{4h}{L}$$

For small sags , $\frac{h}{L} \le 0.1 \quad \Rightarrow \quad \boxed{\ell \simeq L + \frac{8h^2}{3L}}$ Lenght of the parabolic cable

Cable under uniform vertical distributed load

EX. 4 - Given $L = 30m,\ h = 2m,$ find $a,\ H,\ N_{\max},\ \ell$



Answers:

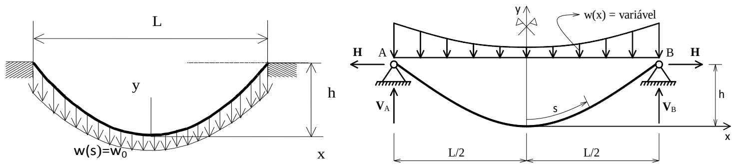
$a = 10.981m,$

$H = 30.15w_0,$

$N_{\max} = 35.64w_0$

$\ell \simeq 33.01m$

Catenary cable (cable under self-weight)



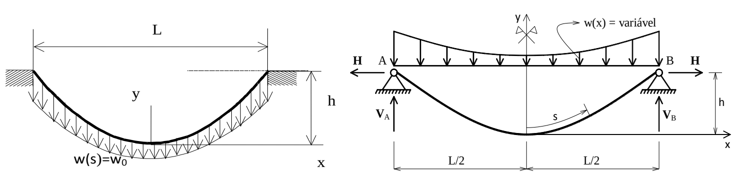
cable of homogeneous material and uniform cros-section , $w(s) = w_0$

Setting the origin ad the cable's midpoint: $s(0) = 0\,;\quad \left.\dfrac{dy}{dx}\right|_{s=0} = 0 \quad ; \quad V(0) = 0$

$$V(x) = \int_0^x w(\xi)d\xi = \int_0^{s(x)} w(s(x))\,ds = \int_0^s w_0 ds = w_0 s$$

$$\frac{dy}{dx} = \frac{V}{H} = \frac{w_0}{H} s$$

Catenary cable (cable under self-weight)



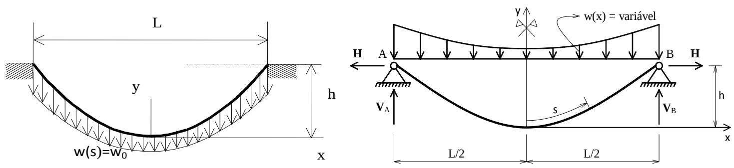
But $\quad ds = dx\left(1+\left(\dfrac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = dx\left(1+\left(\dfrac{w_0 s}{H}\right)^2\right)^{\frac{1}{2}}$

changing variables: $\quad u = \dfrac{w_0}{H} s \quad \Rightarrow \quad du = \dfrac{w_0}{H} ds \quad \Rightarrow \quad ds = \dfrac{H}{w_0} du$

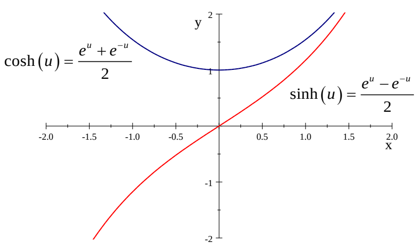
$$\frac{H}{w_0} du = dx\left(1+u^2\right)^{\frac{1}{2}}$$

separating variables: $\quad dx = \dfrac{H}{w_0}\dfrac{du}{\sqrt{1+u^2}}$

Catenary cable (cable under self-weight)



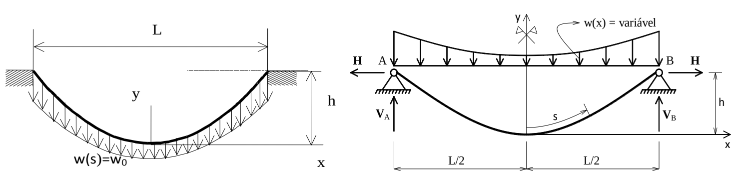
integrating at both sides: $x = \dfrac{H}{w_0}\displaystyle\int \dfrac{du}{\sqrt{1+u^2}} = \dfrac{H}{w_0}\operatorname{arcsinh}(u) + C$



$x = 0 \;\Rightarrow\; s = u = 0 \;\Rightarrow\; C = 0$

$$\boxed{x = \frac{H}{w_0}\operatorname{arcsinh}\left(\frac{w_0}{H}s\right)}$$

Catenary cable (cable under self-weight)



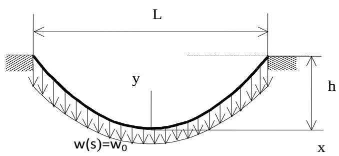
Inverting: $\boxed{s = \dfrac{H}{w_0}\sinh\left(\dfrac{w_0}{H}x\right)}$ lenght of the catenary cable stretch $[0, x]$

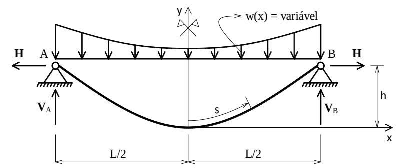
Therefore, $\dfrac{dy}{dx} = \dfrac{w_0}{H}s = \sinh\left(\dfrac{w_0}{H}x\right)$

Integrating, $y = \dfrac{H}{w_0}\cosh\left(\dfrac{w_0}{H}x\right) + D$ and imposing $y(0) = \dfrac{H}{w_0} + D = 0 \;\Rightarrow\; D = -\dfrac{H}{w_0}$

We arrive at the shape of the catenary cable: $\boxed{y = \dfrac{H}{w_0}\left(\cosh\left(\dfrac{w_0}{H}x\right) - 1\right)}$

Catenary cable (cable under self-weight)

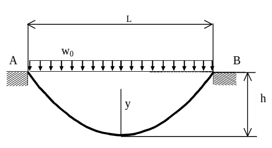




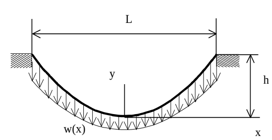
Note that H is still unknow! It can be determined imposing $y\left(\frac{L}{2}\right)=h$

Then numerically solving $\boxed{h=\frac{H}{w_0}\left(\cosh\left(\frac{w_0 L}{2H}\right)-1\right)}$

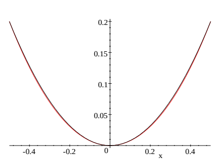
For h~L/10, the parabola and the catenary practically superimpose:



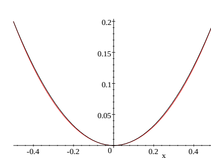
Parabolic cable



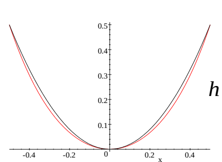
Catenary cable



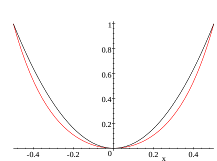
$h=\frac{L}{5}$



$h=\frac{L}{3}$



$h=\frac{L}{2}$



$h=L$

EX. 5 - Given $w_0 = 5kN/m; \quad h = 6m; \quad L = 20m,$

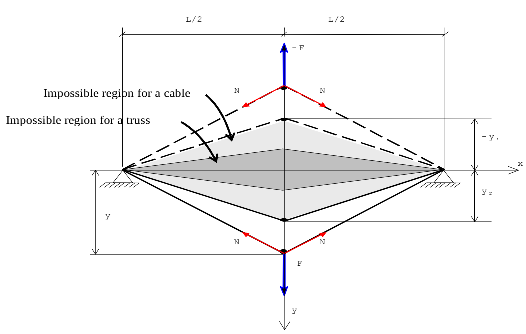
find H , $y(x)$ and ℓ

Answers: $H = 45.945kN;$

$y(x) = 9.189[\cosh(0.10883x) - 1]$

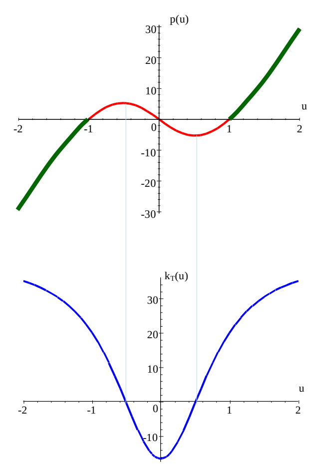
von Mises' Truss ("snap-through system")

$\ell_r > L$

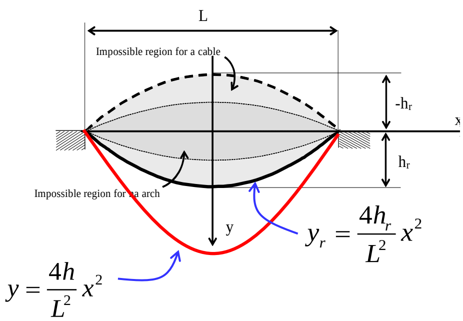


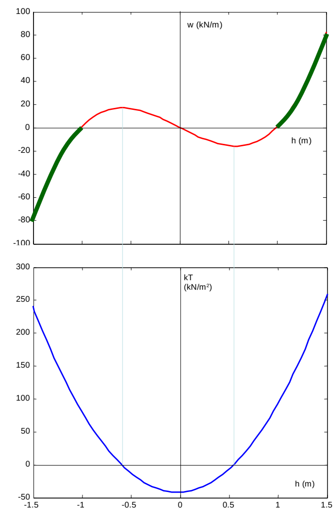
$$f(u) = 4k\left(1 - \frac{\ell_r}{\sqrt{L^2 + 4u^2}}\right)u$$

$$k_t = 4k\left[\left(1 - \frac{\ell_r}{\sqrt{L^2 + 4u^2}}\right) + 4\frac{\ell_r}{\sqrt{\left(L^2 + 4u^2\right)^3}}u^2\right]$$



Deformations of a parabolic cable





Deformations of a parabolic cable

$$y_r = \frac{4h_r}{L^2}x^2 \quad \Rightarrow \quad \ell_r = \int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{1+y_r'^2}\,dx$$

$$y = \frac{4h}{L^2}x^2 \quad \begin{cases} \Rightarrow \quad \ell(h) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{1+y'^2}\,dx \\ \Rightarrow \quad H = \frac{wL^2}{8h} = N(x)\cos\theta(x) = \frac{N(\theta(x))}{\sqrt{1+y'^2}} \end{cases}$$

$$\Delta\ell = \ell - \ell_r$$

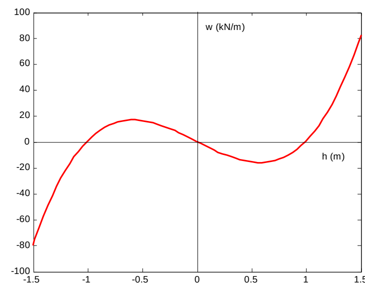
$$\Delta\ell = \int_{-\frac{L}{2}}^{\frac{L}{2}} (1+\varepsilon)\,ds_r = \int_{-\frac{L}{2}}^{\frac{L}{2}} \varepsilon\sqrt{1+y_r'^2}\,dx$$

$$\varepsilon = \frac{N(x)}{EA} = \frac{H}{EA}\sqrt{1+y'^2}$$

$$\Delta\ell = \frac{H}{EA}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{(1+y_r'^2)(1+y'^2)}\,dx$$

Deformations of a parabolic cable

$$w(h) = \frac{8hEA}{L^3} \frac{\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{1+\left(\frac{8h}{L^2}x\right)^2}dx - \ell_r}{\int_{-\frac{L}{2}}^{\frac{L}{2}} \sqrt{\left(1+\left(\frac{8h_r}{L^2}x\right)^2\right)\left(1+\left(\frac{8h}{L^2}x\right)^2\right)}dx}$$



Small deformations for a parabolic cable with small sags:

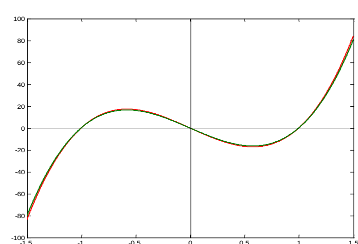
Consider a cable with $h = h_r << L$

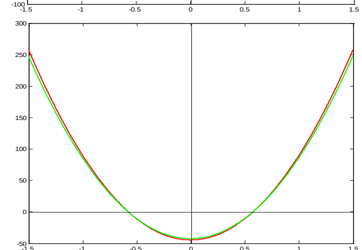
Find the initial stifness of the cable for uniform transversal load.

$$\left.\begin{array}{l} \ell_r \simeq L + \dfrac{8h_r^{\,2}}{3L} \\ \ell \simeq L + \dfrac{8h^2}{3L} \\ \sqrt{1+y_r'^2} \simeq \sqrt{1+y'^2} \simeq 1 \end{array}\right\} \quad \Delta\ell = \frac{HL}{EA}$$

$$\boxed{w \simeq \frac{64}{3}\frac{EA}{L^4}\left(h^3 - h_r^{\,2}h\right)}$$

$$\boxed{k_t(h) \simeq \frac{64}{3}\frac{EA}{L^4}\left(3h^2 - h_r^{\,2}\right)}$$





Ex. 6 – Small deformations for a parabolic cable with small sags:

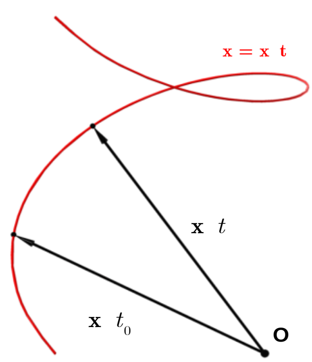
Stiffness around the initial shape $(h = h_r)$:

$$k_0 \simeq \frac{128}{3}\frac{EA}{L^4}h_r^{\;2} \qquad (\text{em N/m}^2)$$

$$\boxed{u \simeq \frac{w}{k_0}}$$

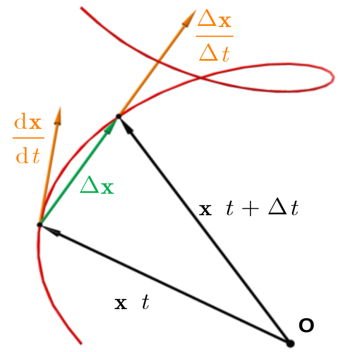
the stiffness drops with the square of the span!

Cables in 3D space



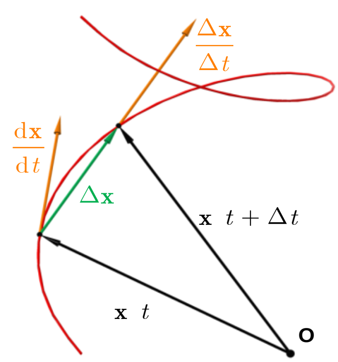
$$\mathbf{x} = \mathbf{x}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

Cables in 3D space



$$\frac{d\mathbf{x}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{x}(t+\Delta t) - \mathbf{x}(t)}{\Delta t} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

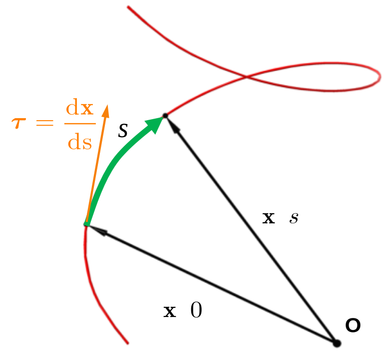
Cables in 3D space



$$\left|\frac{d\mathbf{x}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \neq 1$$

Cables in 3D space

$$t = s \quad \Rightarrow \quad \mathbf{x} = \mathbf{x}(s)$$



$$\|\boldsymbol{\tau}\| = \sqrt{\boldsymbol{\tau}\cdot\boldsymbol{\tau}} = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} = \sqrt{\frac{dx^2 + dy^2 + dz^2}{ds^2}} = \sqrt{\frac{ds^2}{ds^2}} = 1$$

$$\|\boldsymbol{\tau}\| = 1 \quad \forall s$$

Cables in 3D space

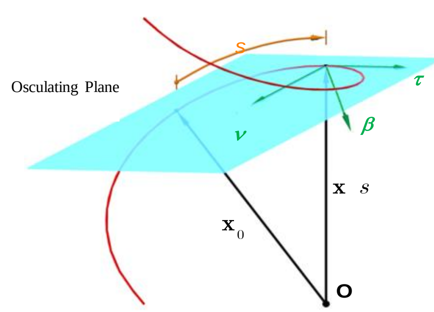
$$\boldsymbol{\tau}\cdot\boldsymbol{\tau} = 1 \quad \forall s$$

$$\frac{d}{ds}(\boldsymbol{\tau}\cdot\boldsymbol{\tau}) = 2\frac{d\boldsymbol{\tau}}{ds}\cdot\boldsymbol{\tau} = 0 \quad \Rightarrow \quad \frac{d\boldsymbol{\tau}}{ds} \perp \boldsymbol{\tau}, \quad \forall s$$

$$\boldsymbol{\nu} = \frac{\frac{d\boldsymbol{\tau}}{ds}}{\left\|\frac{d\boldsymbol{\tau}}{ds}\right\|} = \frac{\frac{d^2\mathbf{x}}{ds^2}}{\left\|\frac{d^2\mathbf{x}}{ds^2}\right\|}$$

$$\boldsymbol{\beta} = \boldsymbol{\tau}\times\boldsymbol{\nu}$$

Frenet-Serret Triedron $\{\boldsymbol{\tau}, \boldsymbol{\nu}, \boldsymbol{\beta}\}$



Cables in 3D space

Curvature: $\kappa = \left\| \frac{d\boldsymbol{\tau}}{ds} \right\| \Rightarrow \frac{d\boldsymbol{\tau}}{ds} = \kappa\,\mathbf{v}$

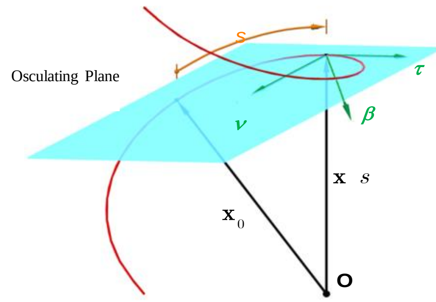
Curvature Radius: $\rho = \frac{1}{\kappa} \Rightarrow \frac{d\boldsymbol{\tau}}{ds} = \frac{\mathbf{v}}{\rho}$

In can be shown that:

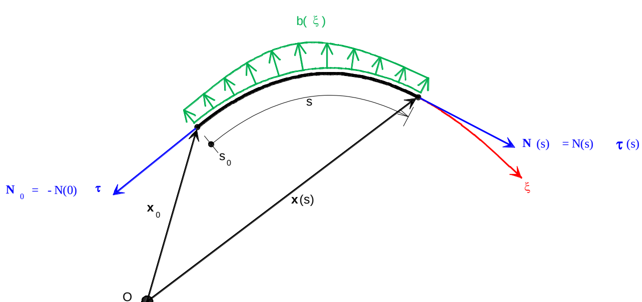
$$\kappa = \frac{\left\| \mathbf{x}' \times \mathbf{x}'' \right\|}{\left\| \mathbf{x}' \right\|^3} \quad \text{where } \mathbf{x}' = \frac{d\mathbf{x}}{d\theta}$$

For plane curves: $y = y(x)$

$$\kappa = \frac{|y''|}{\left(1 + y'^2\right)^{\frac{3}{2}}}$$



Cable Equilibrium in Vectorial Description

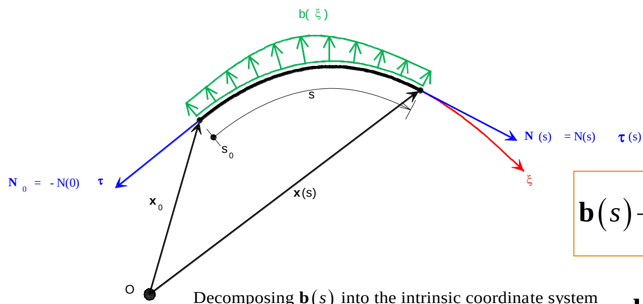


Equilibrium: $\mathbf{N}_0 + \int_0^s \mathbf{b}(\xi)\,d\xi + \mathbf{N}(s) = \mathbf{0},\ \forall s$

Deriving with respect to s: $\mathbf{b}(s) + \frac{d}{ds}\left(N(s)\,\boldsymbol{\tau}(s)\right) = \mathbf{0}$

$$\mathbf{b}(s) + \frac{dN}{ds}\boldsymbol{\tau} + N\frac{d\boldsymbol{\tau}}{ds} = \mathbf{0}$$

Cable Equilibrium in vector Description



$$\mathbf{b}(s)+\frac{dN}{ds}\boldsymbol{\tau}+\frac{N}{\rho}\boldsymbol{\nu}=\mathbf{0}$$

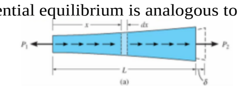
Decomposing $\mathbf{b}(s)$ into the intrinsic coordinate system given the Frenet-Serret triedron $\{\boldsymbol{\tau},\boldsymbol{\nu},\boldsymbol{\beta}\}$:

$$\mathbf{b}=b_\tau\boldsymbol{\tau}-p\boldsymbol{\nu}+b_\beta\boldsymbol{\beta}$$

We arrive at a system of three scalar equilibrium equations:

$$\begin{cases}\dfrac{dN}{ds}+b_\tau=0\\[2mm]\dfrac{N}{\rho}=p\\[2mm]b_\beta=0\end{cases}$$

⊗ Tangential equilibrium is analogous to a axially loaded bar!

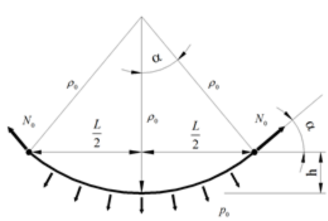


⊗ Transversal loading provoques curvature of the cable!

⊗ The cable adjusts its form in such a way that there is no binormal loading!

Velaria

cable under uniform transversal pressure p_0



$$\mathbf{b}=-p_0\boldsymbol{\nu}$$

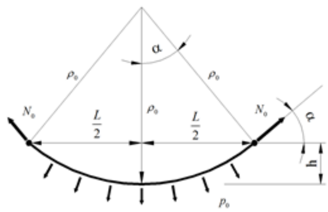
$$\frac{dN}{ds}=0 \quad\Rightarrow\quad N=N_0 \quad \text{constant!}$$

$$\frac{N_0}{\rho}=p_0 \quad\Rightarrow\quad \rho=\frac{N_0}{p_0}=\rho_0 \quad \text{constant!}$$

The velaria is a circular arch!

Velaria

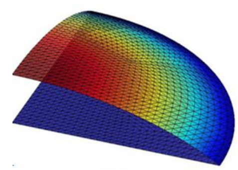
Deformation of infinitely long panels:

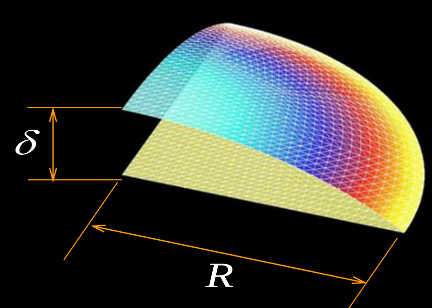


$$p = \frac{64Et}{3L}\left(\frac{\delta}{L}\right)^3$$

Deformation of flat circular membranes of radius R=L/2:

$$p = \frac{3Et}{3(1-\nu)R}\left(\frac{\delta}{R}\right)^3$$





$E = 0.5GPa$

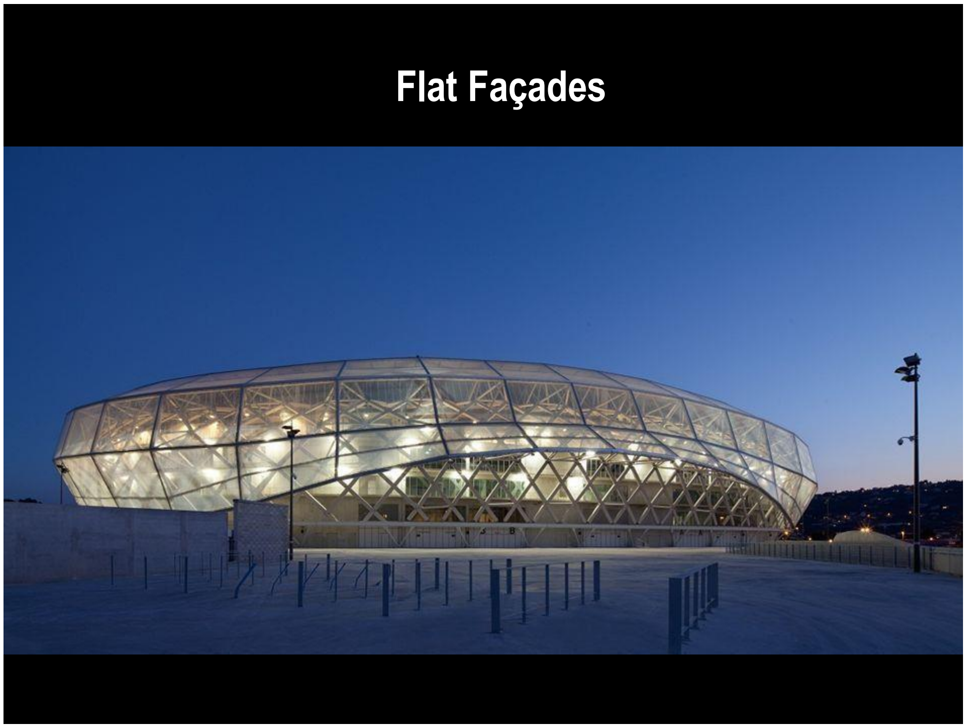
$\nu = 0.2$

$t = 1mm$

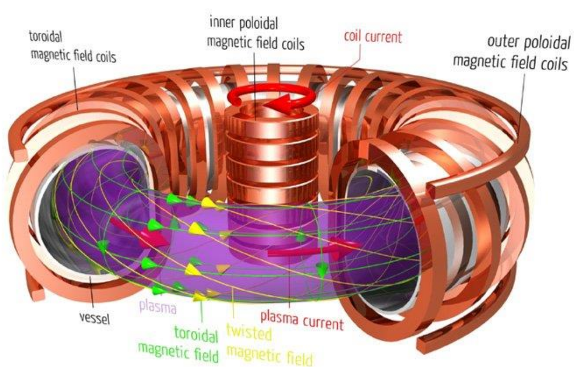
$p = 1\,kN/m^2$

Delta [cm]	S_0 [kN/m]	S_{tot} [kN/m]	w [kN/m]
R=5m			
72.112	0.0	8.7	8.3
70	0.7	8.9	8.5
60	4.4	10.4	10.1
50	8.3	12.5	12.2
40	12.9	15.6	15.4
30	19.3	20.8	20.6
20	30.6	31.2	31.1
10	62.3	62.5	62.0

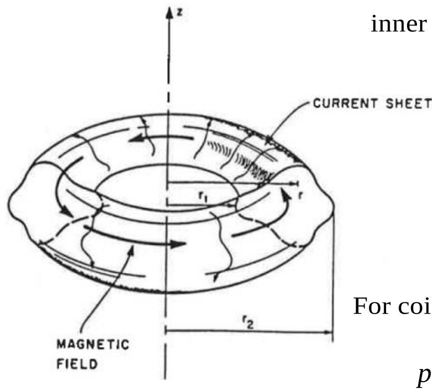
Delta [cm]	S_0 [kN/m]	S_{tot} [kN/m]	w [kN/m]
R=2.5m			
28.618	0.0	8.7	8.3
20	5.1	7.8	7.7
10	15.0	15.6	15.5
5	31.1	31.2	31.2



Princeton-D coils in tokamak machines



Princeton-D coils in tokamak machines



inner toroidal magnetic field:

$$\vec{B}(r) = \frac{\mu_0}{2\pi}\frac{nI}{r}\vec{i}_\phi$$

"inner pressure"

$$p = \frac{1}{2} IB = \frac{\mu_0}{4\pi}\frac{nI^2}{r}$$

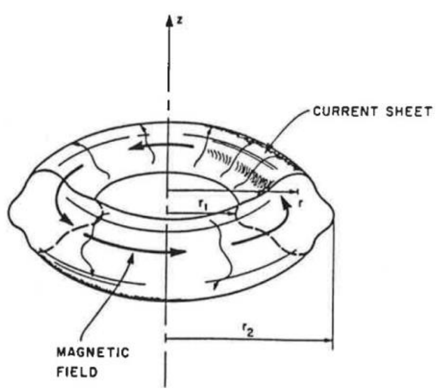
For coils with no bending stiffness:

$$p = \frac{N}{\rho} = \frac{\mu_0}{4\pi}\frac{nI^2}{r}$$

We may adjust curvatures to keep traction $N = N_0$ constant, setting $\rho = \beta r$

$$\beta = \frac{4\pi N_0}{\mu_0 n I^2}$$

Princeton-D coils in tokamak machines



$$\rho = \beta r = \pm \frac{\left(1+\left(\frac{dz}{dr}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2 z}{dr^2}}$$

$$\boxed{r\frac{d^2 z}{dr^2} = \pm\frac{1}{\beta}\left(1+\left(\frac{dz}{dr}\right)^2\right)^{\frac{3}{2}}}$$

The solution in terms of a parameter $\theta = \text{atan}\,\frac{dz}{dr}$ is: $\boxed{\begin{cases} r = r_0 e^{\beta\sin\theta} \\ z = \beta r_0 \beta \int_0^\theta \sin\xi\, e^{\beta\sin\xi} d\xi \end{cases}}$

whose numerical integration is straightforward

Princeton-D coils in tokamak machines