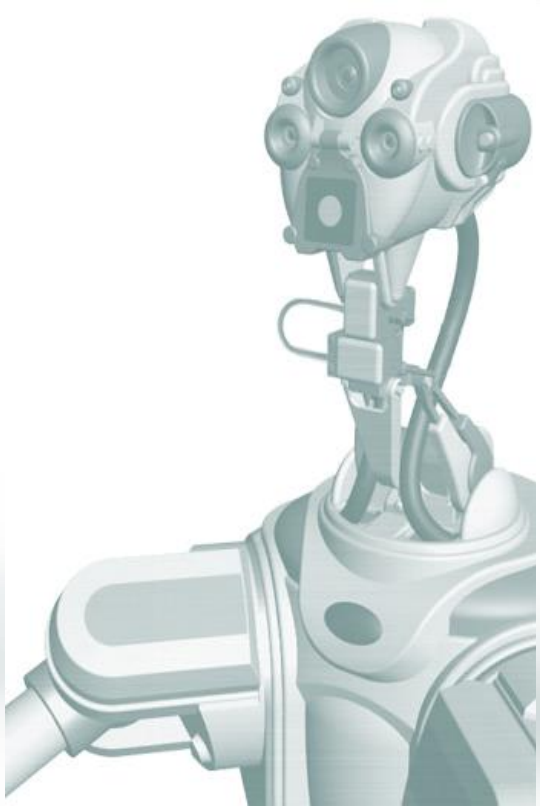
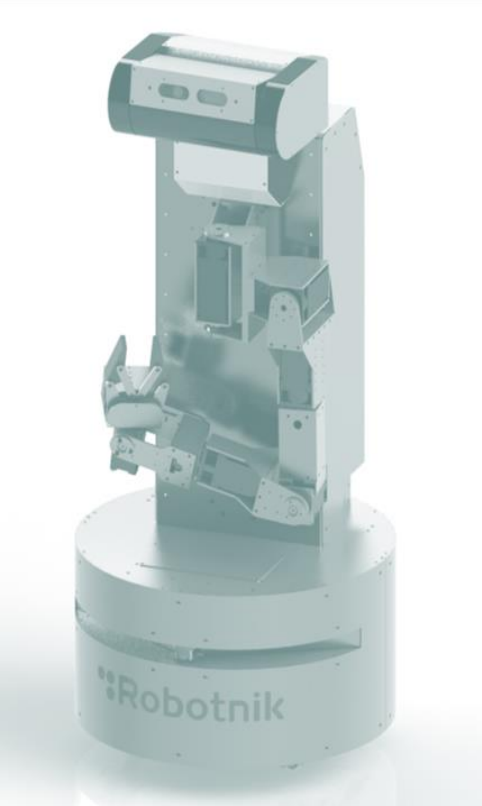
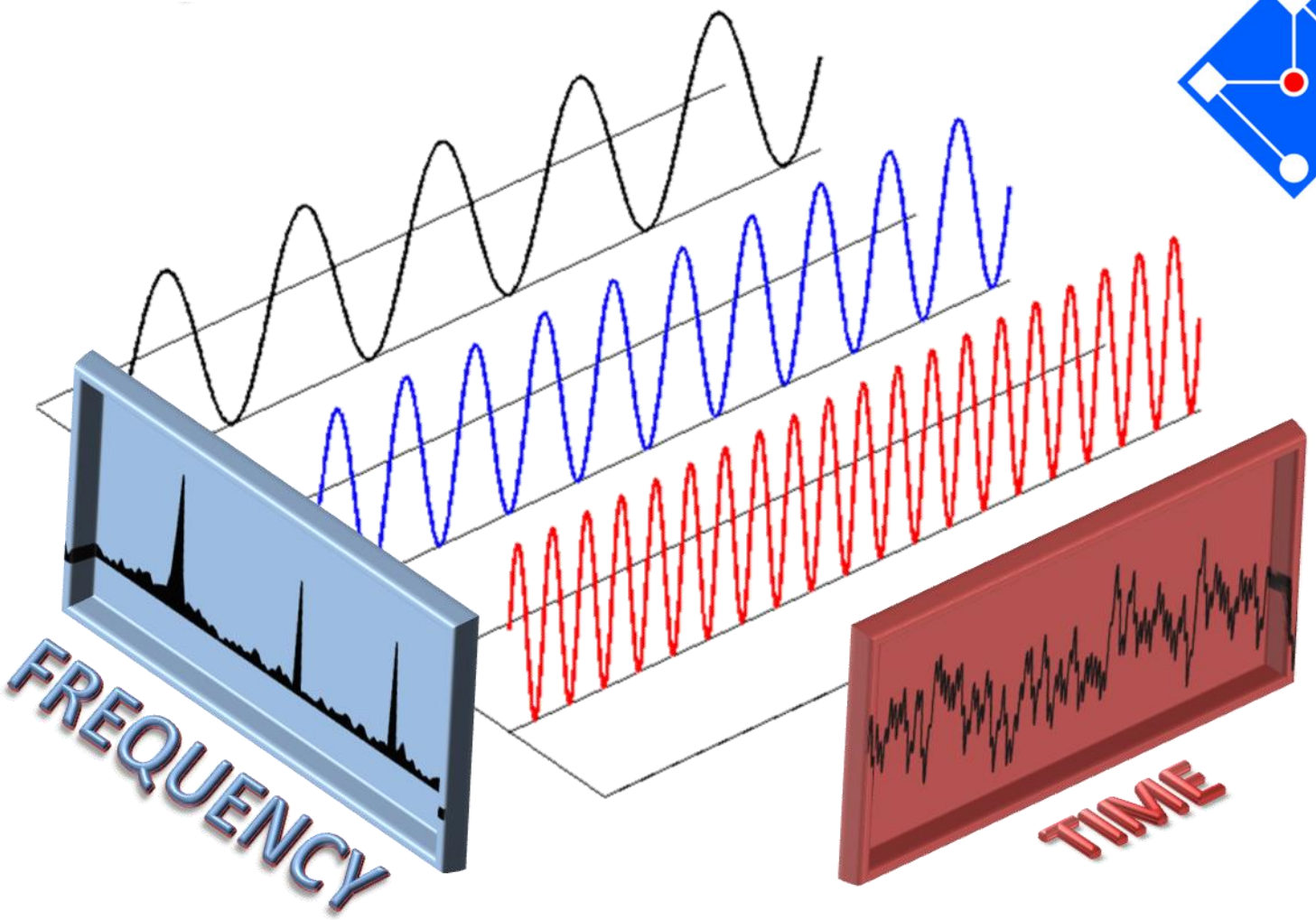
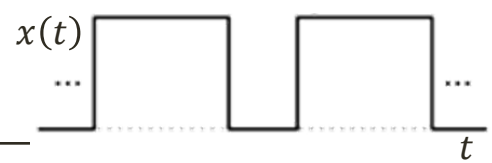
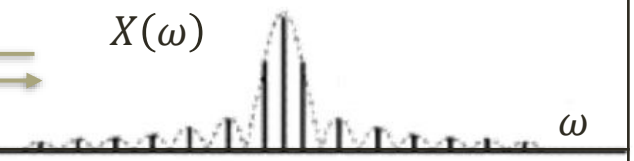
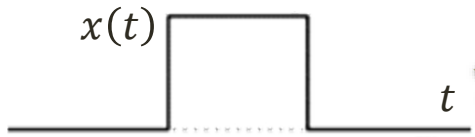
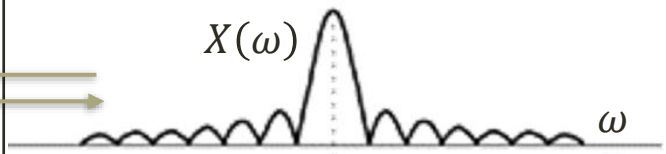
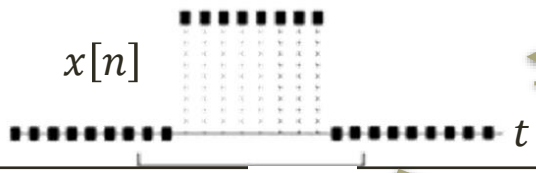
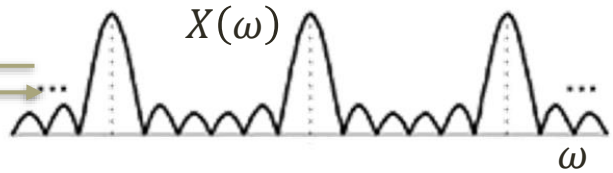
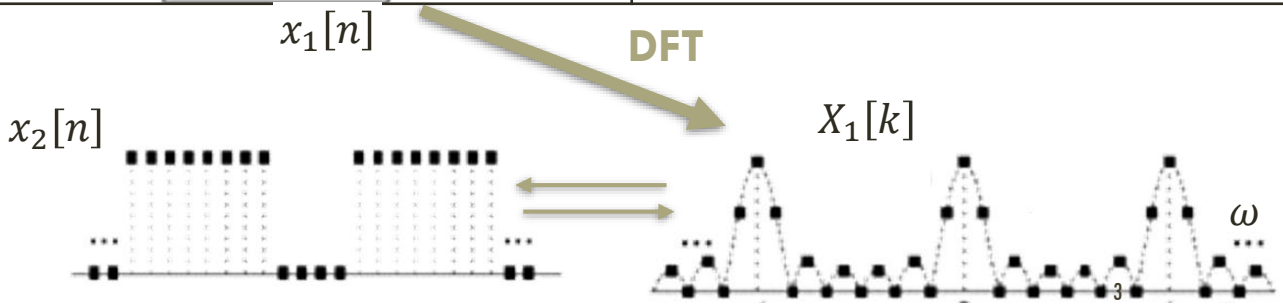




# ANÁLISE DE SINAIS

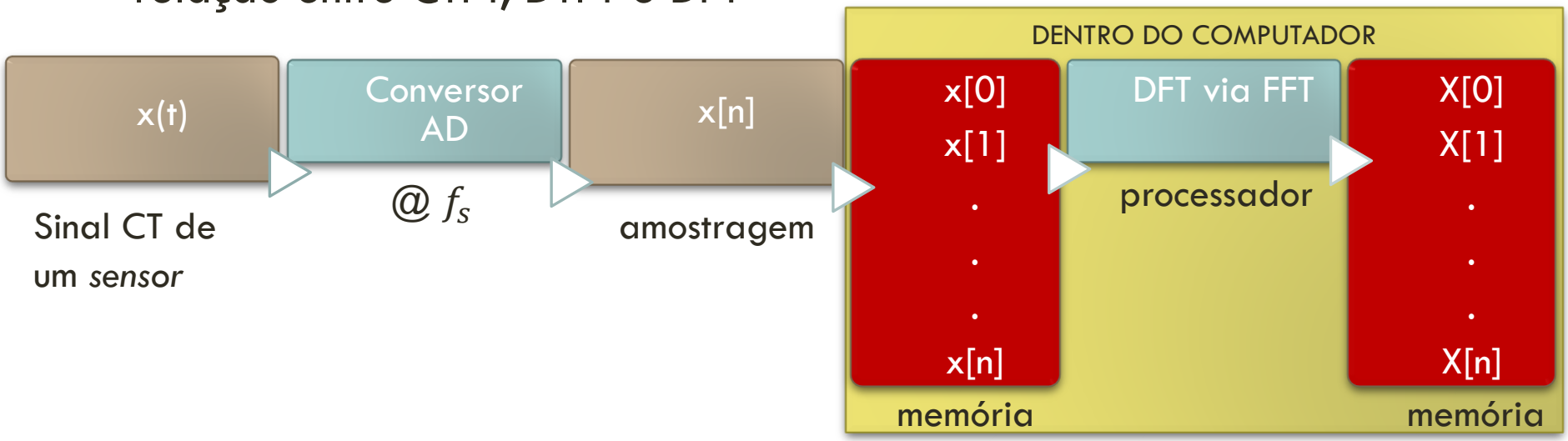
Larissa Driemeier
Marcilio Alves
Rafael T. Moura
Tarcísio H. Coelho


FREQUENCY
TIME

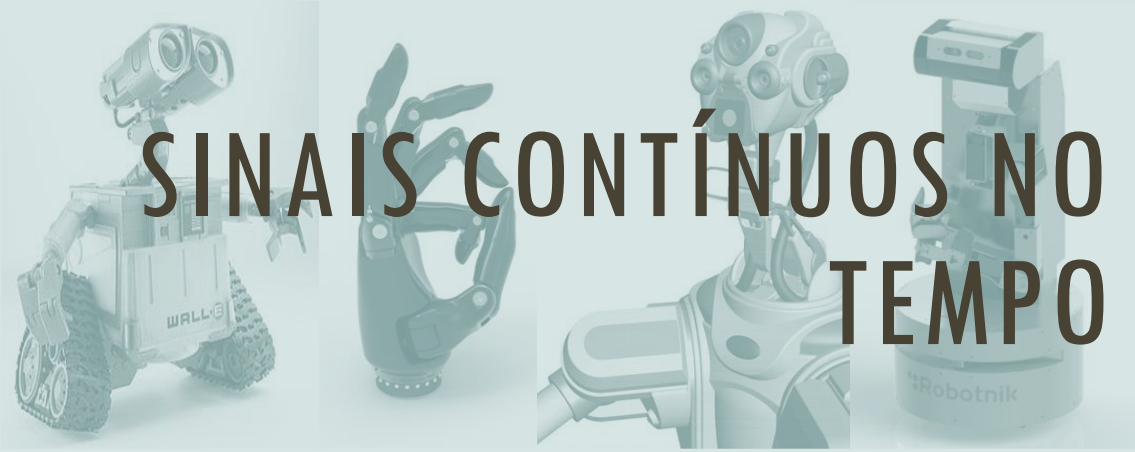
| Tipo de Transformada | Sinal no tempo | Transformada |
|---|---|---|
| **Série de Fourier (FS)** | Sinal contínuo e periódico | Discreta não periódica |
| |   |   |
| **Transformada de Fourier (FT ou CTFT)** | Sinal contínuo não periódico | Contínua não periódica |
| |   |   |
| **Transformada de Fourier em tempo discrete (DTFT)** | Sinal discreto, não periódico, de comprimento infinito | Contínua periódica |
| |   |   |
| **Transformada discreta de Fourier (DFT)** |   | |
| **Fast Fourier Transform (FFT)** | | |

# CTFT, DTFT E DFT

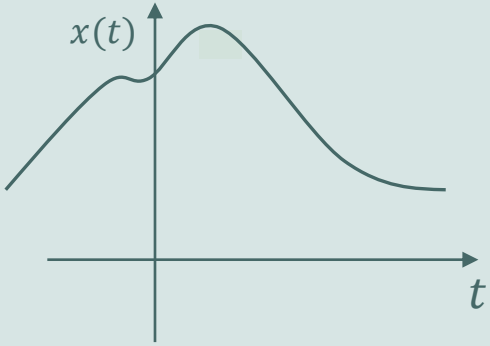
DFT é utilizada para implementar numericamente a FT. Porém, precisamos entender o que os valores da DFT nos dizem sobre os valores da CTFT de $x(t)$. Portanto, precisamos entender a relação entre CTFT, DTFT e DFT



A maioria dos sinais físicos são contínuos, p.ex., posição e velocidade de um corpo, fala ou música captada por um microfone, tensão ou corrente num circuito elétrico...




# SINAIS CONTÍNUOS NO TEMPO



**Um *sinal aperiódico pode ser visto como um sinal periódico com um período infinito.***

$$T \to \infty \text{ e } \frac{1}{T} \longrightarrow \frac{d\omega}{2\pi}$$

# FS

# FT

**Síntese**

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

**Síntese**

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

**Análise**

$$X[k] = \frac{1}{T} \int_{T} x(t) e^{-jk\omega_0 t} dt$$

**Análise**

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Harmônicos $X[k]$ distanciados
$\Delta\omega = \omega_0 = 2\pi/T$

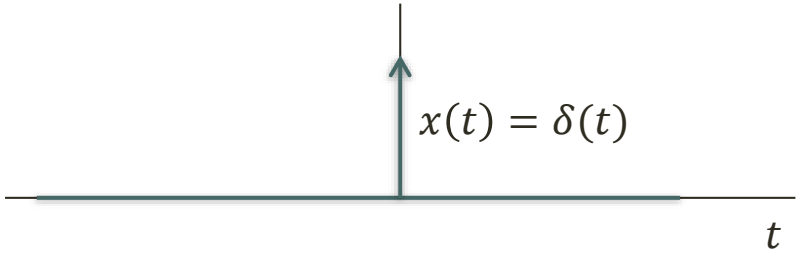
Valores contínuos $X(\omega)$

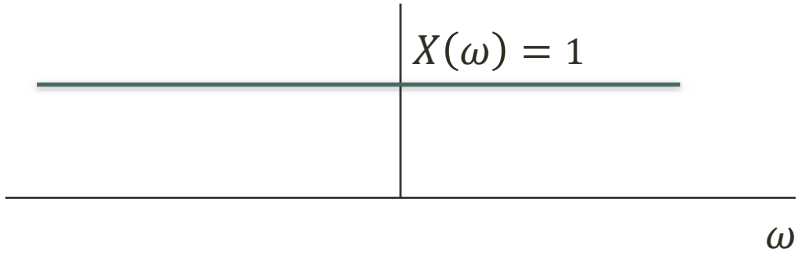
# FT PARA IMPULSO UNITÁRIO $\delta(t)$

Se,

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt = 1$$

**Impulso unitário contém componente em todas as frequências.**



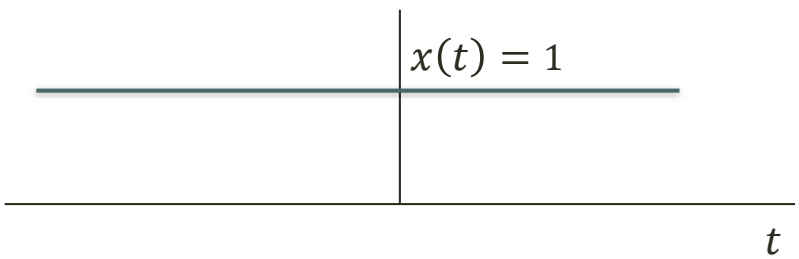
# INVERSA DA FT PARA $\delta(\omega)$

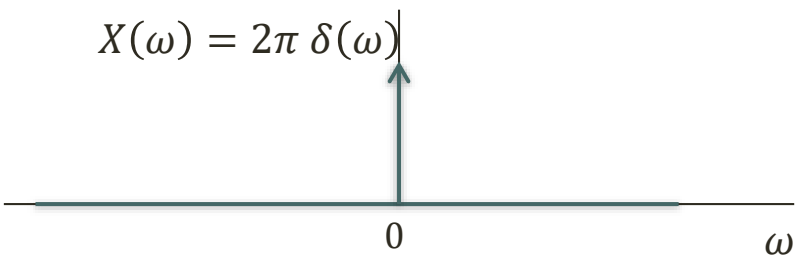
Se,

$$X(\omega) = 2\pi\,\delta(\omega)$$

Então, aplicando-se a equação de síntese da FT,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega t}d\omega = 1$$

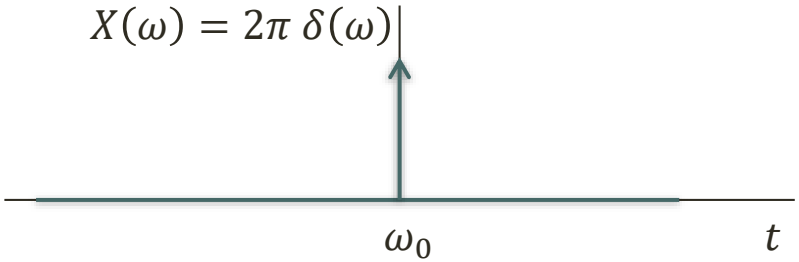


# INVERSA DA FT PARA $\delta(\omega)$

Se,

$$X(\omega) = 2\pi\,\delta(\omega - \omega_0)$$




Então, aplicando-se a equação de síntese da FT,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

Similarmente:

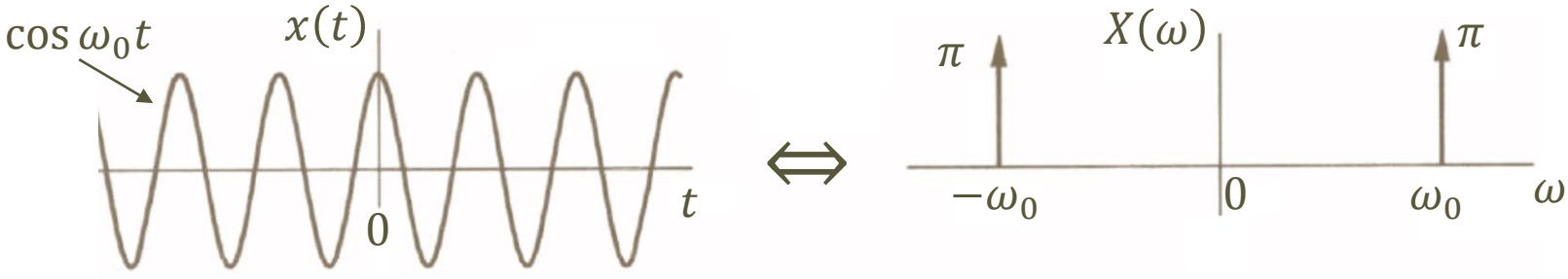
$$X(\omega) = 2\pi\,\delta(\omega + \omega_0) \Leftrightarrow x(t) = e^{-j\omega_0 t}$$

# TRANSFORMADA DE FOURIER DA FUNÇÃO COSSENO $\cos \omega_0 t$

$$x(t) = \cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$X(\omega) = \pi\,[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

Espectro de sinal cosseno tem dois impulsos em frequências positiva e negativa.



# TRANSLAÇÃO NO TEMPO

Transladar um sinal no domínio do tempo faz com que a transformada de Fourier seja multiplicada por uma exponencial complexa.

$$x(t - t_0) \underset{CTFT}{\longrightarrow} e^{-j\omega t_0} X(\omega)$$

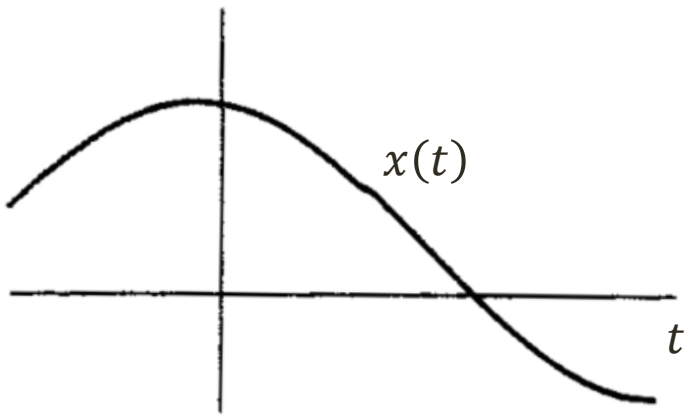
Este resultado mostra que retardar um sinal por $t_0$ segundos não altera o espectro de amplitude. O espectro de fase, no entanto, é alterado por $-\omega t_0$.

# TRANSLAÇÃO NA FREQUÊNCIA

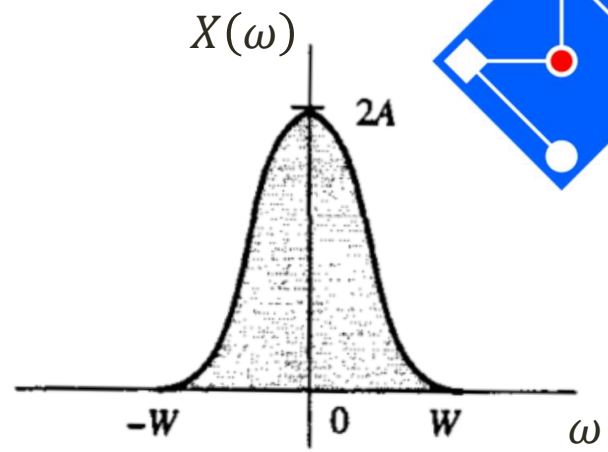
Devido à propriedade da *dualidade*: Para qualquer relação entre $x(t)$ e $X(\omega)$, existe uma relação dual, obtida trocando $x(t)$ e $X(\omega)$ (com pequenas modificações),

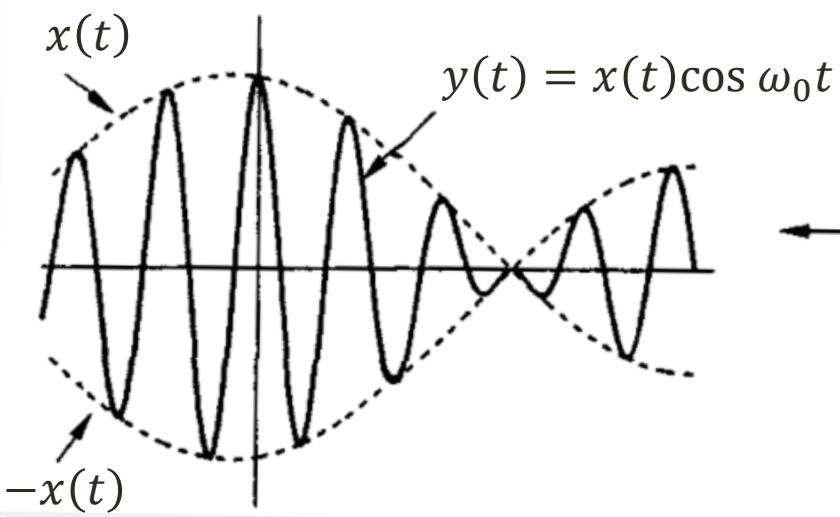
$$x(t) \xrightarrow[CTFT]{} X(\omega) \Rightarrow X(t) \xrightarrow[CTFT]{} 2\pi\, x(-\omega)$$

$$e^{j\omega_0 t} x(t) \xrightarrow[CTFT]{} 2\pi\, X(\omega - \omega_0)$$

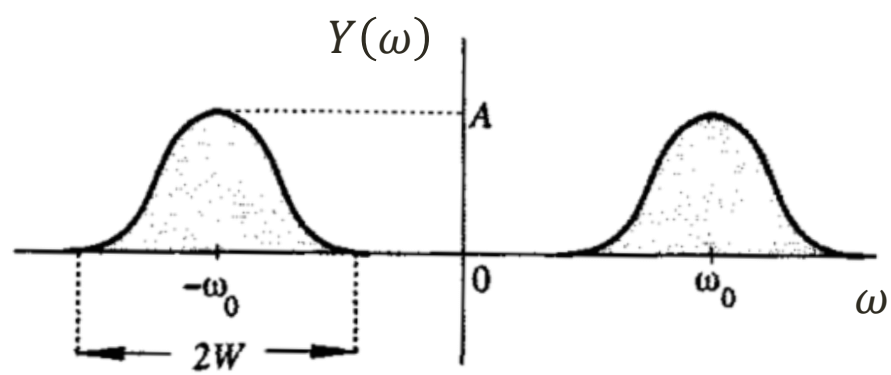


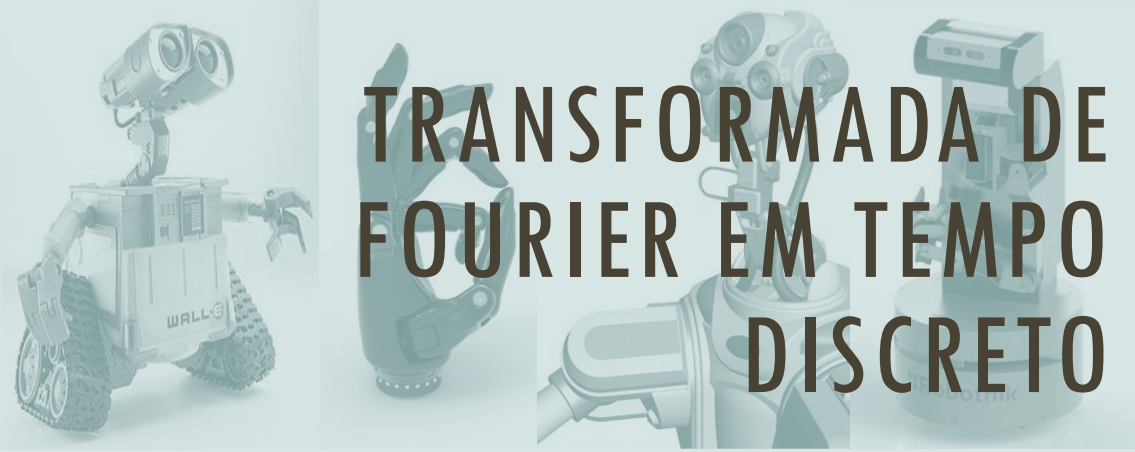

$\longleftrightarrow$



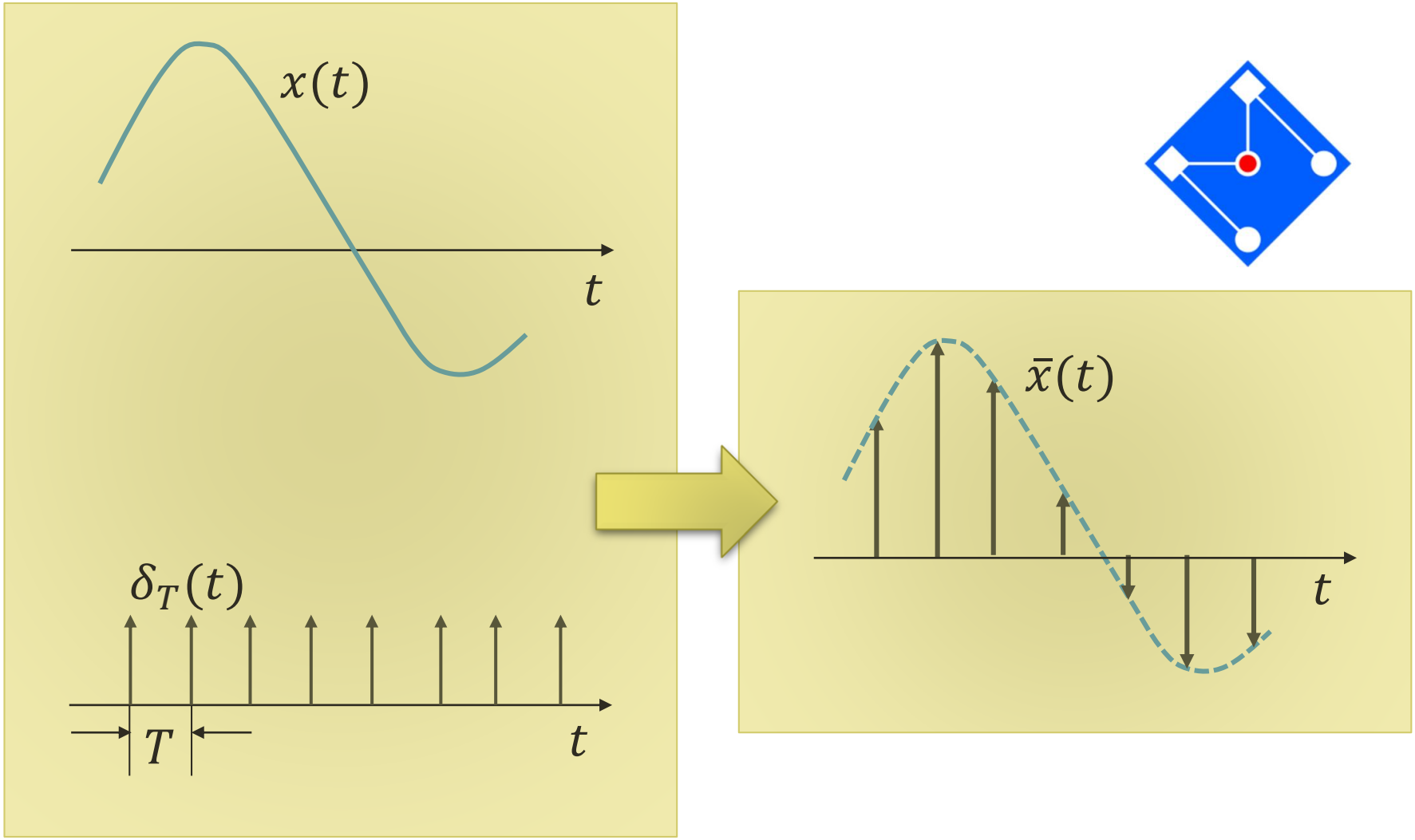




$\longleftrightarrow$



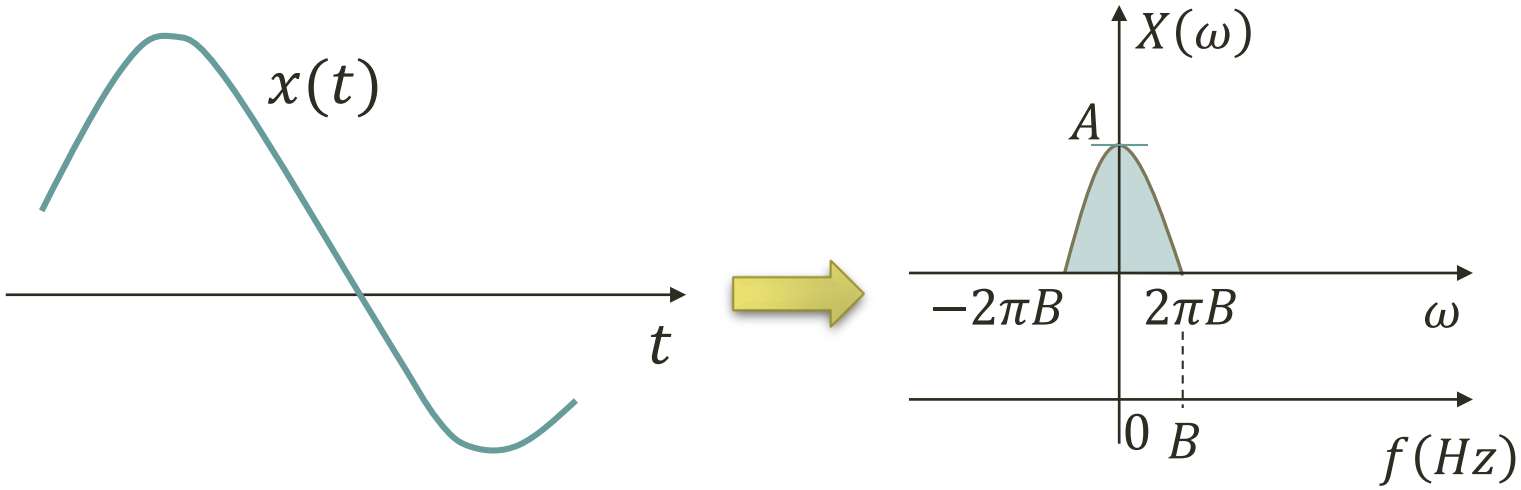


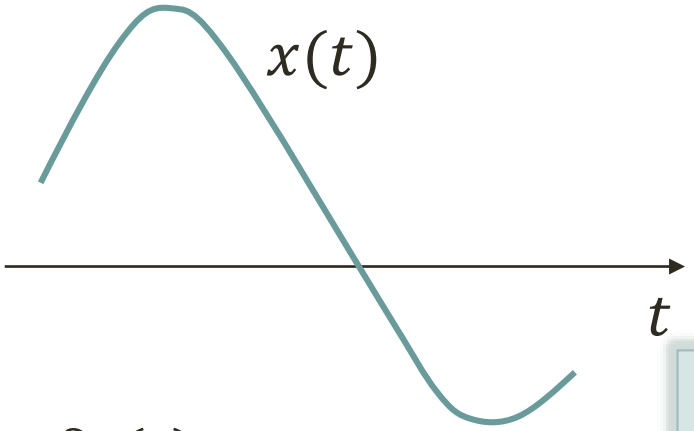

# TRANSFORMADA DE FOURIER EM TEMPO DISCRETO

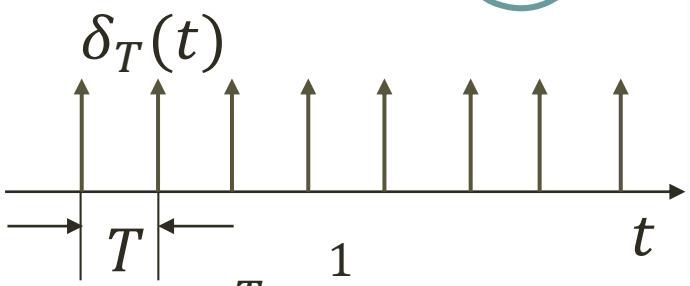



$$\bar{x}(t) = x(nT) = x(t)\delta_T(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

# DOMÍNIO DA FREQUÊNCIA

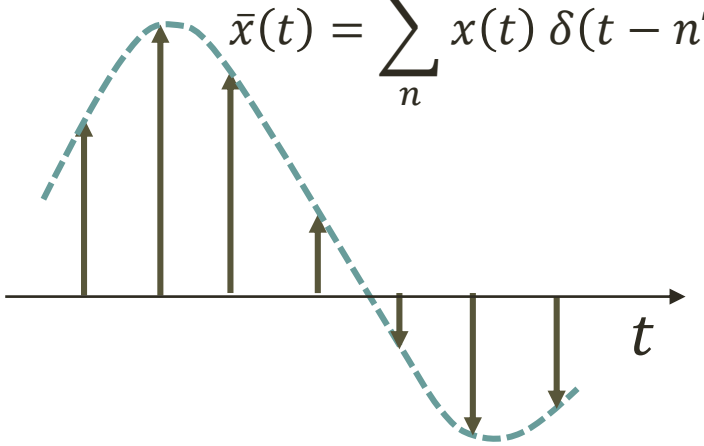





$$T = \frac{1}{f_s}$$

$$\bar{x}(t) = \sum_n x(t)\,\delta(t - nT)$$




$\delta_T(t)$ é um sinal periódico, e, portanto, pode ser escrito em função da série de Fourier,

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} X[n] e^{-jn\omega_0 t}$$

$$X[n] = \frac{1}{T}\int_T \delta_T(t) e^{-jn\omega_0 t}\, dt = \frac{1}{T}$$

$$\therefore\ \delta_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-jn\omega_0 t}$$

$$\delta_T(t) = \frac{1}{T}[1 + 2(\cos\omega_0 t + \cos 2\omega_0 t + \cdots)]$$

$$\bar{x}(t) = \frac{1}{T}[x(t) + 2x(t)\cos\omega_0 t + +2x(t)\cos 2\omega_0 t + \cdots]$$

$$\bar{x}(t)=\frac{1}{T}\left[x(t)+2x(t)\cos\omega_0 t++2x(t)\cos 2\omega_0 t+\cdots\right]$$

$$\bar{X}(\omega)=\frac{1}{T}\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt+\frac{1}{T}\int_{-\infty}^{\infty}2x(t)\cos\omega_0 t\,e^{-j\omega t}dt+\frac{1}{T}\int_{-\infty}^{\infty}2x(t)\cos 2\omega_0 t\,e^{-j\omega t}dt+\cdots$$

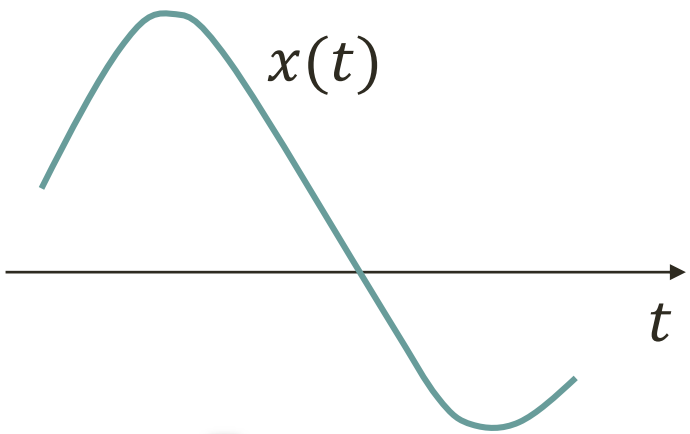
$$\breve{x}(t)=2x(t)\cos\omega_s t$$

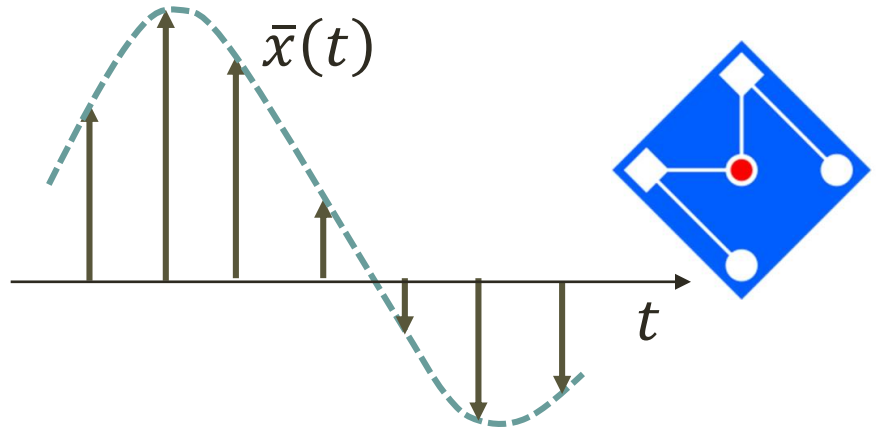
$$\breve{X}(\omega)=X(\omega-\omega_s)+X(\omega+\omega_s)$$

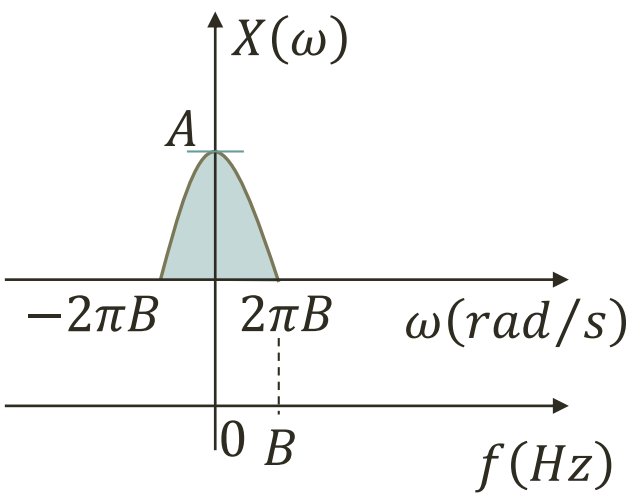
$$\bar{X}(\omega)=X(\omega)+\boxed{X(\omega-\omega_0)+X(\omega+\omega_0)}+\boxed{X(\omega-2\omega_0)+X(\omega+2\omega_0)}+\cdots$$

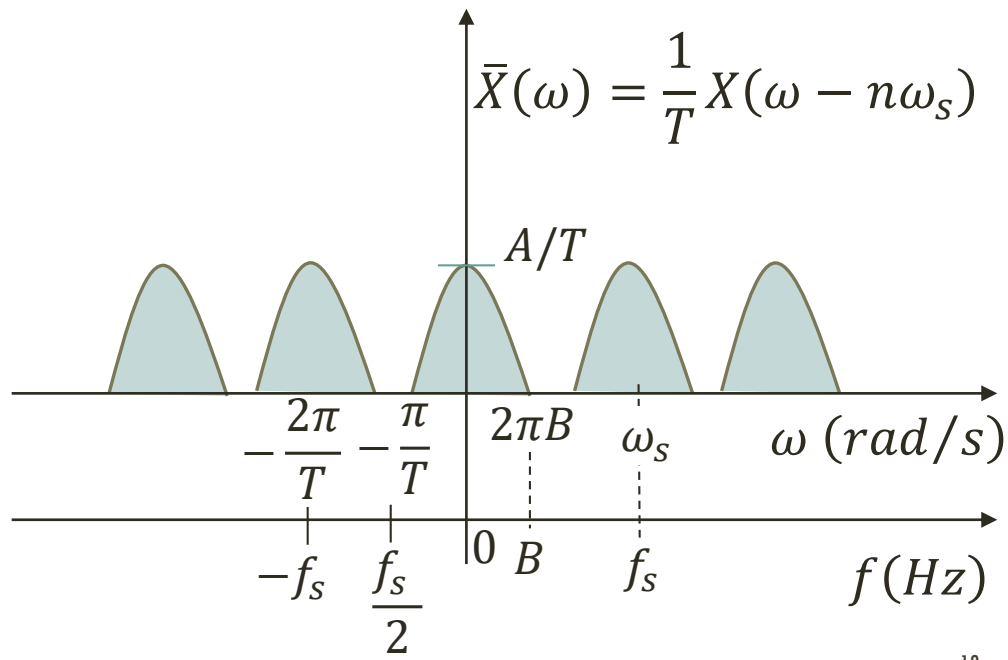
Espectro de $X(\omega)$ deslocado de $\pm\omega_0$

Espectro de $X(\omega)$ deslocado de $\pm\omega_0$


$x(t)$
$t$

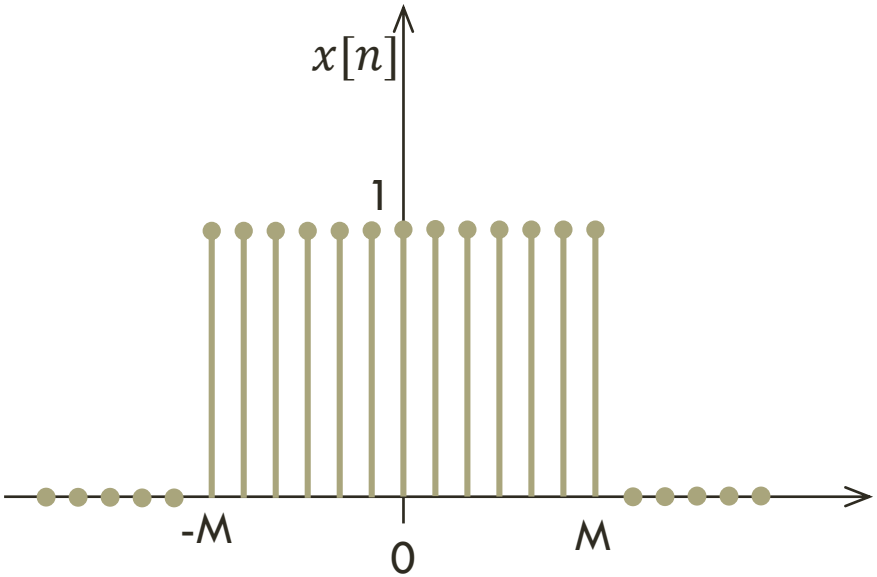

$\bar{x}(t)$
$t$


$X(\omega)$
$A$
$-2\pi B$
$2\pi B$
$\omega(rad/s)$
$0$
$B$
$f(Hz)$

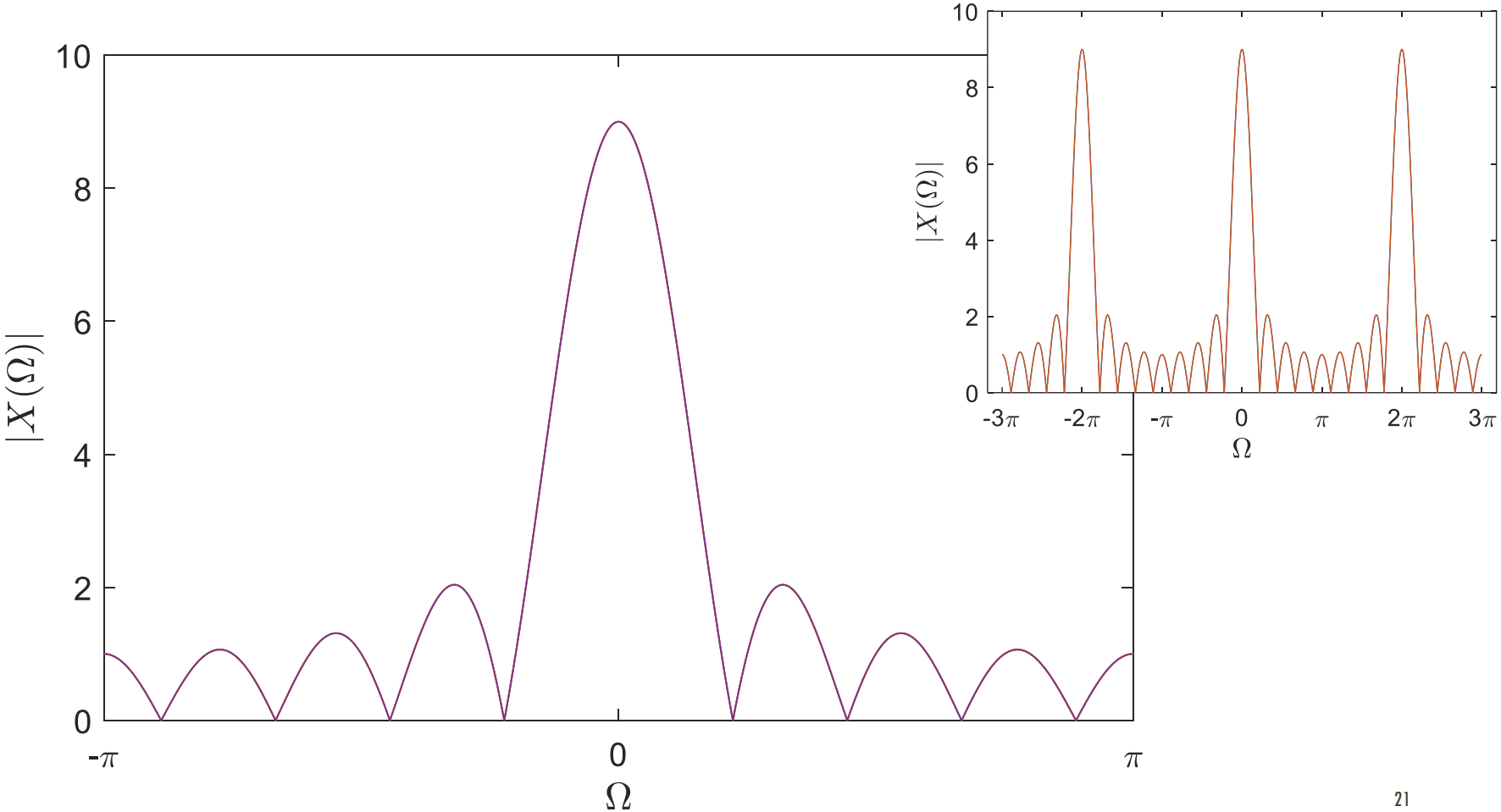

$\bar{X}(\omega) = \frac{1}{T}X(\omega - n\omega_s)$
$A/T$
$-\frac{2\pi}{T}$
$-\frac{\pi}{T}$
$2\pi B$
$\omega_s$
$\omega\ (rad/s)$
$-f_s$
$\frac{f_s}{2}$
$0$
$B$
$f_s$
$f(Hz)$

# EXEMPLO...

$$x[n] = \begin{cases} 1 & \text{para } n = -M, \dots, 0, \dots, M \\ 0 & cc \end{cases}$$



# DTFT


10
8
6
4
2
0
$|X(\Omega)|$
$-\pi$
0
$\pi$
$\Omega$
10
8
6
4
2
0
$|X(\Omega)|$
$-3\pi$
$-2\pi$
$-\pi$
0
$\pi$
$2\pi$
$3\pi$
$\Omega$

# TRANSFORMADA DE FOURIER EM TEMPO DISCRETO

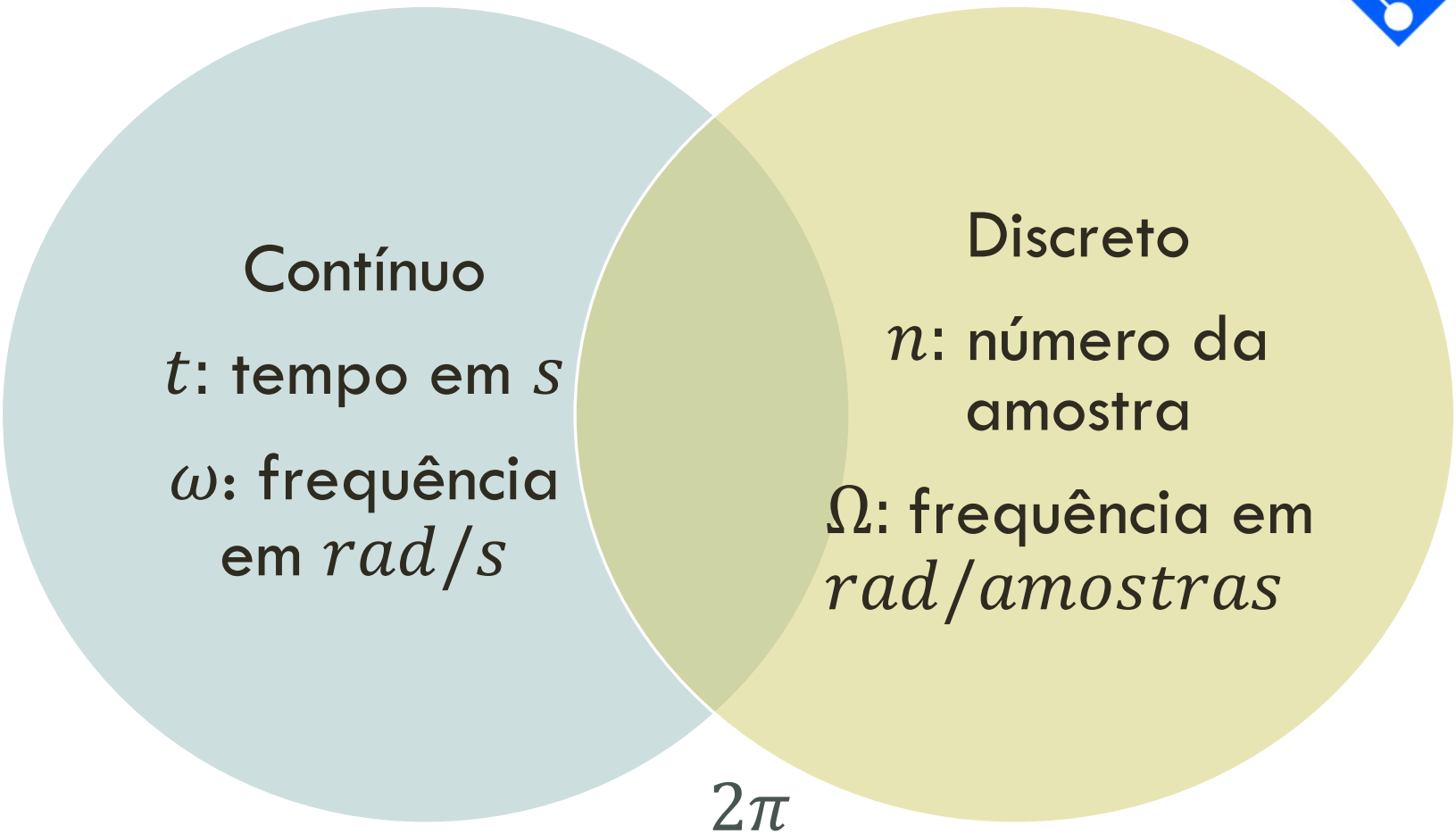
$$\bar{x}(t) = x(kT) = x(t)\delta_T(t) = \boxed{\sum_{k=-\infty}^{\infty} x(t)\delta(t-kT)}$$

$$\bar{X}(\omega) = \int_{-\infty}^{+\infty} \bar{x}(t)e^{-j\omega t}dt \quad + \quad \Omega = \omega T = \omega\frac{2\pi}{\omega_s}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

# FREQUÊNCIA DIGITAL




$$\Omega = \omega \frac{2\pi}{\omega_s}$$

# O QUE É A FREQUÊNCIA DIGITAL?

Se tivermos um sinal de freqüência de $10\ Hz$ e amostrarmos com uma freqüência de amostragem $f_s = 30\ Hz$, então sua freqüência digital é

$$\Omega = \frac{2\pi \times 10}{30} = \frac{2\pi}{3}$$

O que este número significa???

Isso significa que cada amostra move o sinal por em $\Omega$ radianos. Se um ciclo contém $2\pi$ radianos, e cada amostra cobre $\frac{2\pi}{3}$rad, então, 3 amostras são necessárias para completar um ciclo.

# FT ⟷ DTFT

**Síntese**

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$$

**Síntese**

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

**Análise**

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

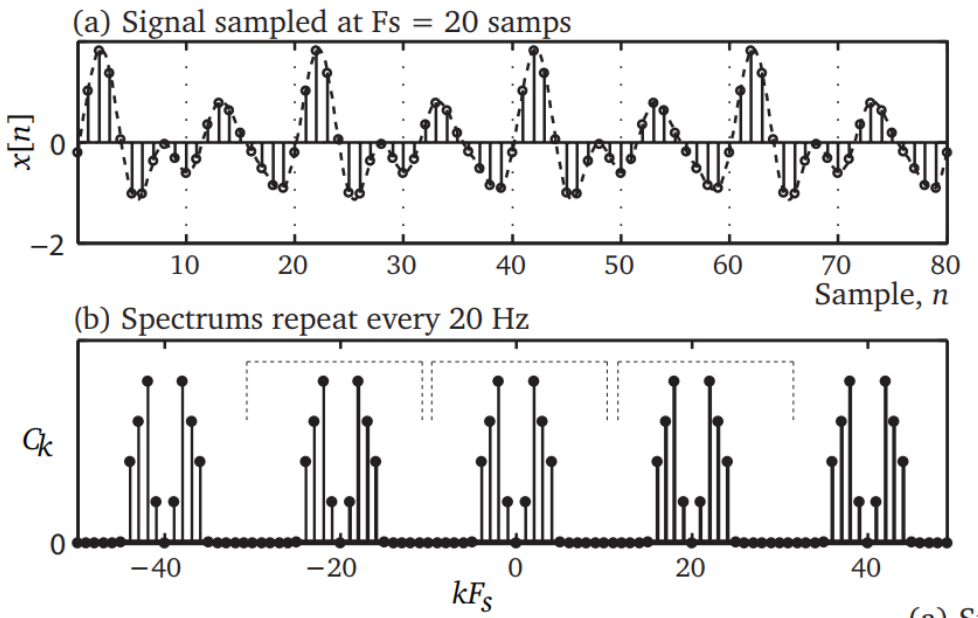
**Análise**

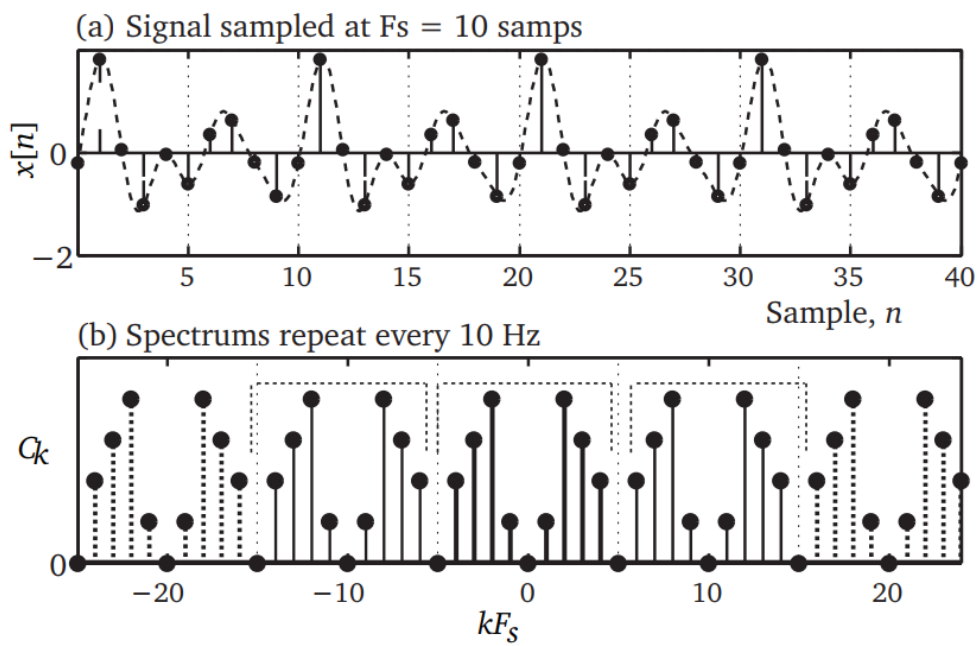
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Valores contínuos $X(\omega)$ do sinal contínuo $x(t)$

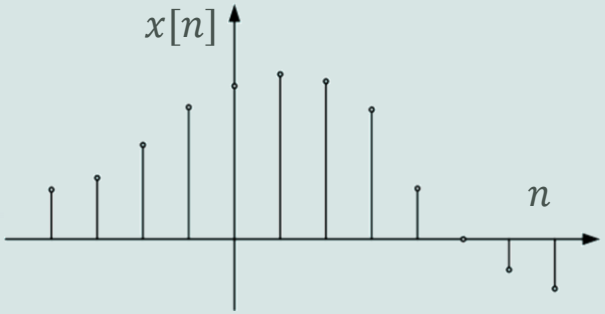
Valores contínuos $X(\Omega)$ do sinal discreto $x[n]$

$$x(t) = 0{,}2\sin(2\pi t) + \sin(4\pi t) + 0.7\cos(6\pi t) + 0.4\cos(8\pi t)$$



Só os sinais discretos podem ser armazenados e processados em computadores digitais.



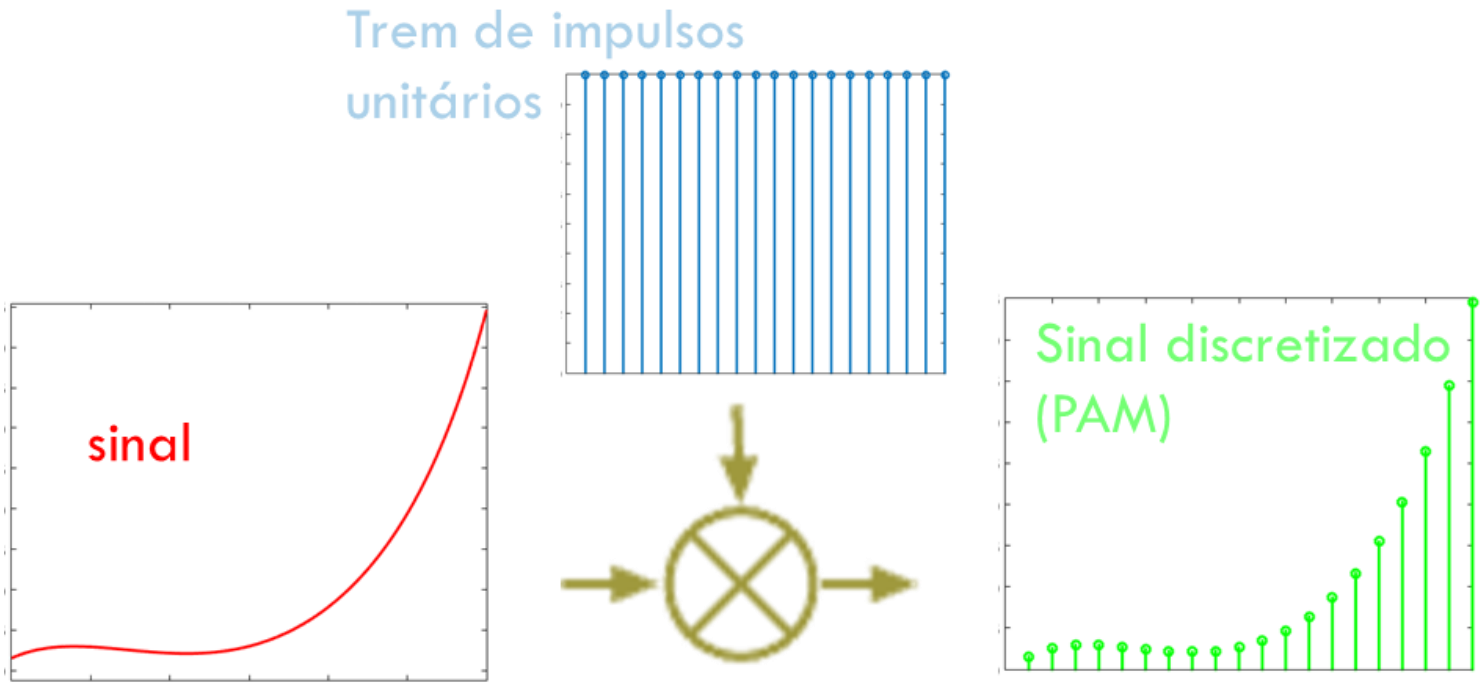
# AMOSTRAGEM

*"Amostragem é a ponte entre os mundos do contínuo e do discreto"*

Lathi, 2007

# AMOSTRAGEM

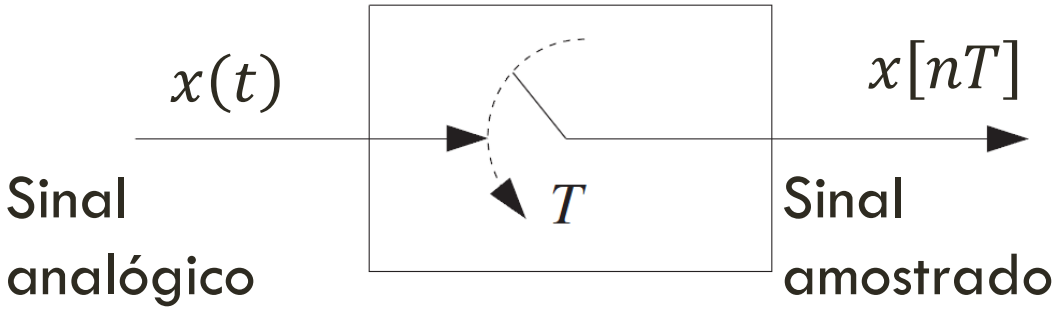
Discretização temporal do sinal analógico original




Impulsos medidos e guardados como sinais amostrados

..., 3.44, 4.67 , 6.39 , 8.69, ...

**Amostrador ideal**




# COMO ESCOLHER A FREQUÊNCIA DE AMOSTRAGEM????

# DOMÍNIO DO TEMPO

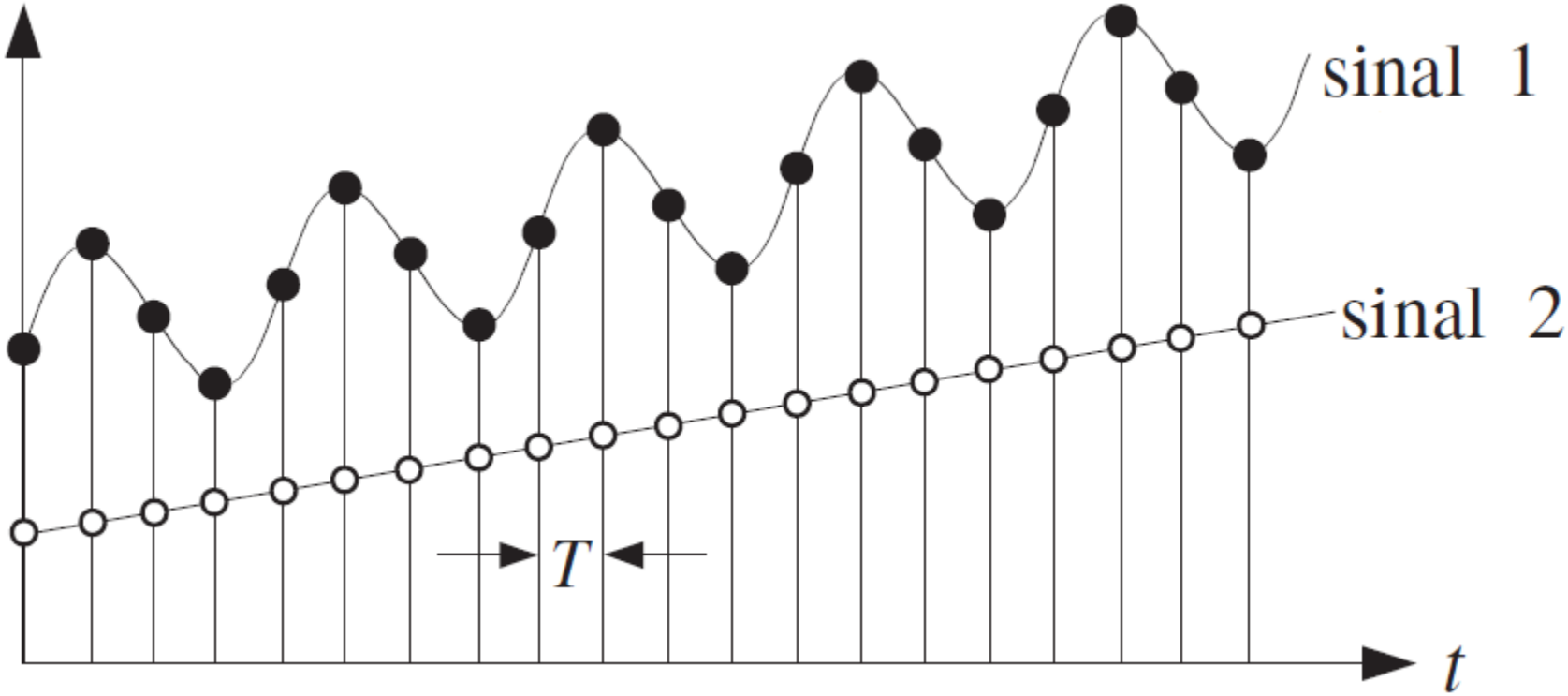
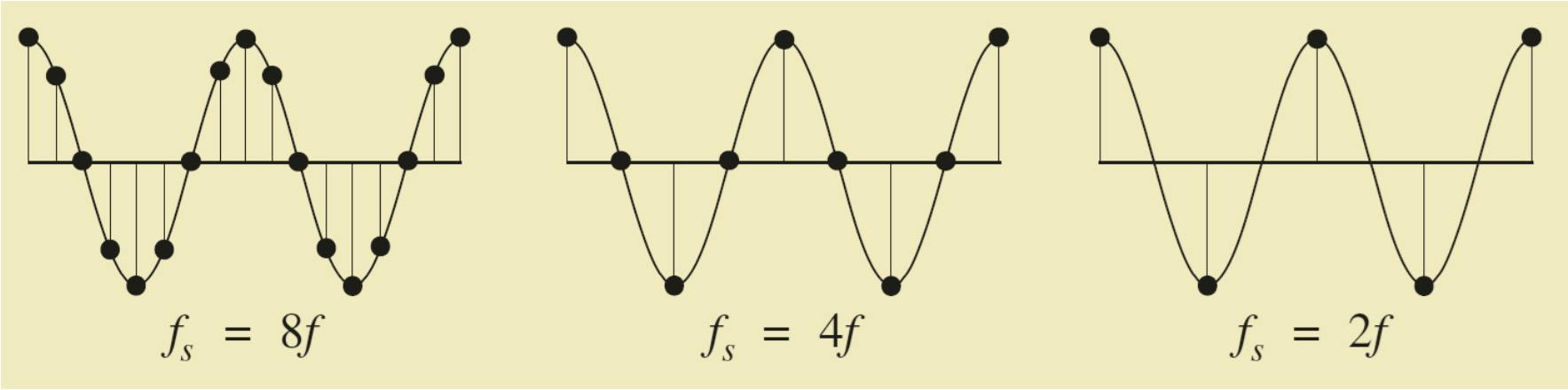


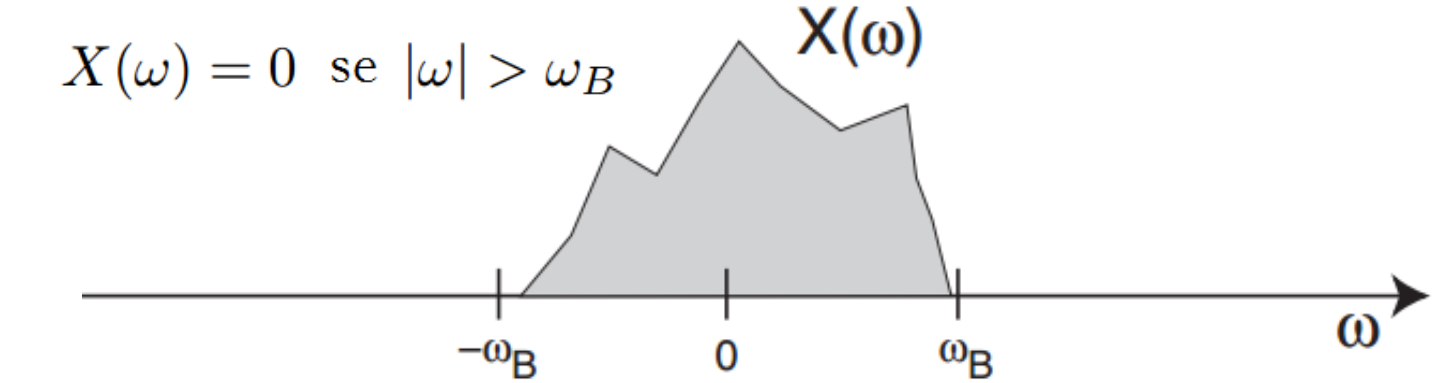

Figura extraída de [3]

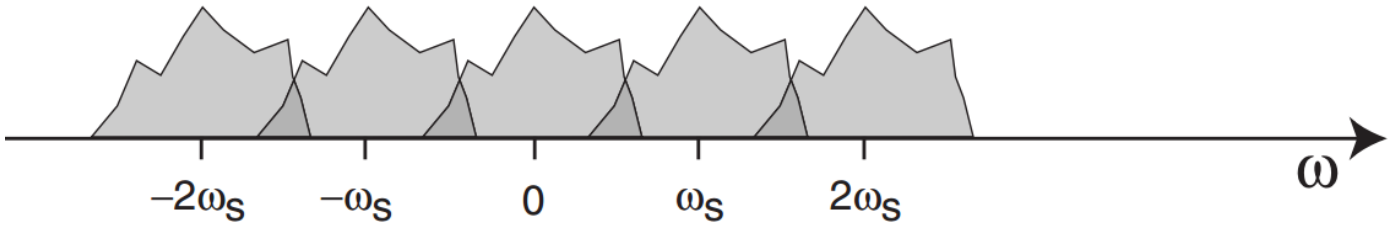
# AMOSTRAGEM DE SENOIDES

$$x(t) = \cos \omega t = \cos 2\pi f t$$

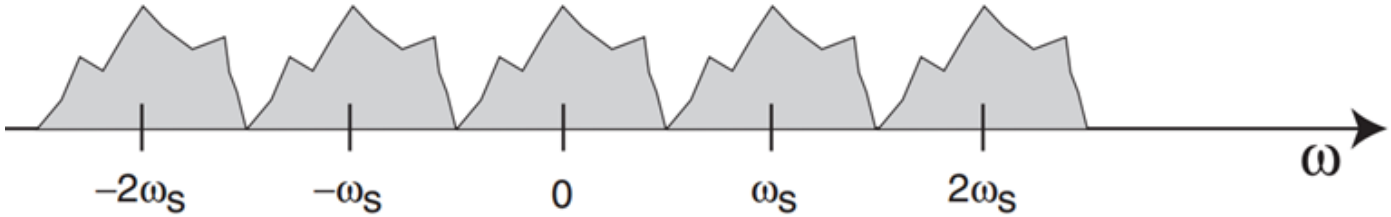


$$X(\omega) = 0 \quad \text{se} \quad |\omega| > \omega_B$$

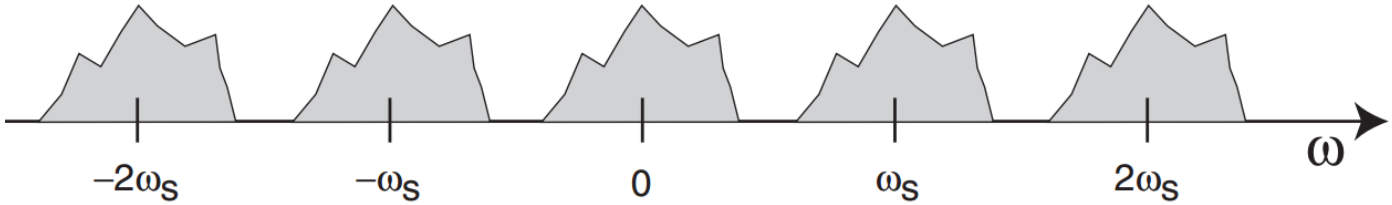



$\omega_s \ll 2\omega_B$
Subamostragem




$\omega_s = 2\omega_B$
Frequência de Nyquist




$\omega_s \gg 2\omega_B$
Superamostragem

# TEOREMA DA AMOSTRAGEM DE NYQUIST–SHANNON

Se um sinal analógico $x(t)$ tem banda limitada, ou seja, se a frequência mais elevada do sinal é $B$ ou seja,

$$X(\omega) = 0 \text{ para } |f| > B,$$

então, é suficiente uma amostragem a qualquer taxa

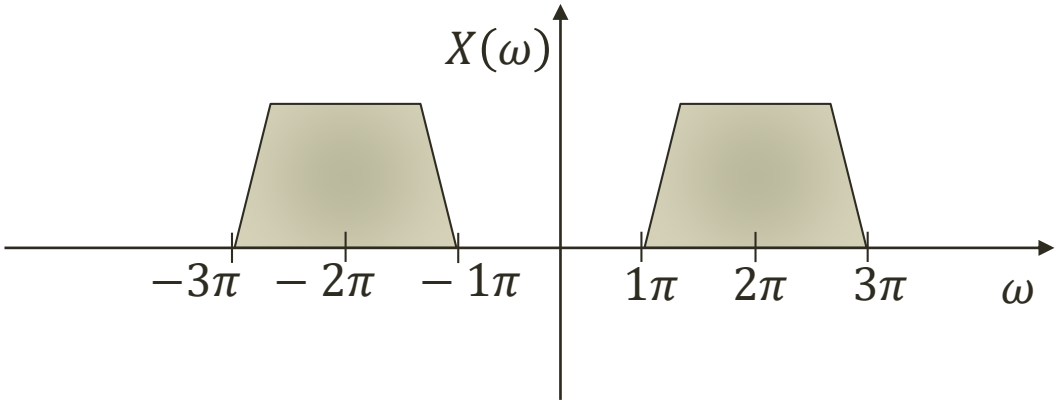
$$f_s > 2B$$

# TESTE...

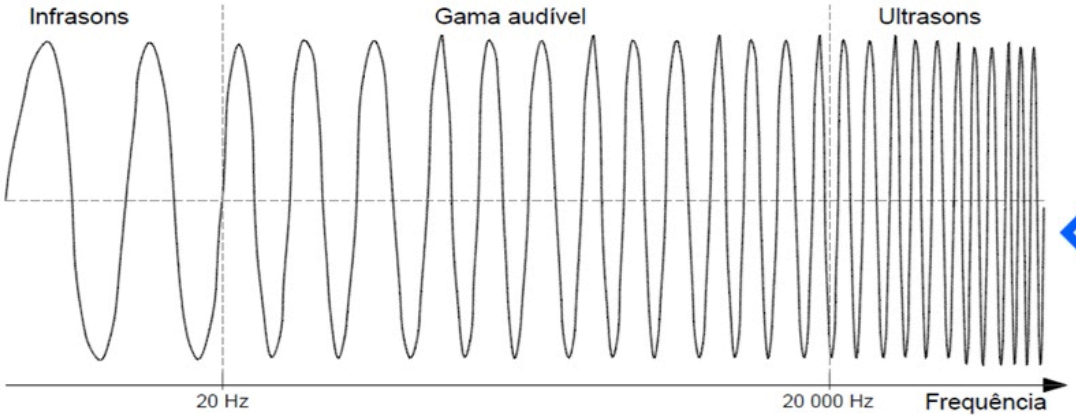
Encontre $T_s$ máximo para correta amostragem de $x(t)$,

A. $x(t) = \sin(2\pi t) + \cos(5\pi t + 0{,}1) + \cos(\pi t)$

B.



# ENTÃO...




Os sons audíveis pelo ouvido humano têm uma frequência entre 20Hz e 20kHz.

Qual o intervalo máximo de amostragem $T_s$ que podemos usar para amostrar um sinal sem perda de informação audível?

A. $100\ \mu s$

B. $50\ \mu s$

C. $25\ \mu s$

D. $100\pi\ \mu s$

E. $50\pi\ \mu s$

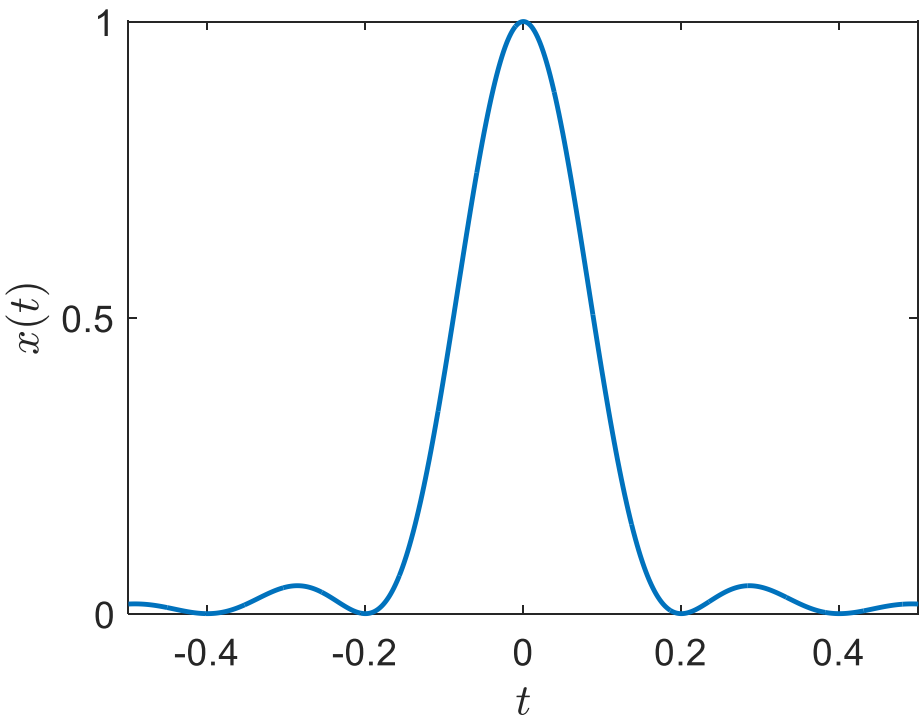
F. $25\pi\ \mu s$

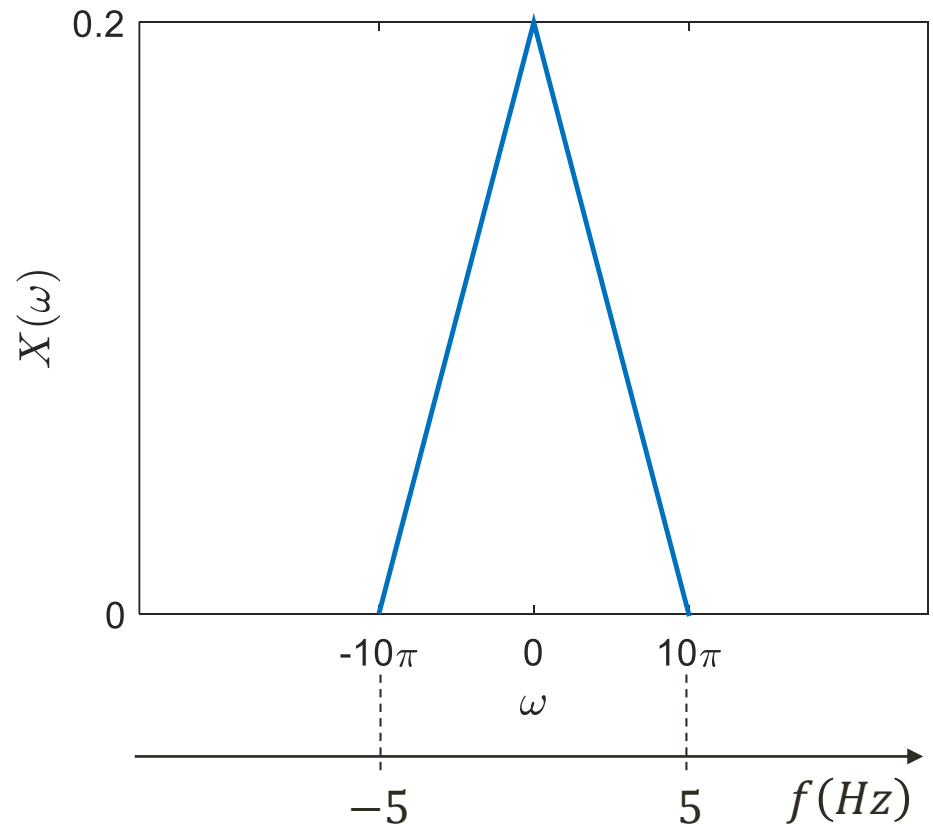
# EXEMPLO

Sonata No 1 In G Minor Presto - Johann Sebastian Bach, amostrada a:

A. 44.1 kHz
B. 22 kHz
C. 11 Hz
D. 5.5 kHz
E. 2.8 kHz

# EXEMPLO



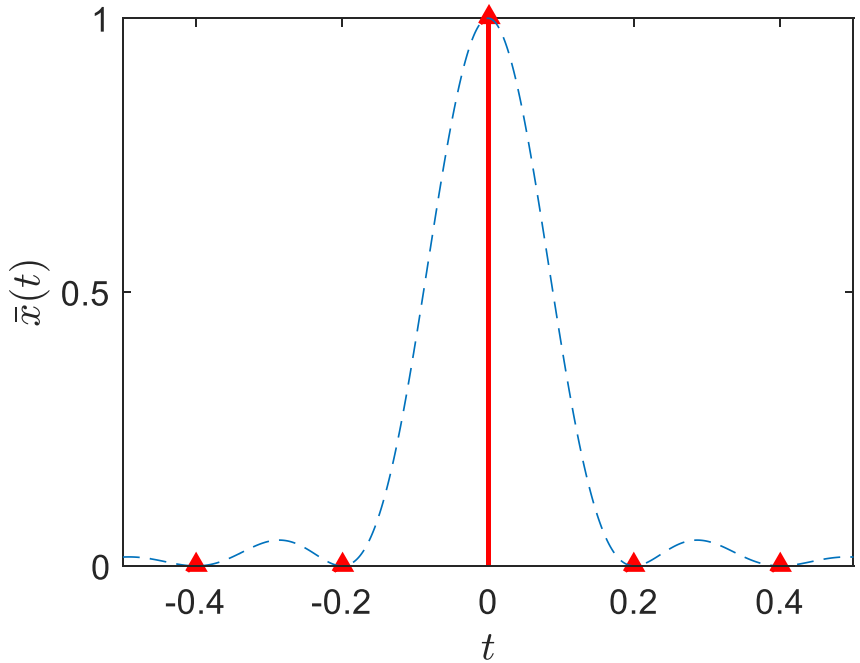

$$x(t) = \mathrm{sinc}^2(5\pi t)$$

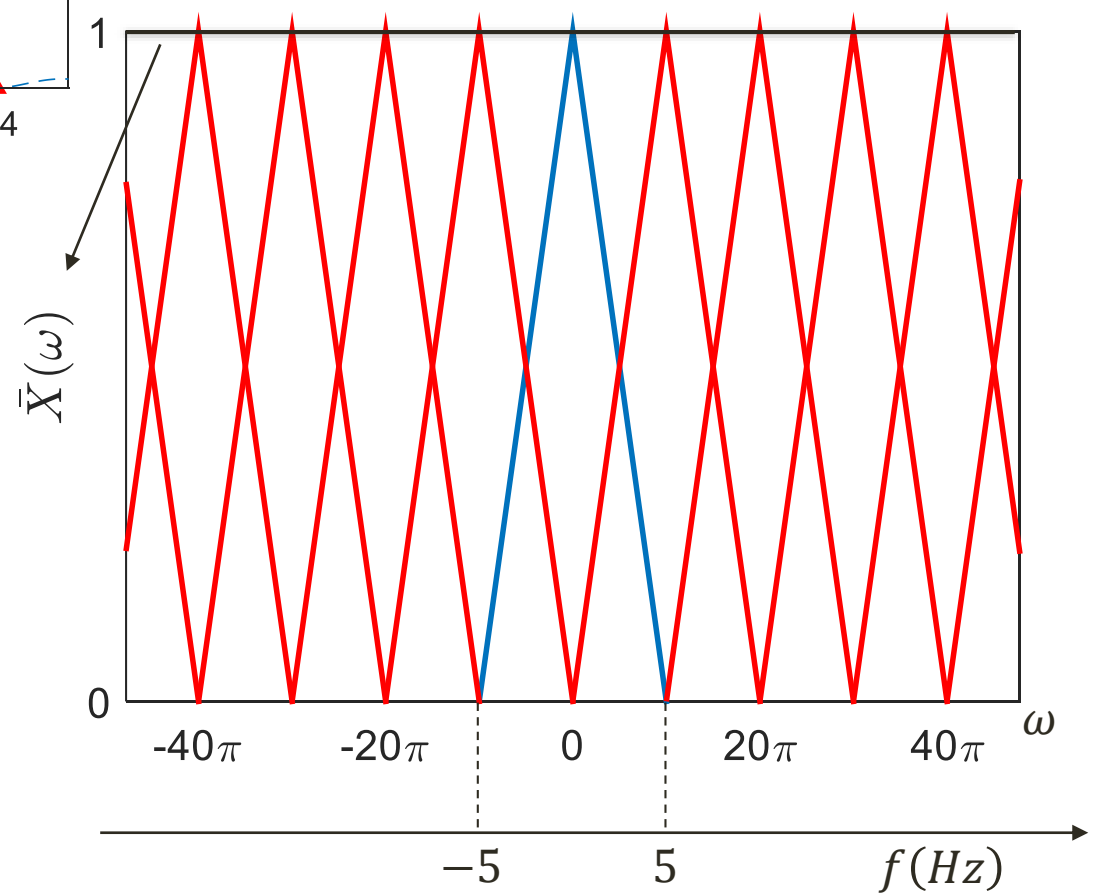
$$X(\omega) = 0{,}2\,\Delta\left(\frac{\omega}{20\pi}\right)$$

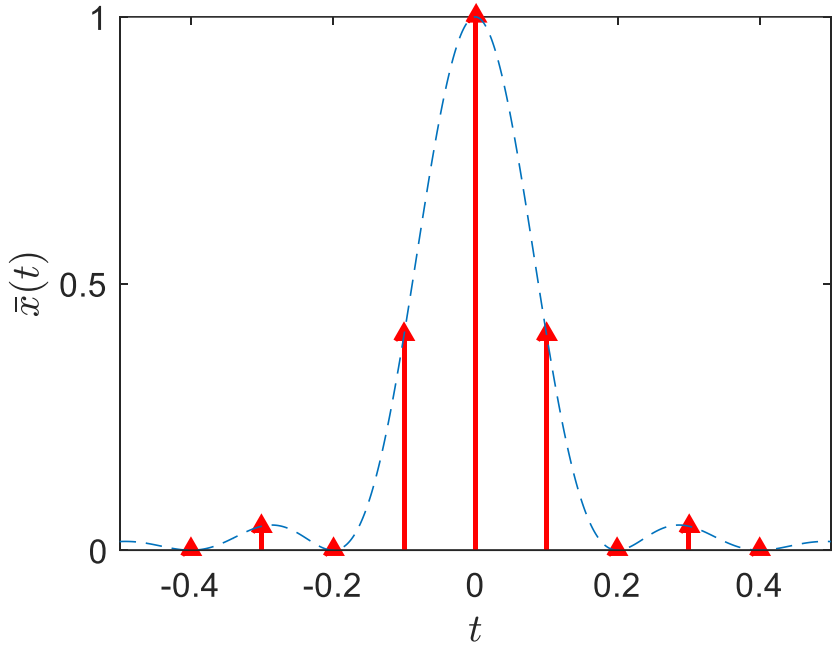
# Frequência de amostragem 5Hz

Intervalo de amostragem: 0,2 s

$$\frac{1}{T}X(\omega) = \Delta\left(\frac{\omega}{20\pi}\right)$$



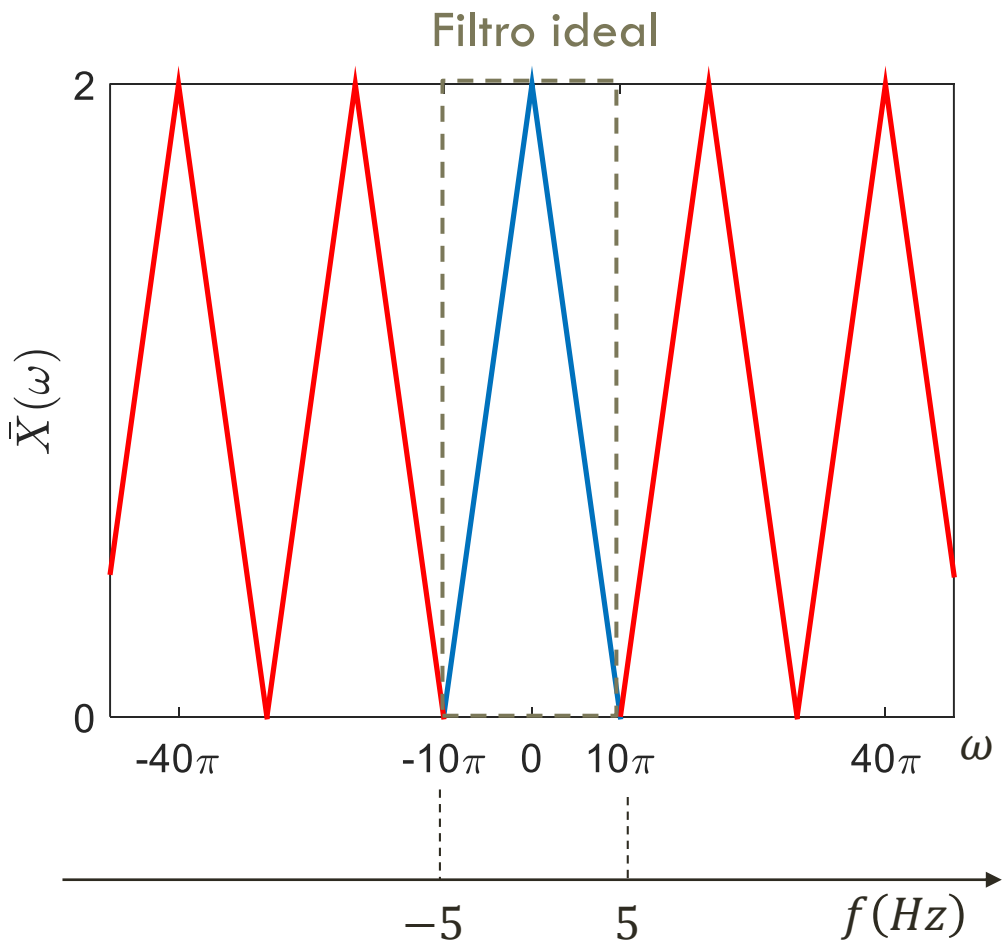



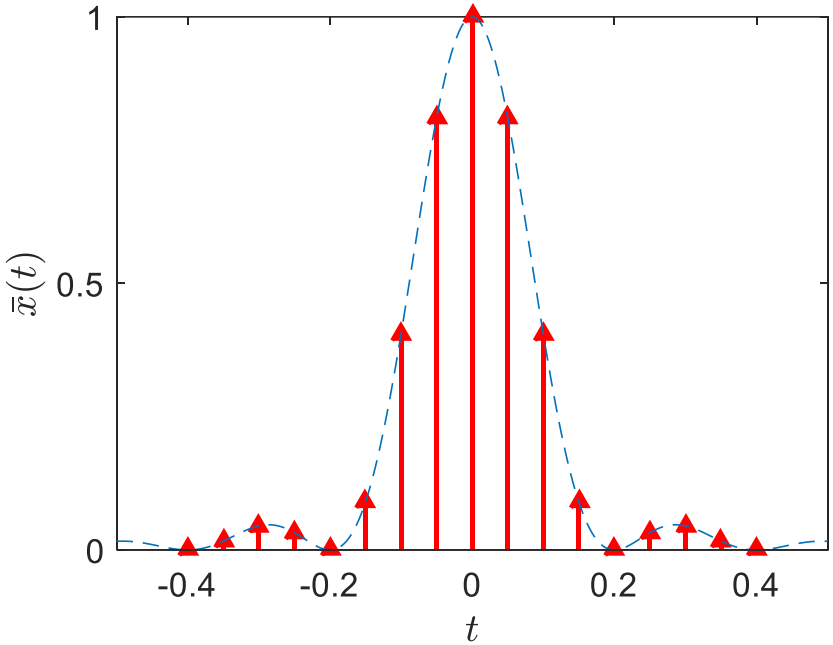



# Frequência de amostragem 10Hz

Intervalo de amostragem: 0,1 s

$$\frac{1}{T}X(\omega) = 2\Delta\left(\frac{\omega}{20\pi}\right)$$



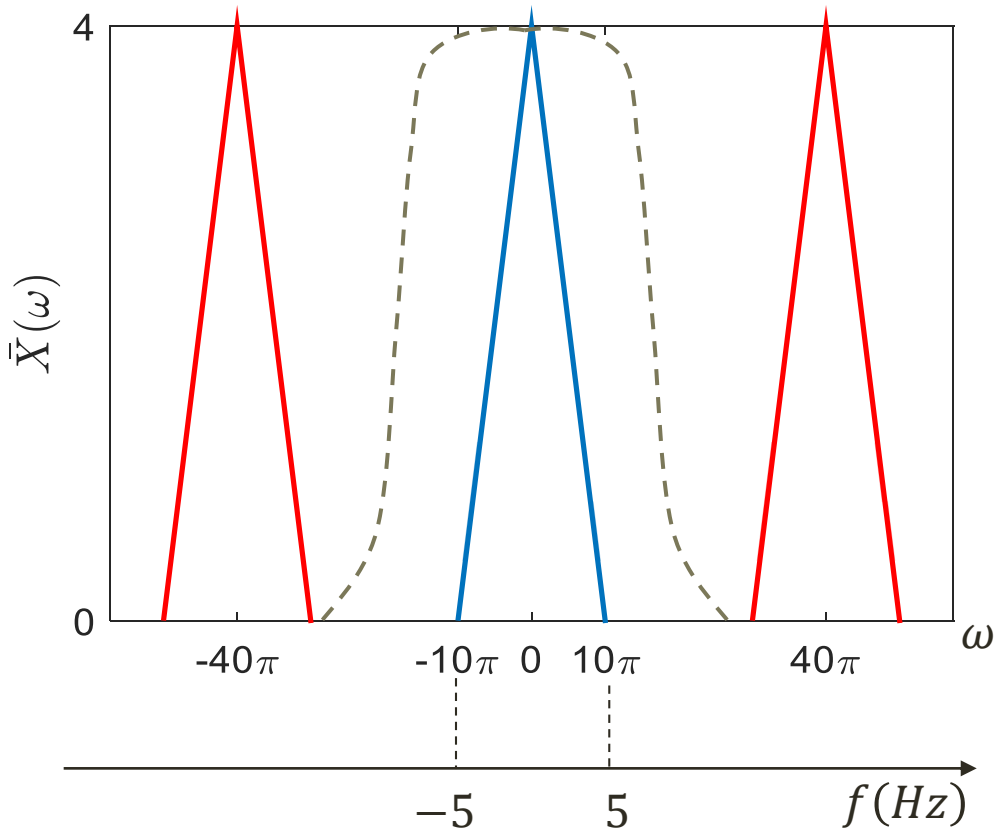



# Frequência de amostragem 20Hz

Intervalo de amostragem: 0,05 s

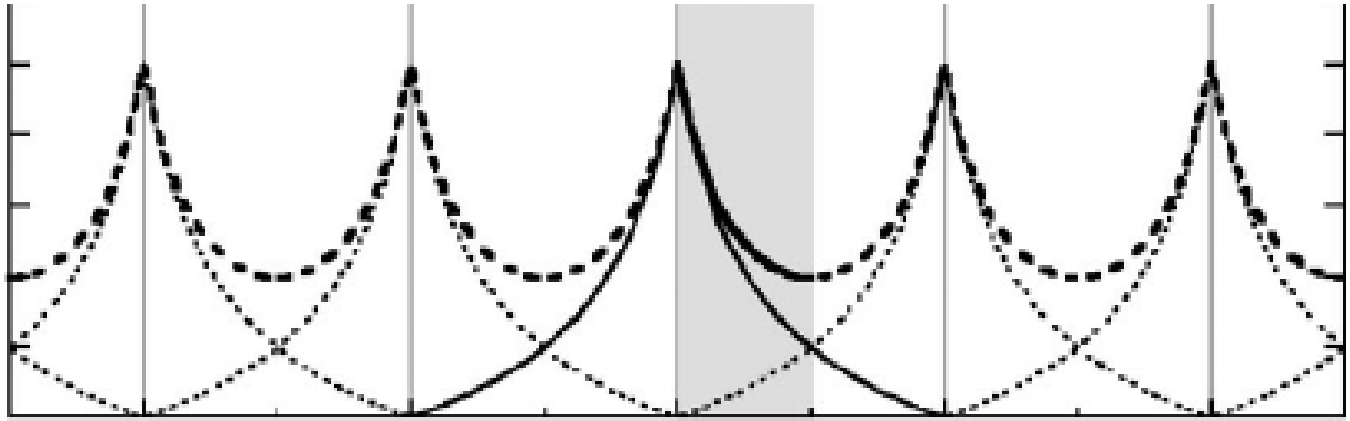
$$\frac{1}{T}X(\omega) = 4\Delta\left(\frac{\omega}{20\pi}\right)$$

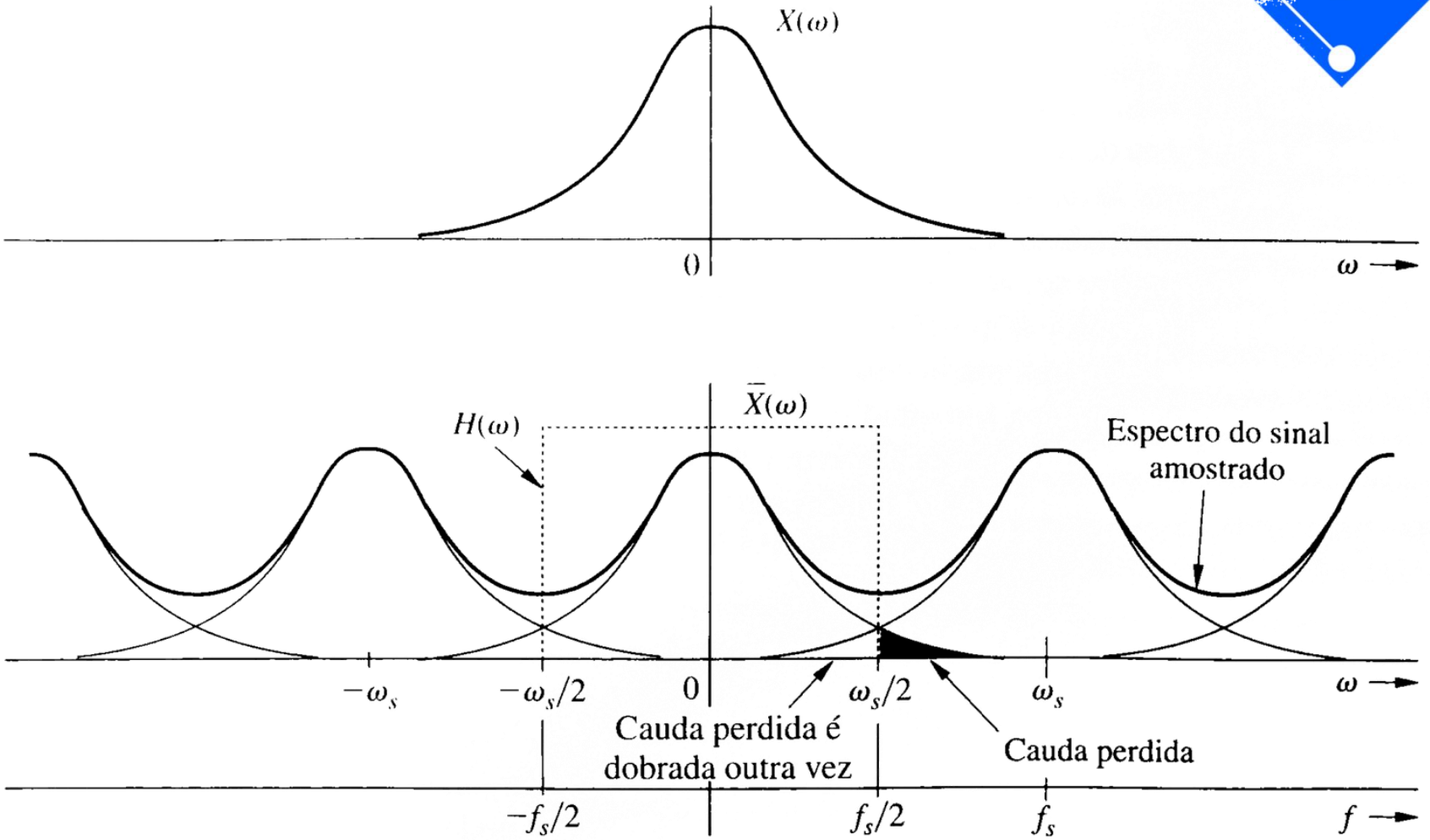
Filtro prático

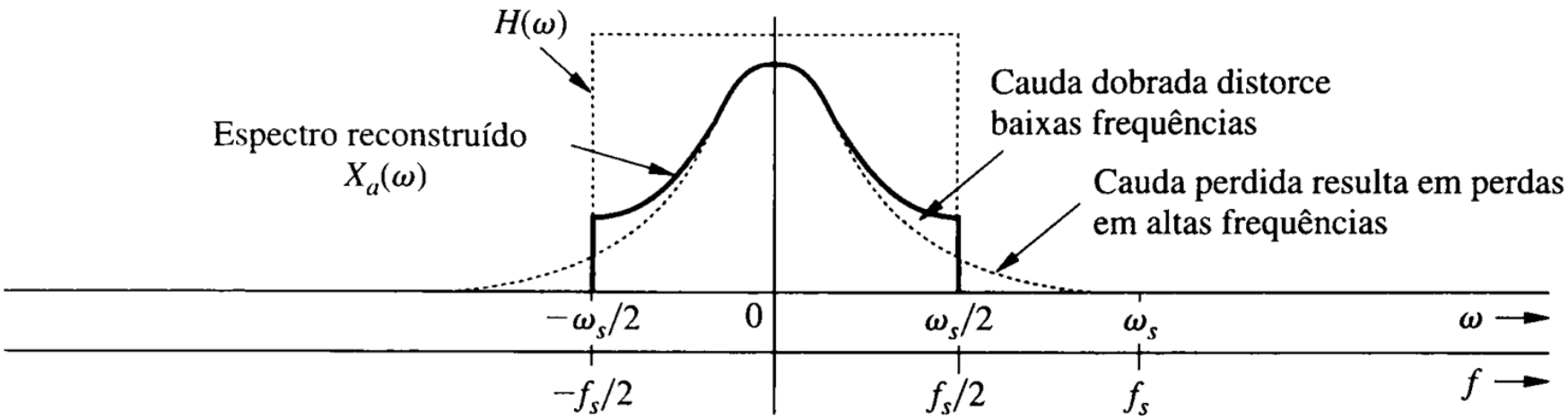


# ALIASING

Muitos sinais não tem largura de banda finita ou não conhecemos. Dessa forma, existe uma grande chance de ocorrer sobreposição nas réplicas da frequência...




$X(\omega)$
0
$\omega$
$\bar{X}(\omega)$
$H(\omega)$
Espectro do sinal amostrado
$-\omega_s$
$-\omega_s/2$
0
$\omega_s/2$
$\omega_s$
$\omega$
Cauda perdida é dobrada outra vez
Cauda perdida
$-f_s/2$
$f_s/2$
$f_s$
$f$


$H(\omega)$
Espectro reconstruído
$X_a(\omega)$
Cauda dobrada distorce
baixas frequências
Cauda perdida resulta em perdas
em altas frequências
$-\omega_s/2$
0
$\omega_s/2$
$\omega_s$
$\omega$
$-f_s/2$
$f_s/2$
$f_s$
$f$

# SENOIDE E ALIASING

Os seguintes sinais são amostrados a uma frequência de 4 Hz,

$$-\sin 14\pi t \quad -\sin 6\pi t \quad \sin 2\pi t \quad \sin 10\pi t \quad \sin 18\pi t$$

Mostre que eles representam todos o mesmo sinal amostrado...

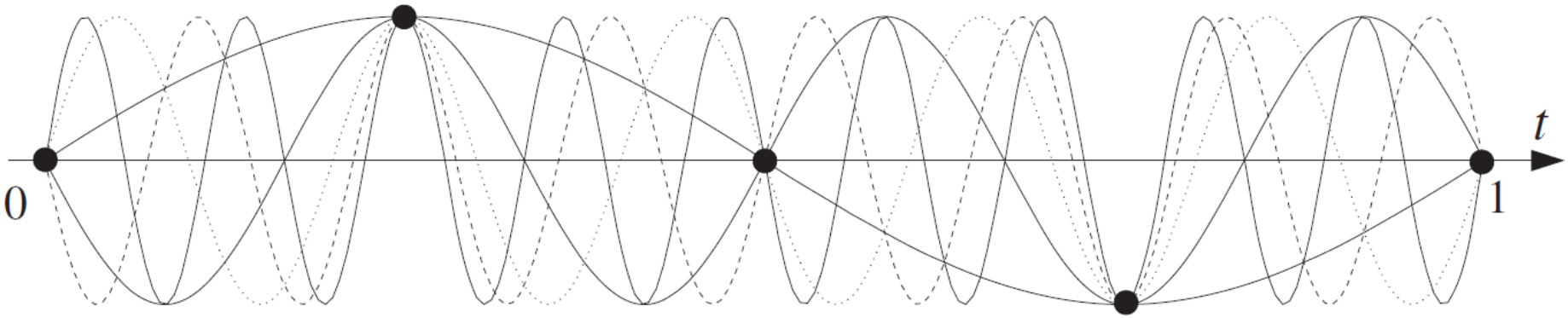


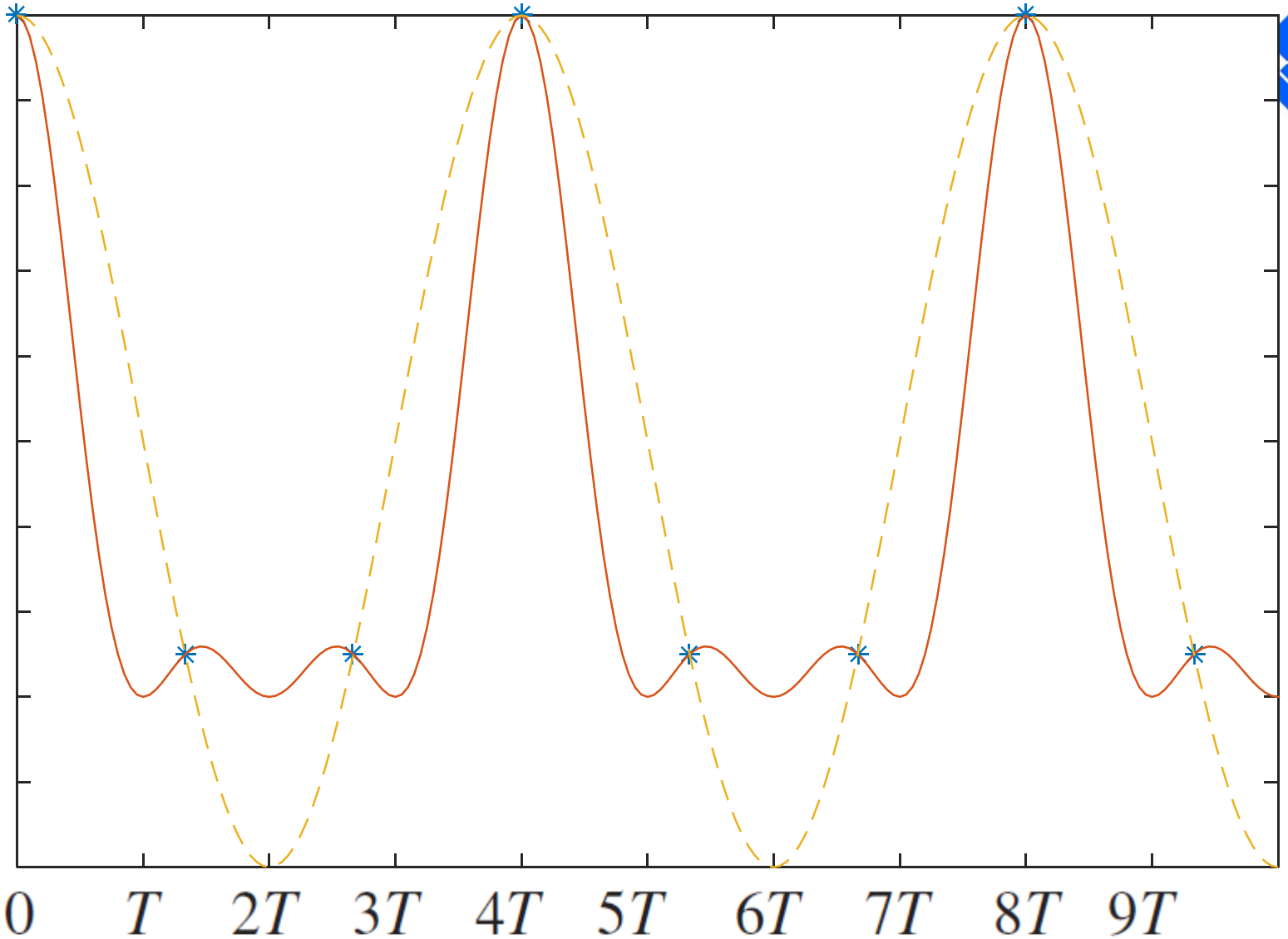

Figura extraída de [3]

# EXEMPLO 2

Considere $x(t)$ como a soma de sinais senoidais,

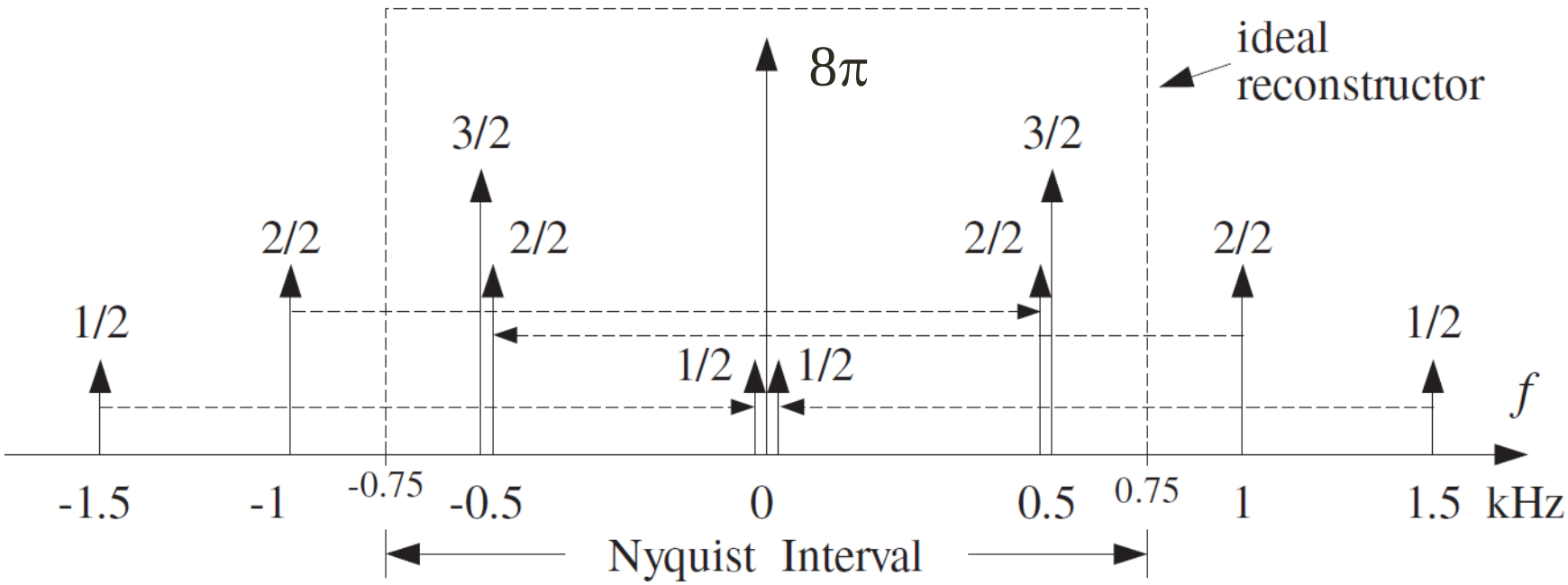
$$x(t) = 4 + 3\cos \pi t + 2\cos 2\pi t + \cos 3\pi t$$

onde $t$ está em milisegundos. Determine a frequência mínima de amostragem para que não ocorra aliasing, i. é, a frequência de Nyquist ($f_N$). Para observação dos efeitos do aliasing, suponha que o sinal é amostrado à $f_N/2$. Determine o sinal recuperado $x_a(t)$.


0
T
2T
3T
4T
5T
6T
7T
8T
9T

$$x(t) = \cos \omega_0 t \Longrightarrow X(\omega) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\bar{x}(t) = \sum_n x(nT)\,\delta(t - nT) \Longrightarrow \bar{X}(\omega) = \frac{1}{T} X(\omega - n\omega_s)$$




$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t}$$

$$\boxed{x(t) = 4{,}16 + 5\cos \pi t}$$

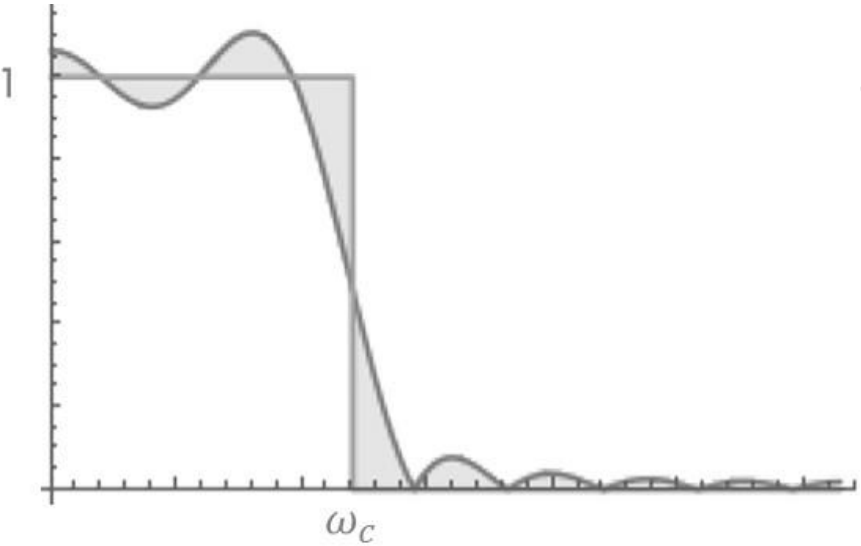
# FILTRO

Para extrair corretamente a informação fundamental do sinal analisado é necessário selecionar as frequências de interesse que compõe esse sinal. Como fazer isso?

Com um **FILTRO**. Filtros são SLIT capazes de modificar as características dos sinais de entrada de tal modo que apenas uma parcela específica dos seus componentes de frequência chega à saída do filtro.

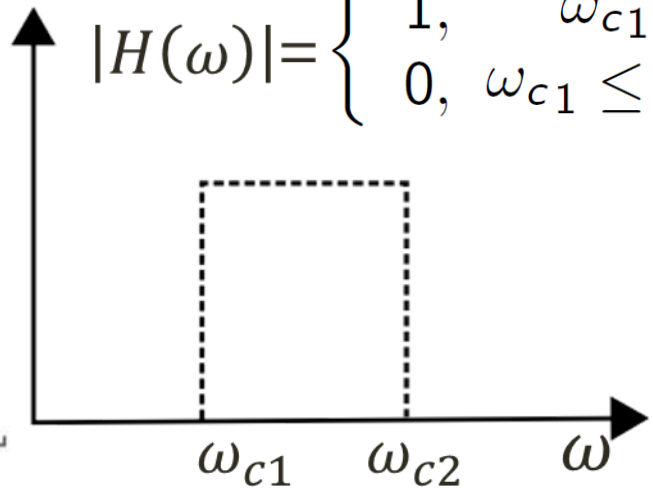
A resposta em frequência do filtro é caracterizada por **uma faixa de passagem** e **uma faixa de rejeição**, separadas por uma **faixa de transição ou faixa de guarda**

## Passa baixa



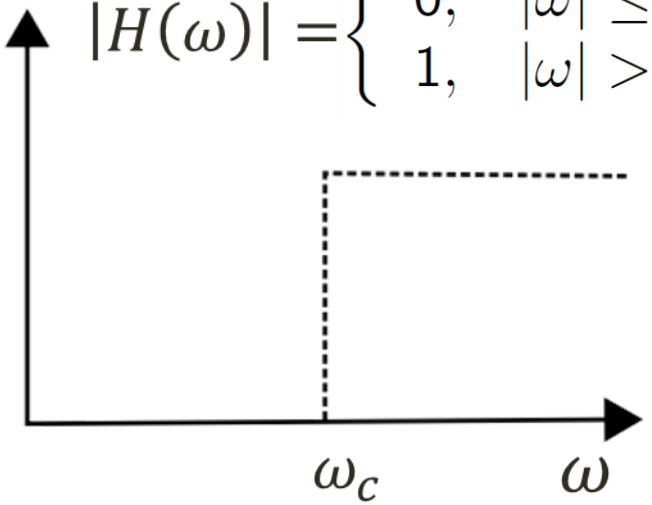

## Passa banda (faixa)

$$|H(\omega)| = \begin{cases} 1, & \omega_{c1} < |\omega| \leq \omega_{c2} \\ 0, & \omega_{c1} \leq |\omega| \text{ e } |\omega| > \omega_{c2} \end{cases}$$



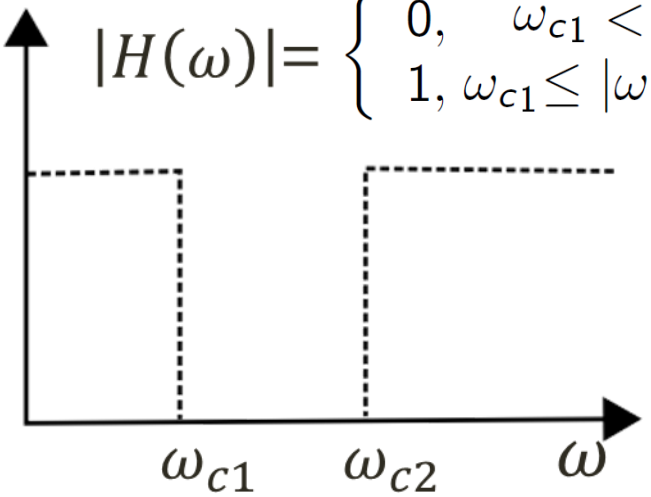

## Passa alta

$$|H(\omega)| = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$




## Rejeita banda (faixa)

$$|H(\omega)| = \begin{cases} 0, & \omega_{c1} < |\omega| \leq \omega_{c2} \\ 1, & \omega_{c1} \leq |\omega| \text{ e } |\omega| > \omega_{c2} \end{cases}$$



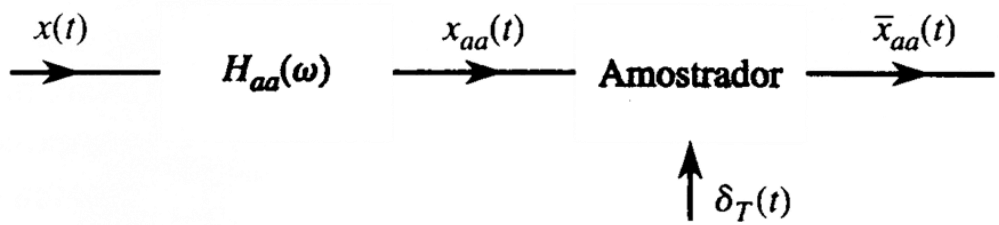
# SOLUÇÃO: FILTRO ANTIALIASING

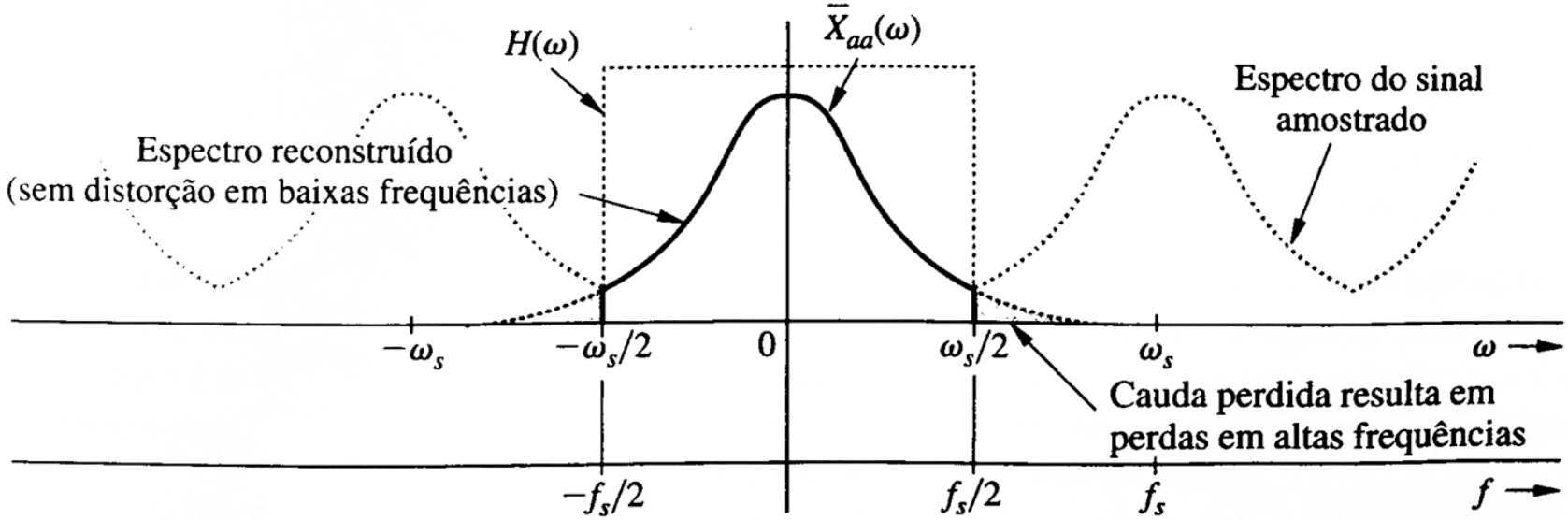
As frequências altas devem ser eliminadas ANTES da amostragem do sinal. Como???

Emprega-se um filtro *passa-baixa* com frequência de corte $f_s/2$.

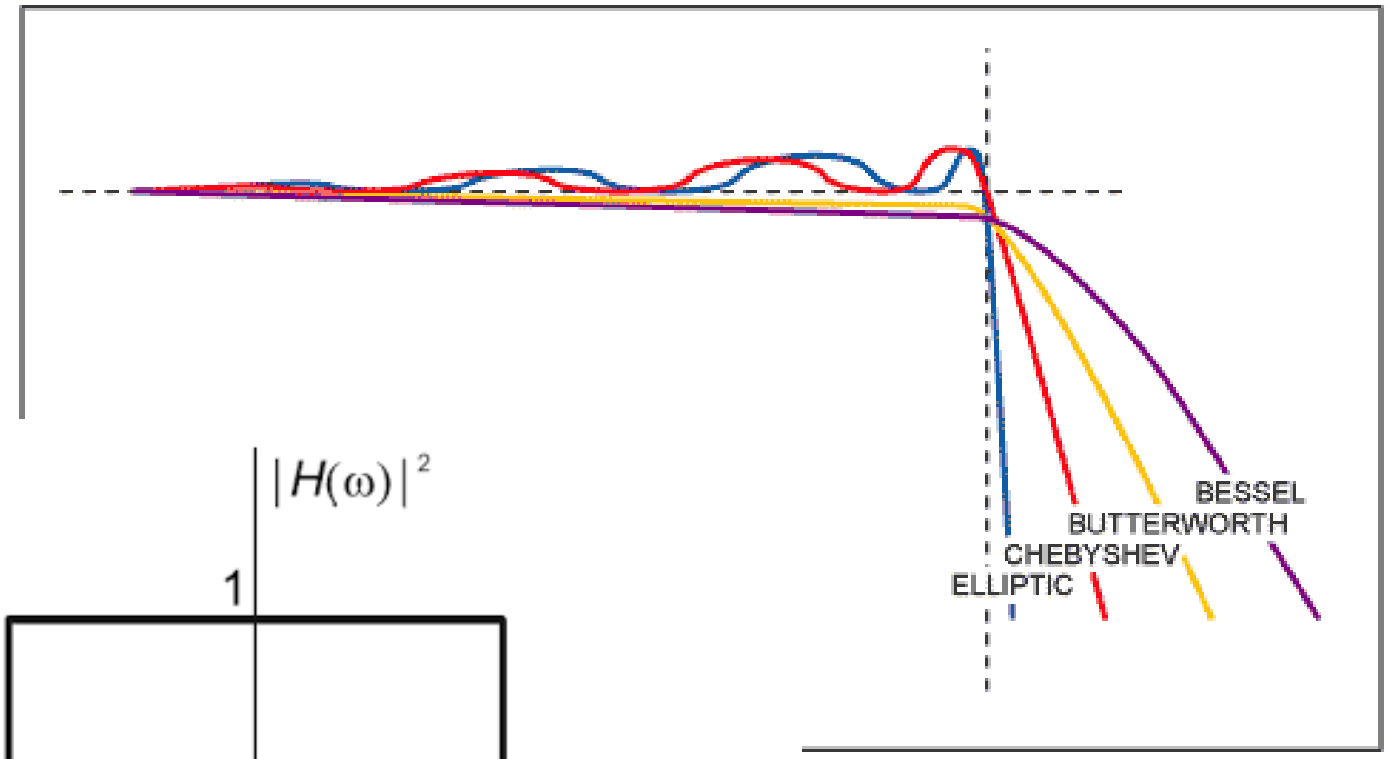
Esse filtro é chamado de filtro *anti-aliasing*.

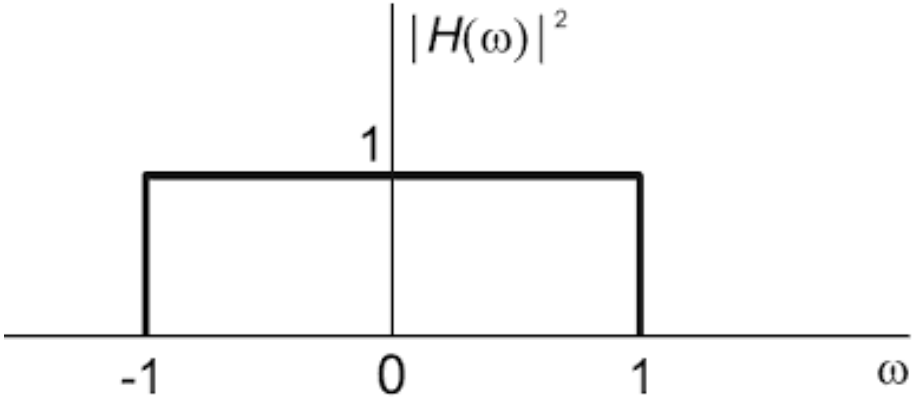
O espectro das componentes de baixa frequência permanece intacto. Como perdemos as componentes de alta frequência, determina-se a frequência de corte com base nas frequências de interesse do sinal e na frequência de amostragem.

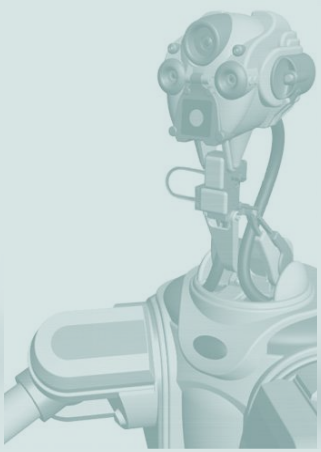

$x(t)$
$H_{aa}(\omega)$
$x_{aa}(t)$
Amostrador
$\bar{x}_{aa}(t)$
$\delta_T(t)$


$H(\omega)$
$\bar{X}_{aa}(\omega)$
Espectro do sinal amostrado
Espectro reconstruído
(sem distorção em baixas frequências)
$-\omega_s$
$-\omega_s/2$
0
$\omega_s/2$
$\omega_s$
$\omega$
Cauda perdida resulta em perdas em altas frequências
$-f_s/2$
$f_s/2$
$f_s$
$f$

# FILTROS





# FIM!