Design Methods for Control Systems

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Schedule

November 13	MSt	
November 20	MSt	Homework # 1
November 27	MSt	
December 4	MSt	Homework # 2
December 18	GM	
January 8	GM	Homework # 3
January 15	GM	
January 22	GM	Homework # 4

Overview

Scope and features

Mature review of “classical” and
“modern” control system design techniques

- Linear time-invariant systems
- 70% SISO- 30% MIMO
- Continuous-time
- MATLAB exercises
  - Control toolbox
  - Mu-Tools and Robust Control toolboxes

Overview

Ch. 1. Introduction to feedback control theory

Introduction

- Types of control systems
- Design issues
- Configurations
- High-gain feedback

Stability

- Closed-loop characteristic polynomial
- Nyquist criterion
- Stability margins

Performance

- System functions
- Low and high frequencies

Robustness

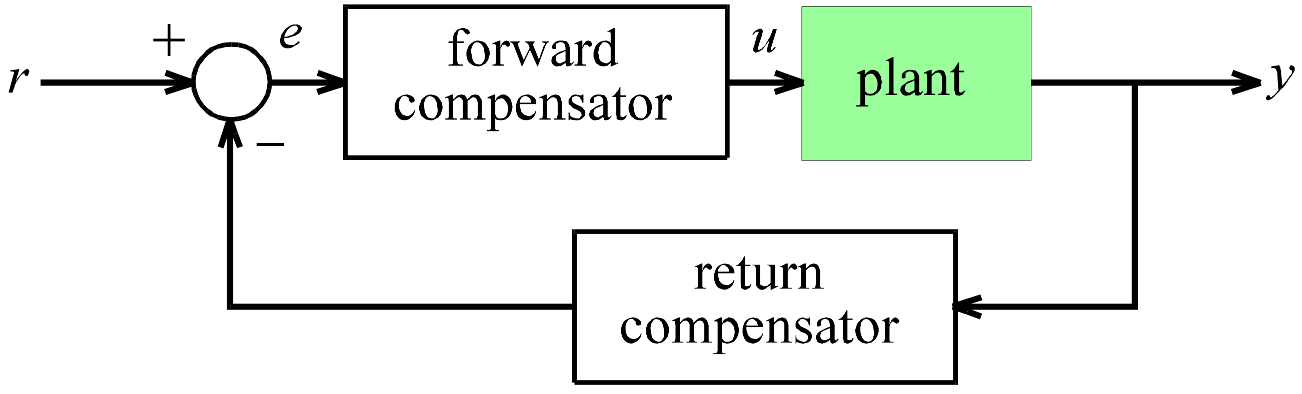
- Robustness functions

Loop shaping

Limits of performance

Two-degree-of-freedom control systems

Types of control systems

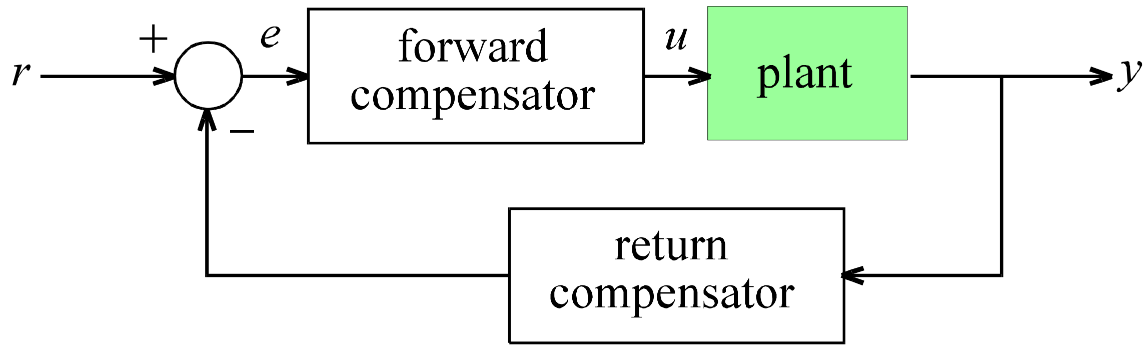


- Regulator systems
- Servo or positioning systems
- Tracking systems

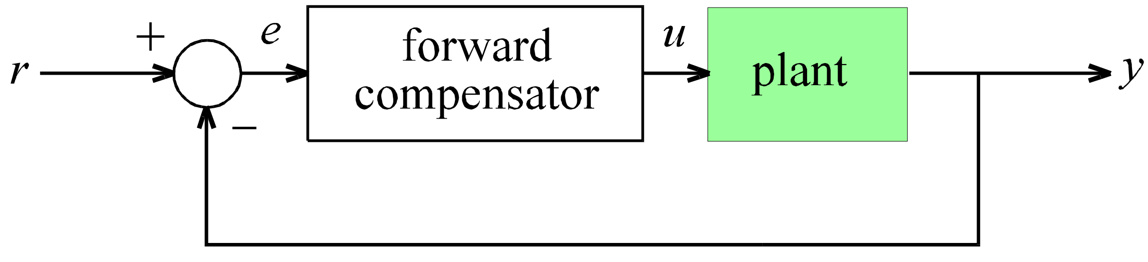
Design issues

Targets	***Limitations***
Closed-loop stability	Plant capacity
Disturbance attenuation	Measurement noise
Good command response	
Robustness	

Configurations

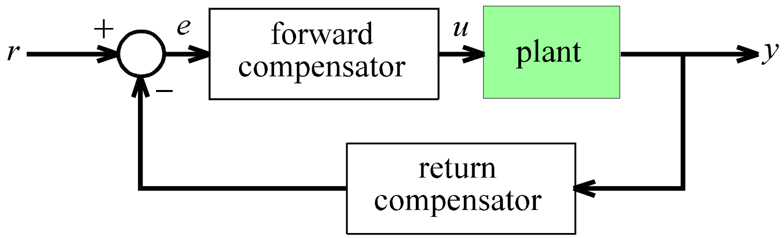


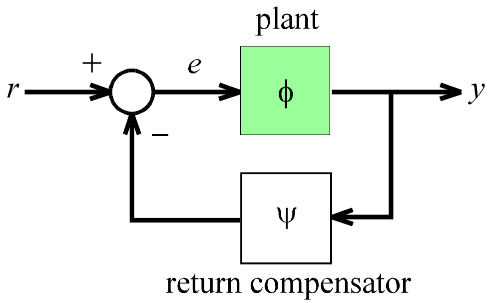
Two degrees of freedom



One degree of freedom

High-gain feedback-1



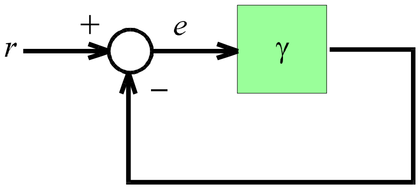


Loop gain

$$\gamma = \psi \circ \phi$$

Signal balance

$$e = r - \gamma(e)$$



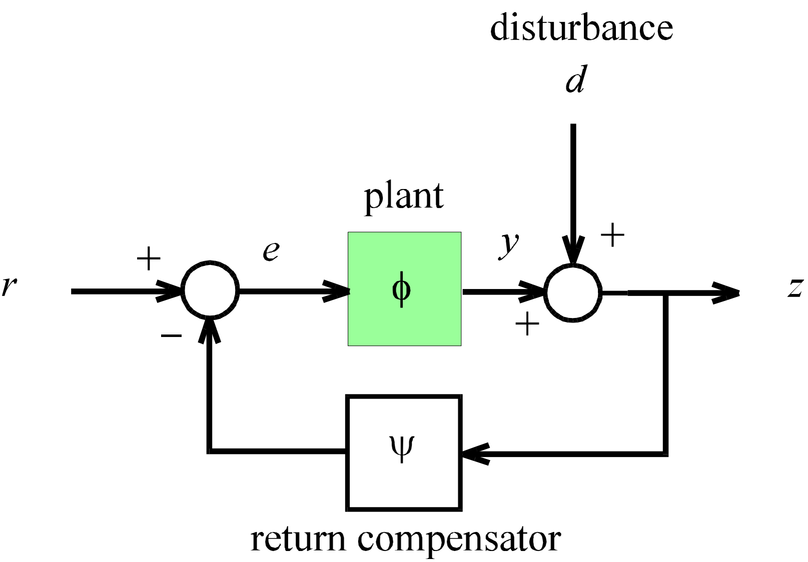
Feedback equation

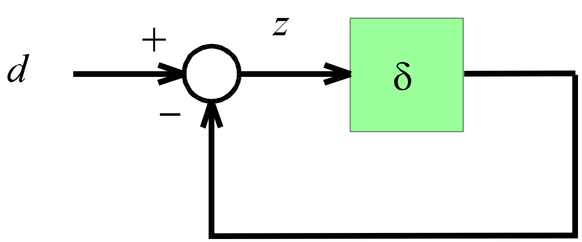
$$e + \gamma(e) = r$$

High-gain feedback-2

Feedback equation	$e + \gamma(e) = r$
High gain:	$\|\gamma(e)\| >> \|e\|$
Implies:	$\gamma(e) \approx r$
Hence:	$\|e\| << \|r\| \quad \Rightarrow \quad r \approx \psi(y)$
So that	$y \approx \psi^{-1}(r)$
In case of unit feedback	$y \approx r$ ***Good tracking***

High-gain feedback-3





Loop gain $\delta = (-\phi) \circ (-\psi)$

Signal balance: $z = d - \delta(z)$

High gain: $|\,\delta(z)\,| >> |\,z\,|$

Hence: $|\,z\,| << |\,d\,|$

Good disturbance reduction

High-gain feedback-4

Need *closed-loop stability*

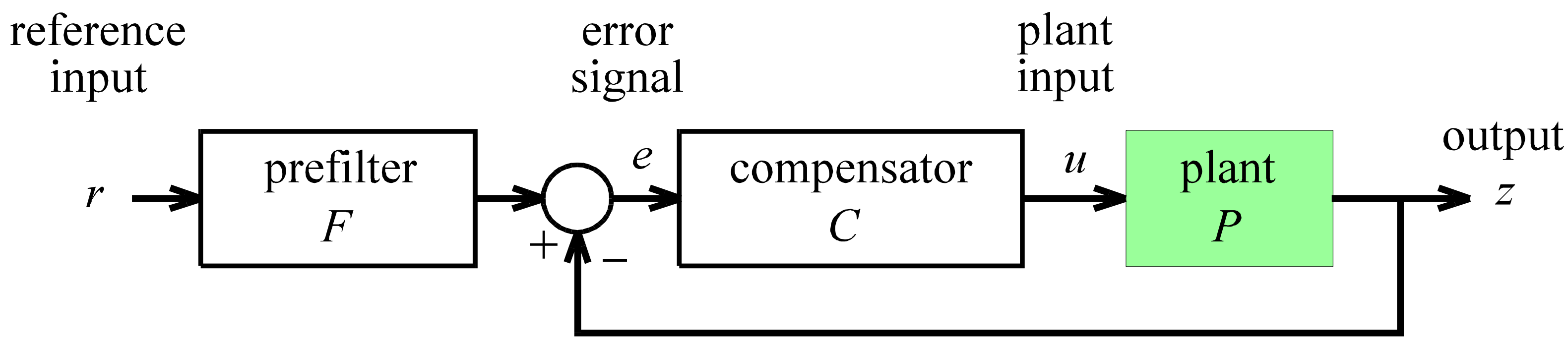
- Good tracking and disturbance attentuation are retained as long as
- the closed-loop system remains stable
- the gain remains high

Under these conditions high-gain feedback implies **robustness** with respect to loop uncertainty

Pitfalls of high-gain feedback

- High-gain feedback has pitfalls:
- Naively making the gain large easily results in an *unstable* feedback system
- Even if the feedback system is stable overly large plant inputs may occur that exceed the plant capacity
- Measurement noise causes loss of performance

Stability-1



State space representation:

$$\dot{x}(t) = Ax(t) + Br(t)$$

$$\begin{bmatrix} e(t) \\ u(t) \\ z(t) \end{bmatrix} = Cx(t) + Dr(t)$$

Stability-2

$$\dot{x}(t) = Ax(t) + Br(t)$$

$$\begin{bmatrix} e(t) \\ u(t) \\ z(t) \end{bmatrix} = Cx(t) + Dr(t)$$

The closed-loop system is *stable* if its state space representation is *asymptotically stable*

Equivalent statements:

- $x(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$ for every solution of $\dot{x}(t) = Ax(t)$
- All eigenvalues of A have strictly negative real parts
- All roots of $\det(sI - A)$ have strictly negative real parts

Stability-3

The control system is *BIBO stable* if every bounded input signal r results in bounded output signals e, u and z.

BIBO = "bounded input bounded output"

Asymptotic stability $\Rightarrow$ BIBO stability

The converse is *not* true

Stability-4

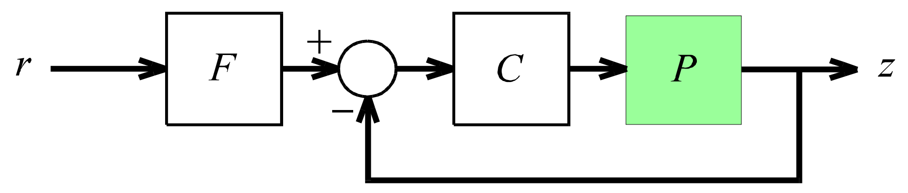
Internal stability

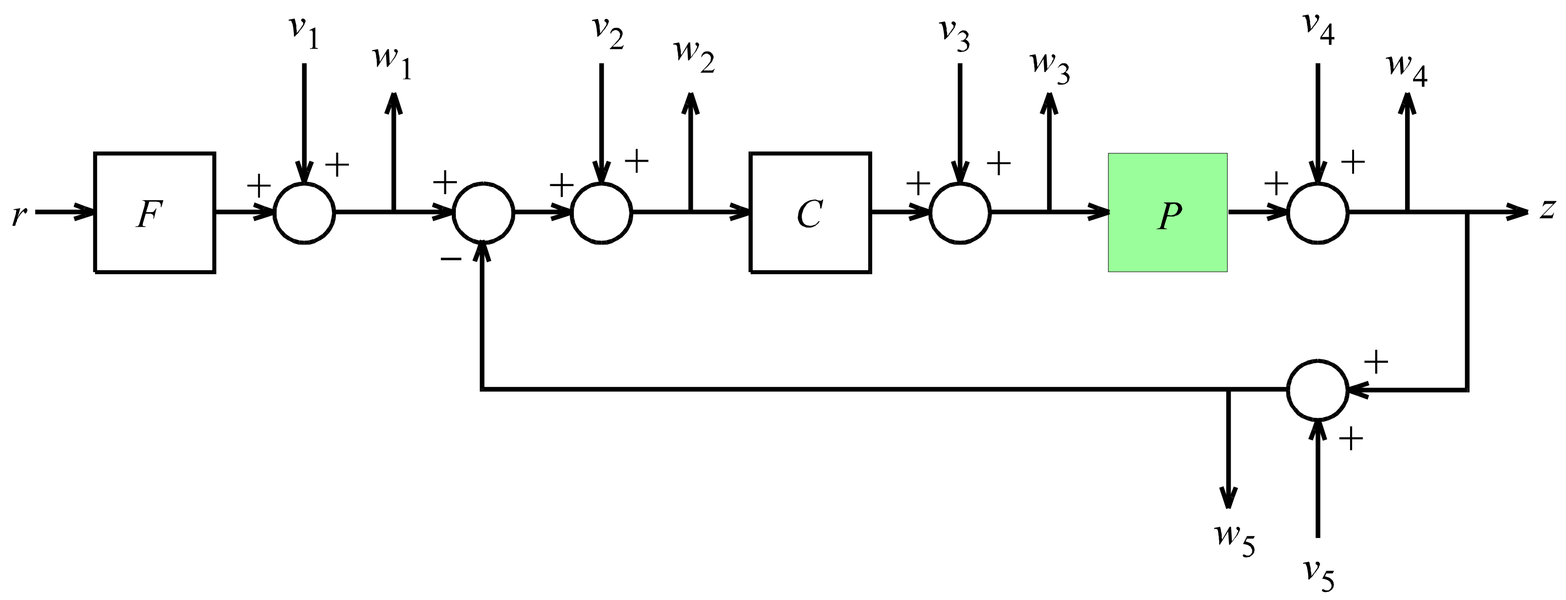
Inject "internal" signals into each "exposed interconnection" of the system, and define additional "internal" output signals after each injection point

Then the system is *internally stable* if it is BIBO stable with respect to all inputs (external and internal) and all (external and internal) outputs

Stability-5

Example





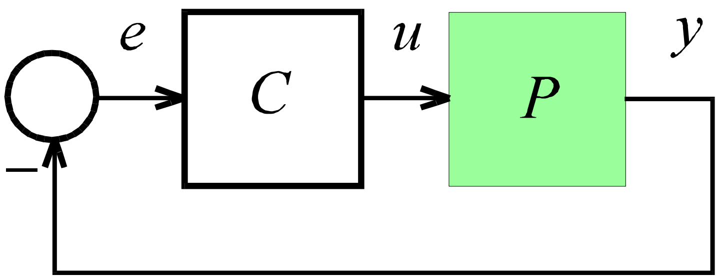
Stability-6

If each component system is stabilizable and detectable ("has no hidden unstable modes") then

Stability $\Leftrightarrow$ Internal stability

When input-output descriptions are used (such as transfer functions) internal stability is often easier to check than asymptotic stability

Closed-loop characteristic polynomial-1



State space representation of the open-loop system:

$$\dot{x}(t) = Ax(t) + Be(t)$$
$$y(t) = Cx(t) + De(t)$$

Characteristic polynomial:

$$\chi(s) = \det(sI - A)$$

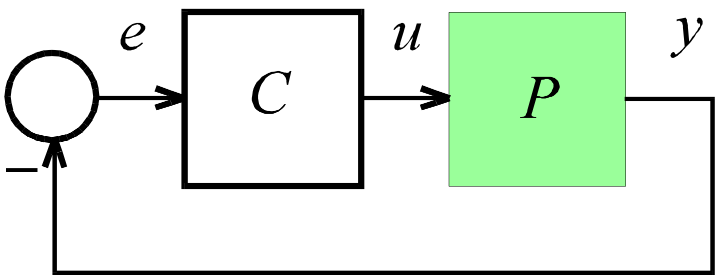
State space representation of the closed-loop system:

$$\dot{x}(t) = A_{cl}\, x(t),$$
$$A_{cl} = A - B(I + D)^{-1} C$$

Characteristic polynomial:

$$\chi_{cl}(s) = \det(sI - A_{cl})$$

Closed-loop characteristic polynomial-2



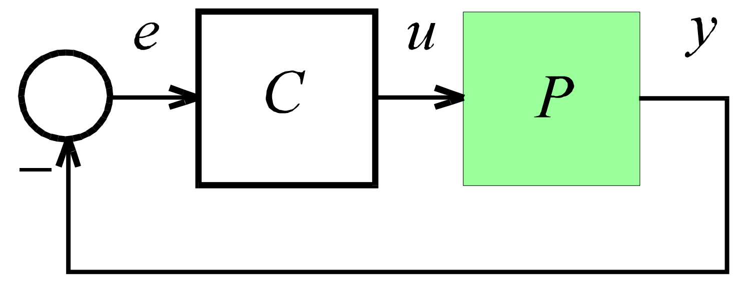
$P(s)$	plant transfer matrix
$C(s)$	compensator transfer matrix
$L(s)=P(s)C(s)$	loop gain transfer matrix

Then

$$\chi_{cl}(s) = \chi(s)\frac{\det(I+L(s))}{\det(I+L(\infty))}$$

Closed-loop characteristic polynomial-3

$$\chi_{cl}(s) = \chi(s)\frac{\det(I+L(s))}{\det(I+L(\infty))}$$



SISO case: $$L(s) = P(s)C(s) = \frac{N(s)}{D(s)}\cdot\frac{Y(s)}{X(s)}$$

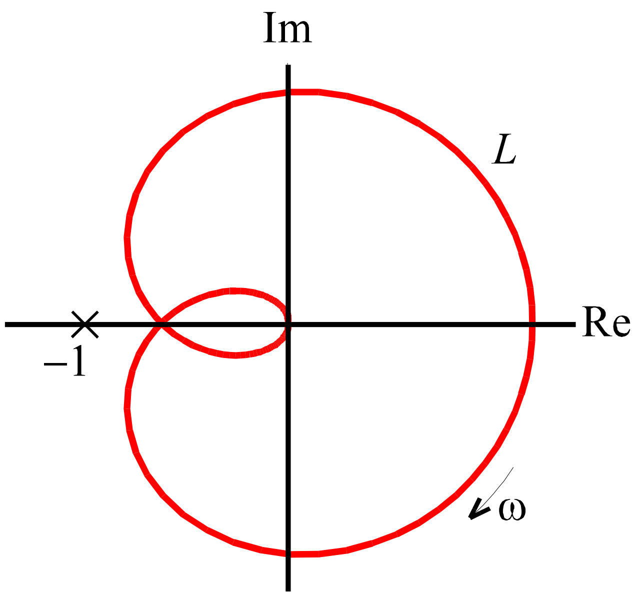
Then (within a nonzero constant factor)

$$\chi(s) = D(s)X(s) \qquad \chi_{cl}(s) = D(s)X(s) + N(s)Y(s)$$

Nyquist criterion

Consider the SISO case

The locus of $L(j\omega), \quad \omega \in \mathbb{R}$ in the complex plane is called the *Nyquist plot* of the loop gain



The number of unstable closed-loop poles

=

The number of times the Nyquist plot encircles the point –1

+

The number of unstable open-loop poles

Generalized Nyquist criterion

Consider the MIMO case

The number of unstable closed-loop poles

=

The number of times the locus of

$$\det(I + L(j\omega)), \quad \omega \in \mathrm{R}$$

encircles the origin

+

The number of unstable open-loop poles

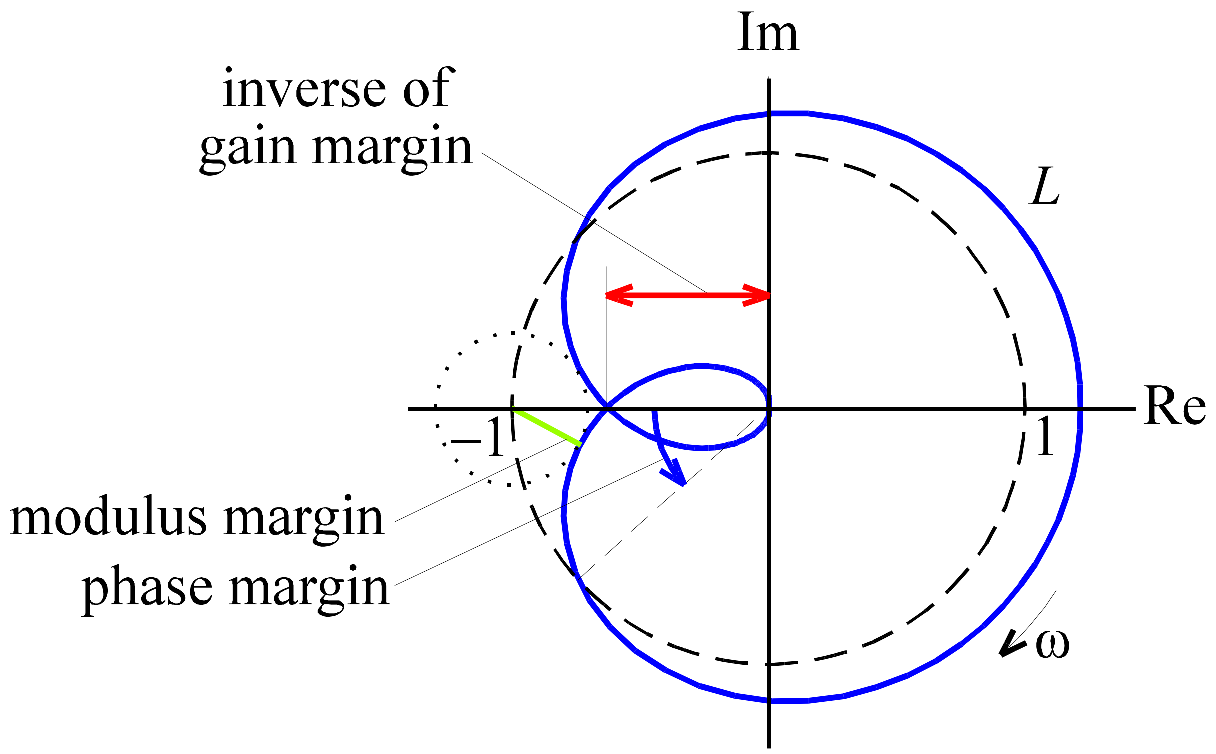
(Principle of the argument)

Stability margins-1

- In the SISO case, the point –1 is a *critical point* for the Nyquist plot of the closed-loop system. If the Nyquist plot is changed so that it crosses the point –1 then the system becomes unstable
- If the closed-loop system is stable but the Nyquist plot passes closely by –1 then
- the system is near-unstable, that is, has an oscillatory response
- the system may become unstable by small perturbations of the plant, that is, the system is not robust

Stability margins-2

There exist various *stability margins.* They measure how close the Nyquist plot gets to –1

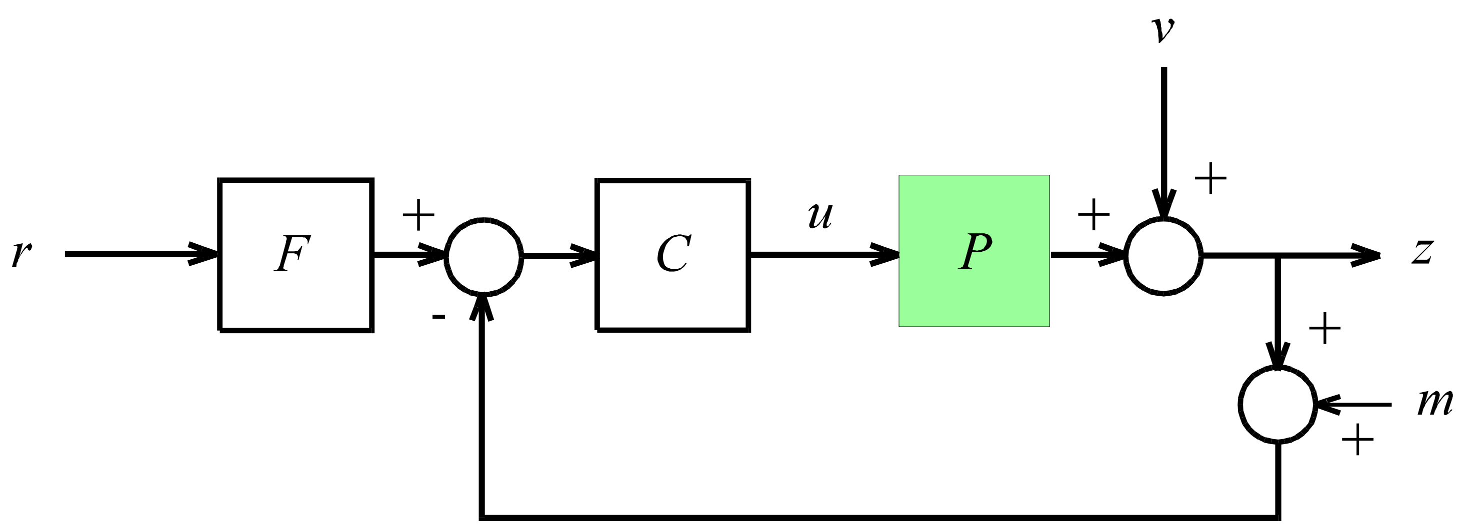


Gain margin k_m

Phase margin ϕ_m

Modulus margin τ_m
(Landau)

System functions: *L* and *S*



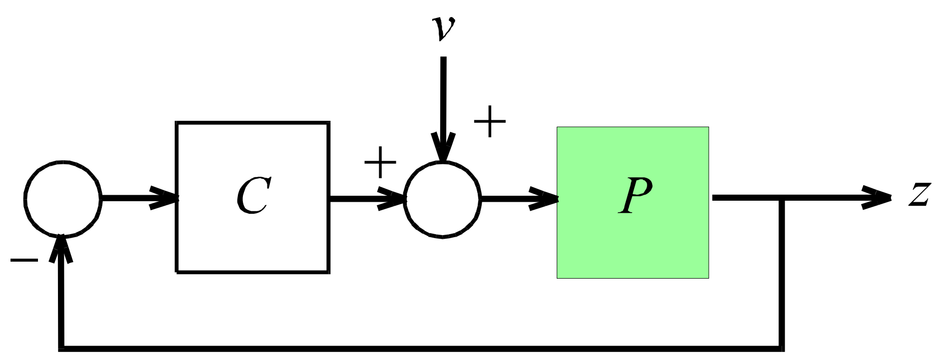
Loop gain L

Sensitivity function S

$$L = PC$$

$$z = \underbrace{\frac{1}{1+L}}_{S} v$$

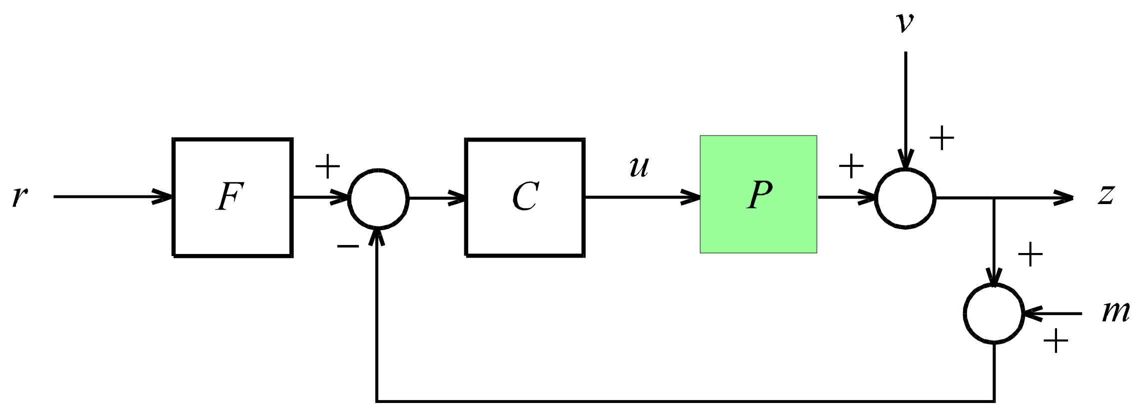
System functions: *R*



Input disturbance ('proces') sensitivity function R

$$z = \frac{1}{1+L} P v = \underbrace{SP}_{R} r$$

System functions: *H* and *T*



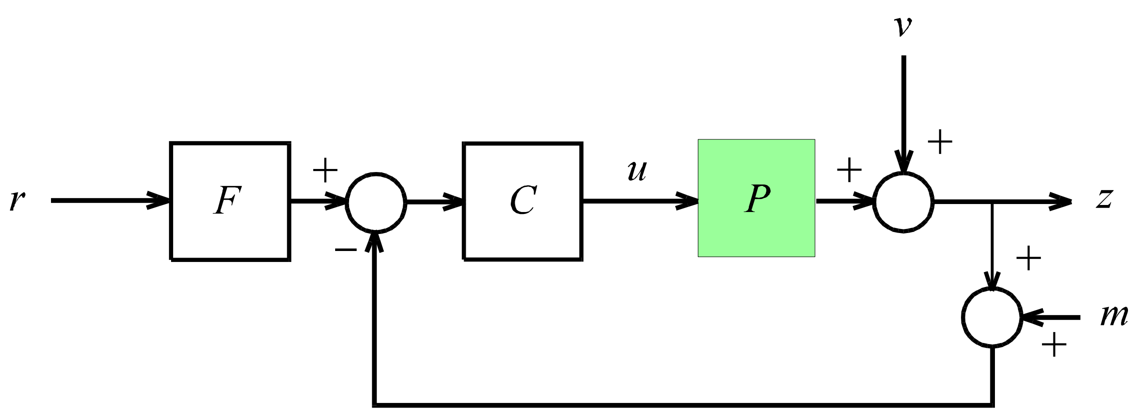
Closed-loop transfer function H

$$z = \underbrace{\frac{L}{1+L}F}_{H}\, r$$

Complementary sensitivity function T

$$H = \underbrace{\frac{L}{1+L}}_{T} F$$

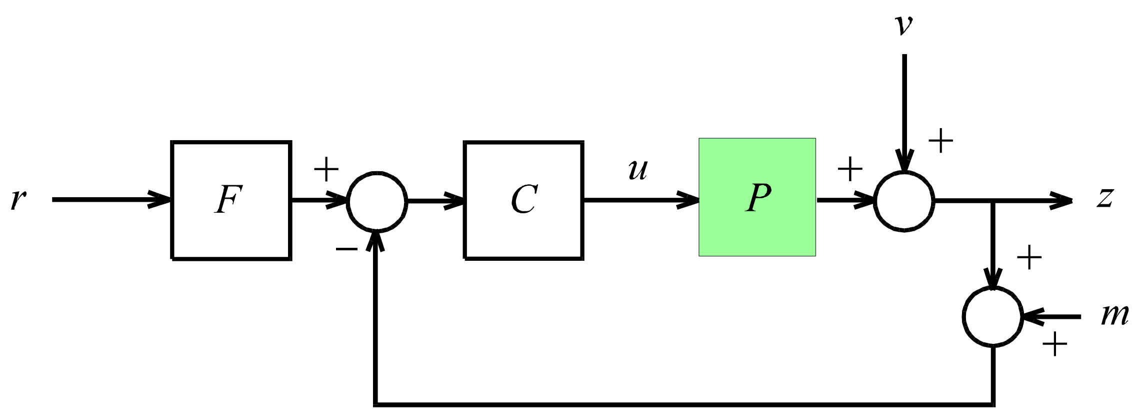
System functions: *U*



Input ('control') sensitivity function U

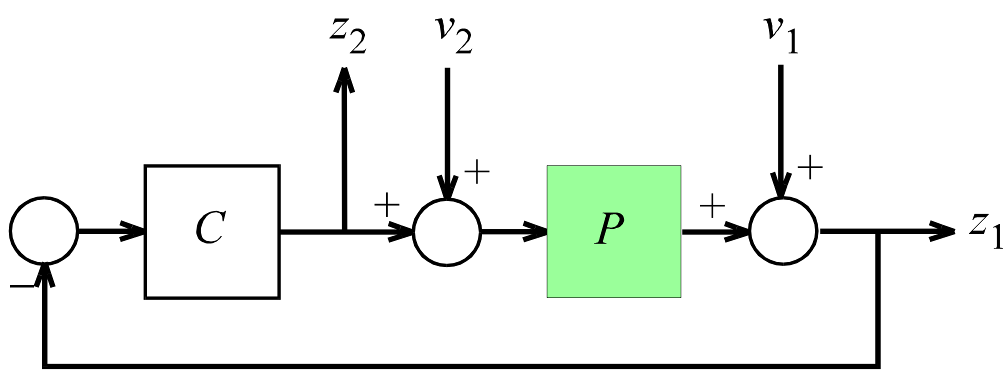
$$u = \underbrace{\frac{C}{1+CP}}_{U}(Fr - m - v)$$

Measurement noise



$$z = \underbrace{\frac{1}{1+PC}}_{S} v + \underbrace{\frac{PC}{1+PC}}_{T} Fr - \underbrace{\frac{PC}{1+PC}}_{T} m$$

S, R, U and T



$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+PC} & \dfrac{P}{1+PC} \\ -\dfrac{C}{1+PC} & -\dfrac{PC}{1+PC} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} S & R \\ -U & -T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Design interrelations

$$S = \frac{1}{1+L}$$

$$T = \frac{L}{1+L}$$

$$U = T/P$$

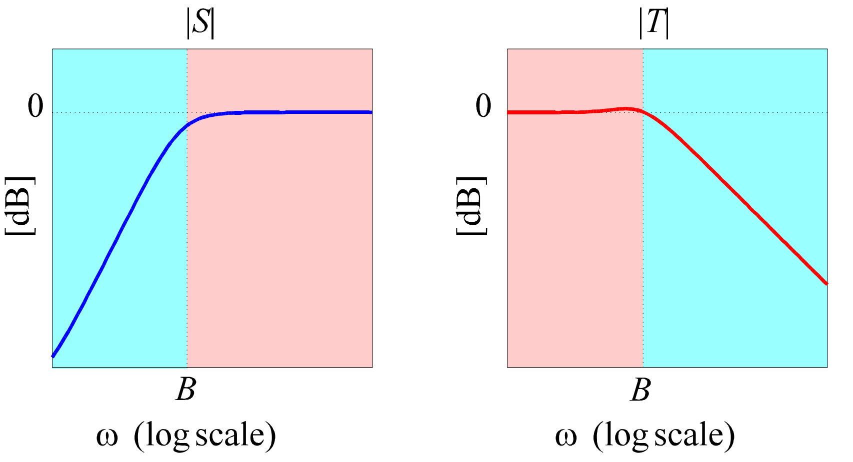
$$R = SP$$

$$H = TF$$

S and T are suitable objects for manipulation

Low and high frequencies-1

Typical shapes for S and T



	S	T
low frequencies	small	$\cong 1$
high frequencies	$\cong 1$	small

Low and high frequencies-2

Loop gain L

- large at low frequencies: $|L(j\omega)|>>1, \quad S\approx 1/L, \quad T\approx 1$
- small at high frequencies: $|L(j\omega)|<<1, \quad S\approx 1, \quad T\approx L$

- *Crossover* region: $|L(j\omega)|\approx 1$

Low and high frequencies-3

input sensitivity

$$U = T/P = \frac{C}{1+PC} \approx \begin{cases} 1/P & \text{for low frequencies} \\ C & \text{for high frequencies} \end{cases}$$

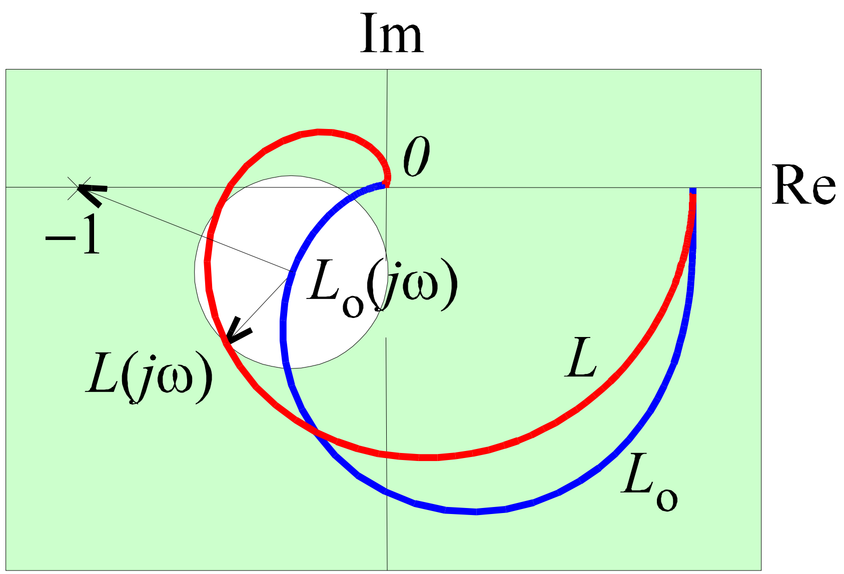
input disturbance sensitivity

$$R = SP = \frac{P}{1+PC} \approx \begin{cases} 1/C & \text{for low frequencies} \\ P & \text{for high frequencies} \end{cases}$$

closed-loop transfer function

$H = TF$ F corrects T

Robustness functions-1



Sufficient condition for stability under perturbation:

$$\left|L(j\omega) - L_o(j\omega)\right| < \left|1 + L_o(j\omega)\right|,$$

$$\omega \in \mathbb{R}$$

Robustness functions-2

Equivalently,

$$\left|\frac{L(j\omega)-L_o(j\omega)}{L_o(j\omega)}\right| < \left|\frac{1+L_o(j\omega)}{L_o(j\omega)}\right|, \quad \omega \in \mathbb{R}$$

or

$$\left|\frac{L(j\omega)-L_o(j\omega)}{L_o(j\omega)}\right| < \frac{1}{\left|T_o(j\omega)\right|}, \quad \omega \in \mathbb{R}$$

Robustness functions-3

Bound on the relative size of perturbations:

$$\left|\frac{L(j\omega)-L_o(j\omega)}{L_o(j\omega)}\right| \leq |W_1(j\omega)|, \quad \omega \in \mathbb{R}$$

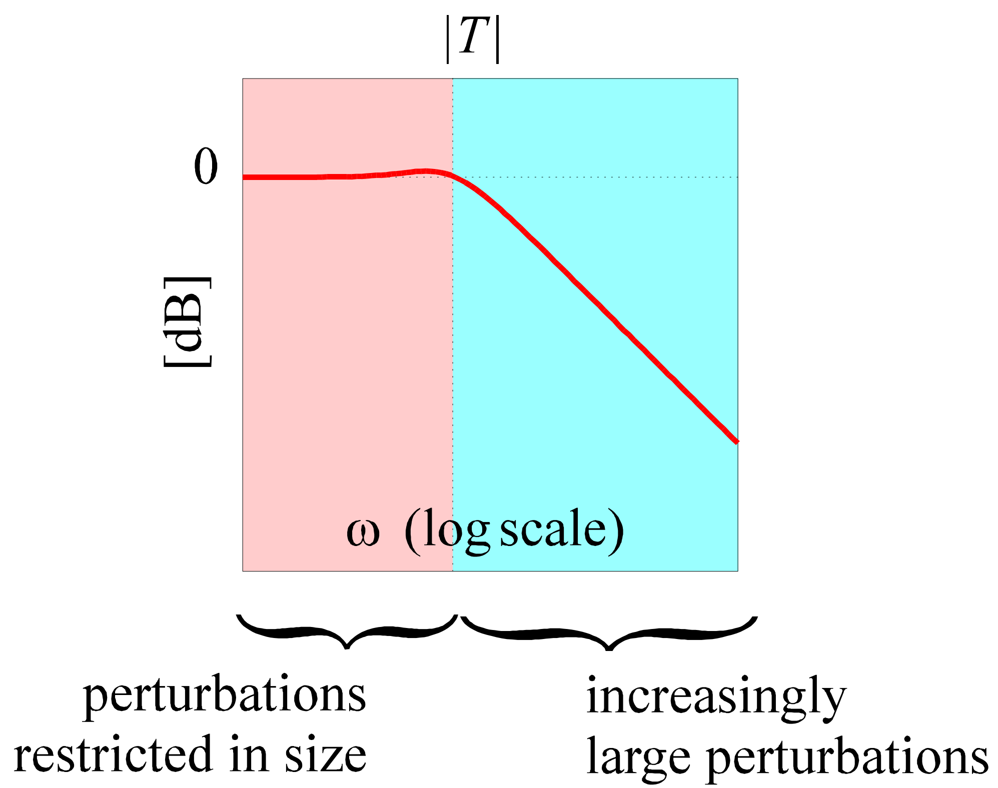
Sufficient and necessary condition for stability under all perturbations that satisfy the bound:

$$|W_1(j\omega)| < \frac{1}{|T_o(j\omega)|}, \quad \omega \in \mathbb{R}$$

Robustness functions-4

Size of the smallest perturbation that may destabilize the system:

$$|W_1(j\omega)| = \frac{1}{|T_o(j\omega)|}, \quad \omega \in \mathbb{R}$$



Robustness functions-5

The preceding discussion focuses on preventing the Nyquist plot of the loop gain L from crossing the point -1. Preventing the *inverse Nyquist plot* – that is, the Nyquist plot of $1/L$ – from crossing the point -1 also guarantees stability.

Sufficient condition:

$$\left| \frac{1}{L(j\omega)} - \frac{1}{L_o(j\omega)} \right| < \left| 1 + \frac{1}{L_o(j\omega)} \right|, \quad \omega \in \mathbb{R}$$

Robustness functions-6

Equivalently,

$$\left| \frac{\frac{1}{L(j\omega)} - \frac{1}{L_o(j\omega)}}{\frac{1}{L_o(j\omega)}} \right| < \frac{1}{|S_o(j\omega)|}, \quad \omega \in \mathbb{R}$$

Robustness functions-7

Consider perturbations such that

$$\left| \frac{\dfrac{1}{L(j\omega)} - \dfrac{1}{L_o(j\omega)}}{\dfrac{1}{L_o(j\omega)}} \right| \leq \left| W_2(j\omega) \right|, \quad \omega \in \mathbb{R}$$

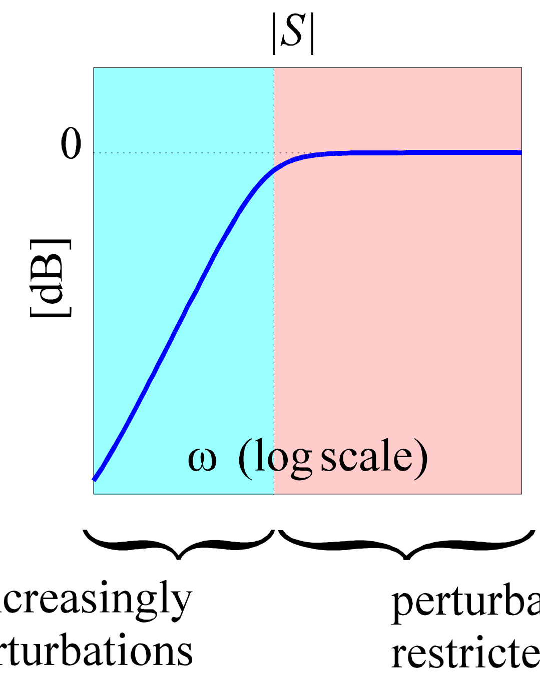
Sufficient and necessary condition for robust stability:

$$\left| W_2(j\omega) \right| < \frac{1}{\left| S_o(j\omega) \right|}, \quad \omega \in \mathbb{R}$$

Robustness functions-8

Size of the smallest perturbation that may destabilize the system:

$$|W_2(j\omega)| = \frac{1}{|S_o(j\omega)|}, \quad \omega \in \mathbb{R}$$



Combined robustness test-1

Define

$$\delta_L(j\omega) = \left|\frac{L(j\omega) - L_o(j\omega)}{L_o(j\omega)}\right|$$

$$\delta_{L^{-1}}(j\omega) = \left|\frac{\dfrac{1}{L(j\omega)} - \dfrac{1}{L_o(j\omega)}}{\dfrac{1}{L_o(j\omega)}}\right|$$

Combined robustness test-2

Then the perturbed closed-loop system is stable if

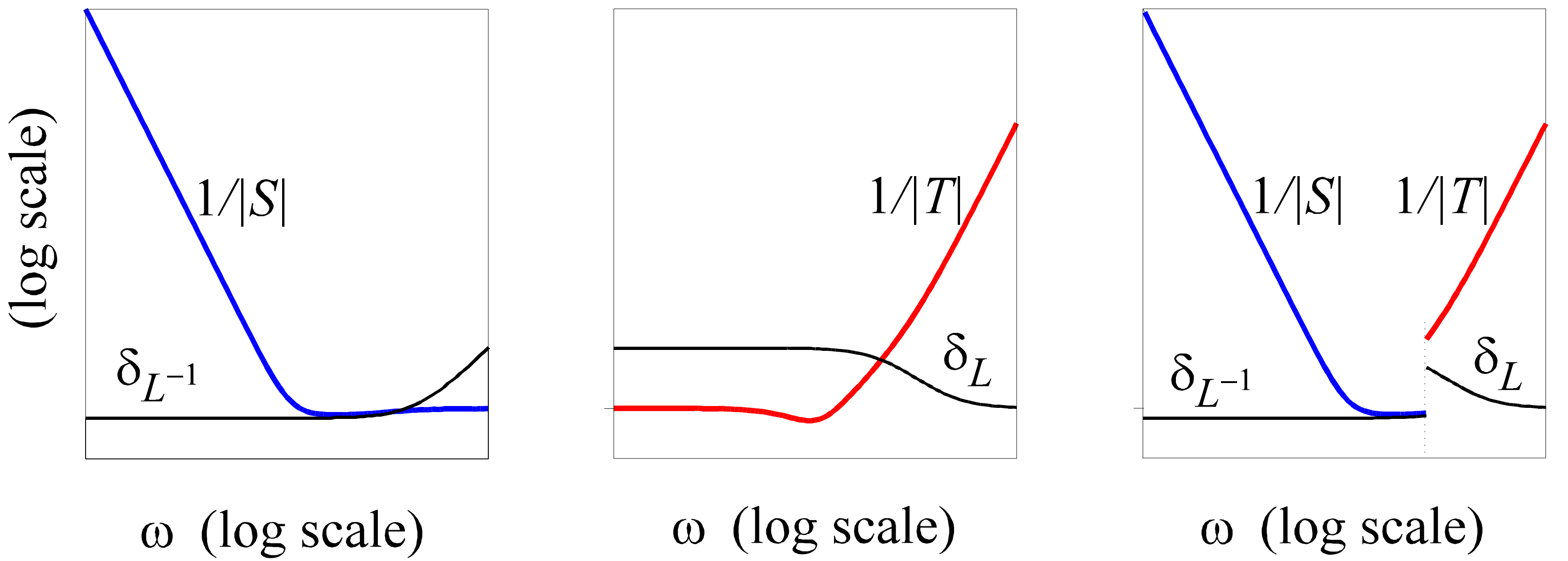
$\forall \omega \in \mathbb{R}$

$$\left|\delta_{L^{-1}}(j\omega)\right| < \frac{1}{\left|S(j\omega)\right|} \qquad \textbf{\textit{or}} \qquad \left|\delta_L(j\omega)\right| < \frac{1}{\left|T(j\omega)\right|}$$

typically satisfied at ***low*** frequencies

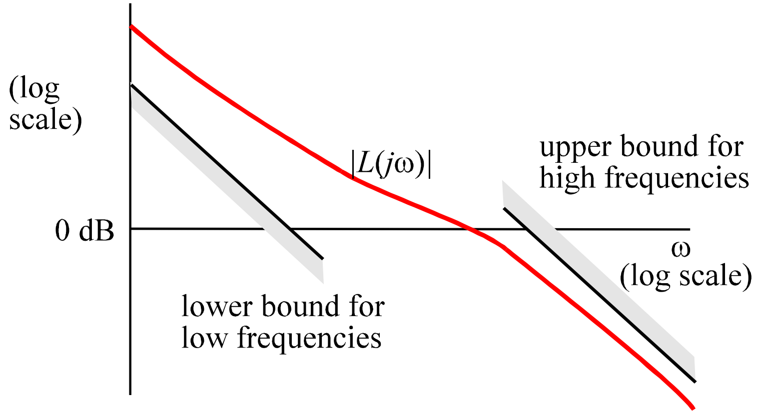
typically satisfied at ***high*** frequencies

Combined robustness test-3



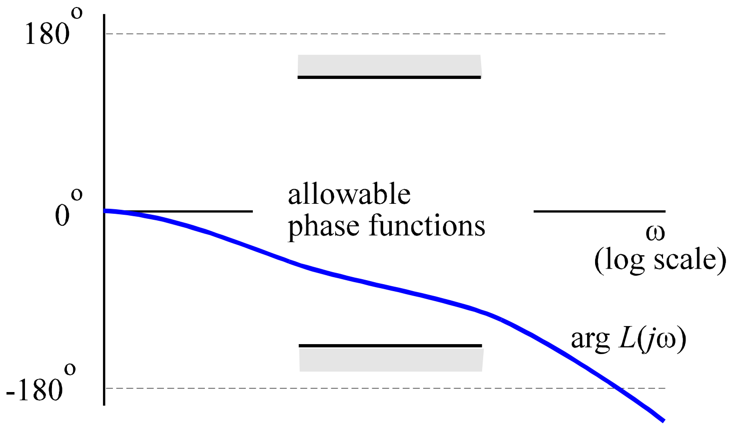
Critical frequency region: crossover area

Loop shaping



Low frequencies: large loop gain

High frequencies: small loop gain



In the crossover region the phase is constrained because of stability

Bode's gain-phase relationship-1

Between break frequencies the loop gain behaves as

$$L(j\omega) \approx c(j\omega)^n$$

Hence

$$|L(j\omega)| \approx c\omega^n$$

$$\arg L(j\omega) \approx n \times \frac{\pi}{2}$$

Phase and magnitude do not behave independently

Bode's gain-phase relationship describes the relation more accurately

Bode's gain-phase relationship-2

The limitations imposed by stability on the phase in the crossover region by Bode's gain-phase relationship limit the rate at which the loop gain decreases:

If, say,

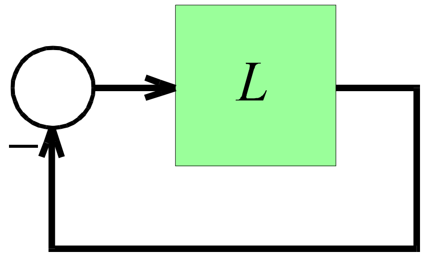
$$\arg L(j\omega) \approx -\frac{\pi}{2} \quad \text{in the crossover region}$$

then

$$|L(j\omega)| \approx c\,\omega^{-1} \quad \text{in the crossover region}$$

Limits of performance

- Bode's integral
- The Freudenberg-Looze equalities



Limitations are imposed by

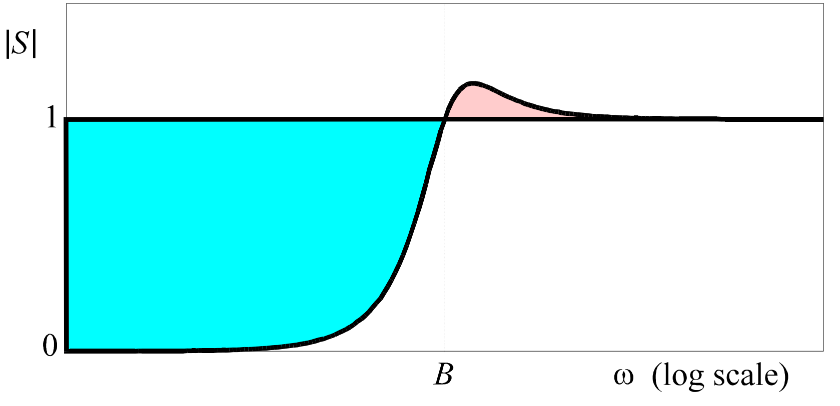
- causality
- the pole-zero configuration

Bode's integral-1

If L has at least two more poles than zeros then

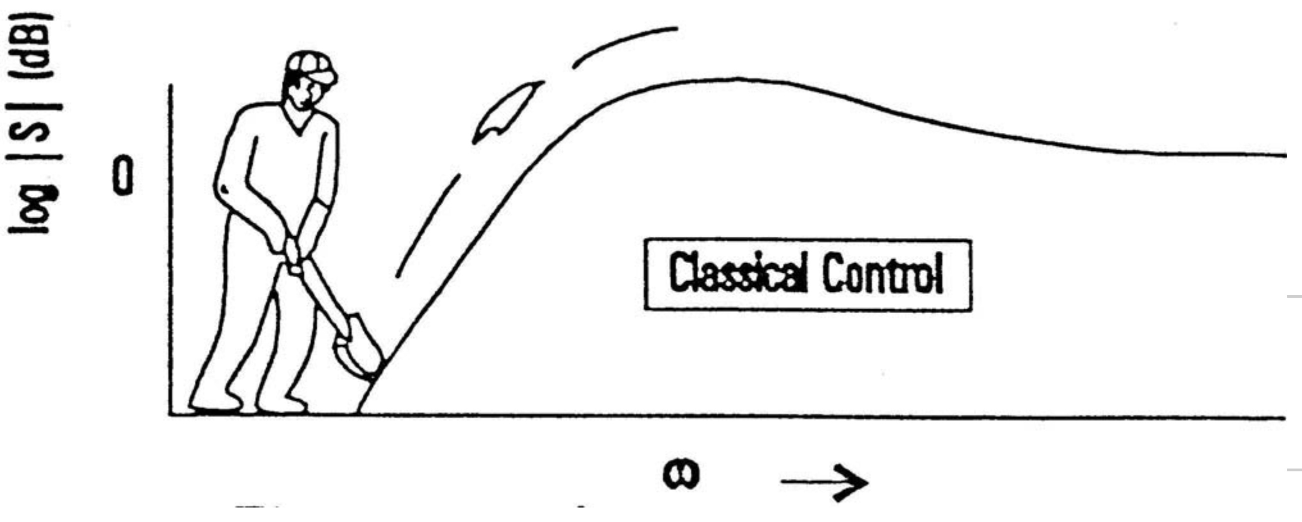
$$\int_0^\infty \log|S(j\omega)|\,d\omega = \pi \sum_i \mathrm{Re}\, p_i \geq 0$$

The p_i are the right-half plane poles of the loop gain.

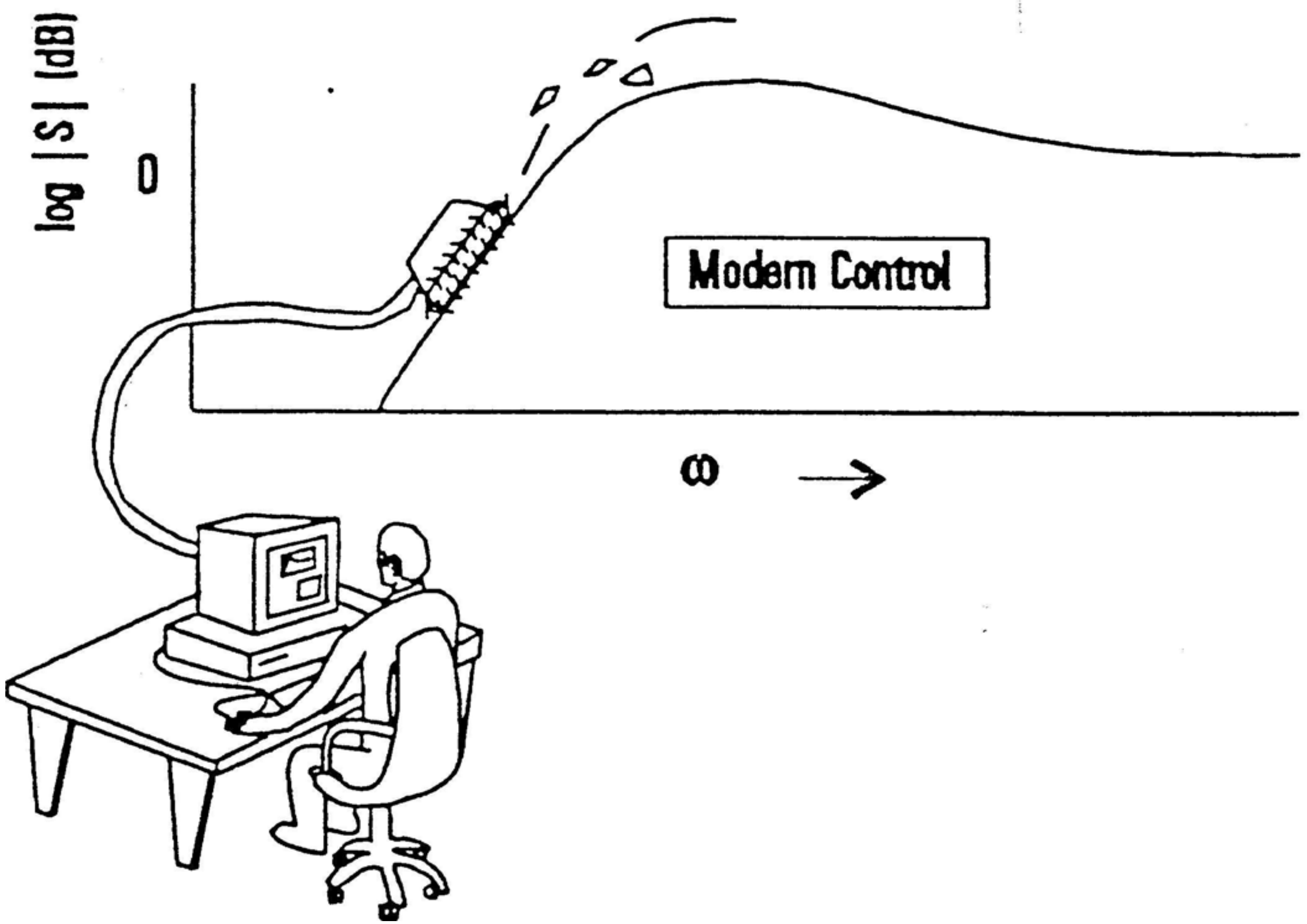


Proof: Use the Poisson integral from complex function theory

Bode's integral-1



Bode's integral-1

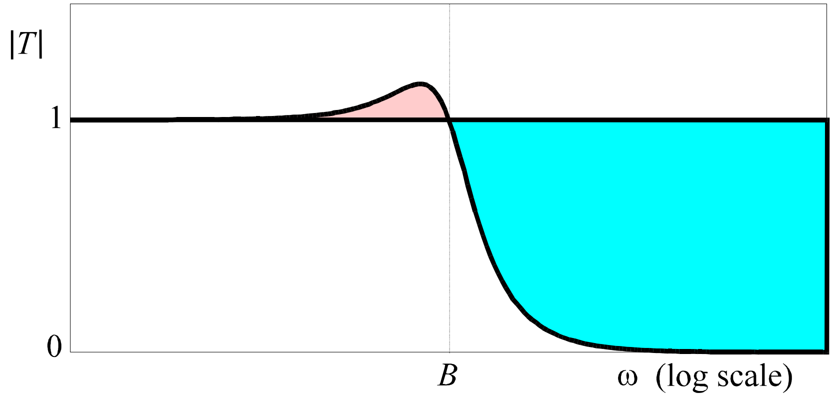


Bode's integral-2

"Dual" result: Suppose that the loop has integrating action of at least order 2. Then

$$\int_0^\infty \log\left|T(1/j\omega)\right| d\omega = \pi \sum_i \operatorname{Re}\frac{1}{z_i} \geq 0$$

The z_i are the right-half plane zeros of the loop gain.



Freudenberg-Looze equality-1

Let z be any right-half plane zero of the loop gain. Poisson's formula of complex function theory leads to the equality

$$\int_0^\infty \log(|S(j\omega)|)\, dW_z(\omega) = \log\left|B_{\text{poles}}^{-1}(z)\right| \geq 0$$

Strengthens Bode's integral

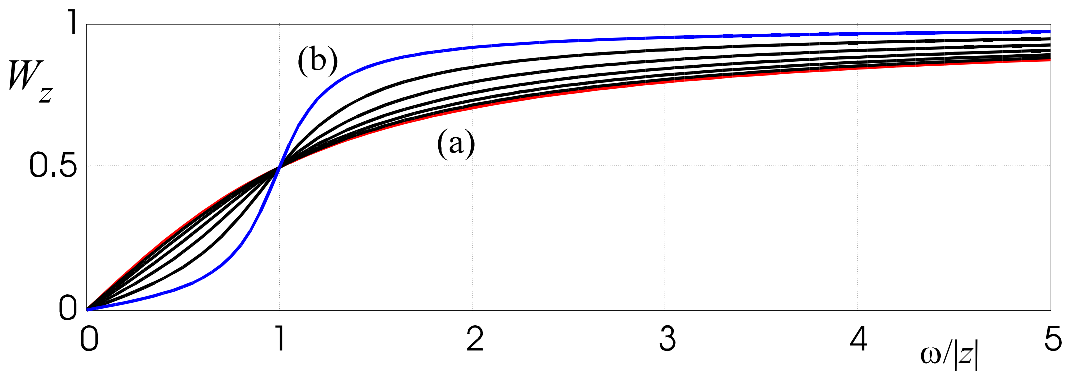
$$W_z(\omega) = \frac{1}{\pi}\arctan\frac{\omega - \operatorname{Im} z}{\operatorname{Re} z} + \frac{1}{\pi}\arctan\frac{\omega + \operatorname{Im} z}{\operatorname{Re} z}$$

Increasing function.
Rises most steeply at |z|.

$$B_{\text{poles}}(s) = \prod_i \frac{p_i - s}{\overline{p}_i + s}$$

Blaschke product

Freudenberg-Looze equality-2



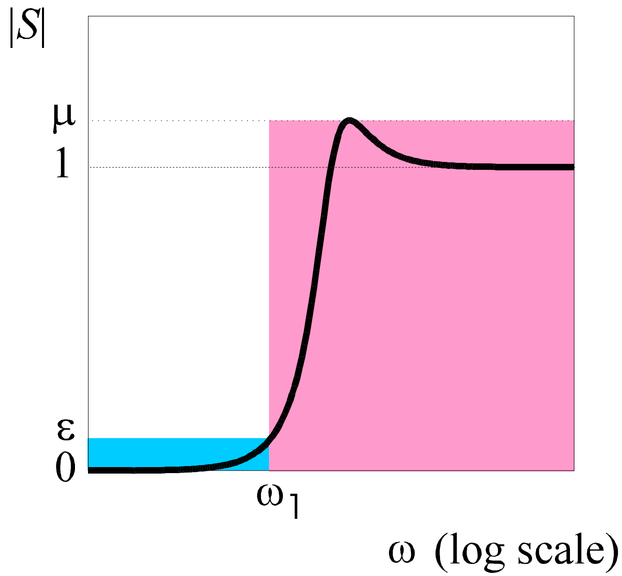
W_z for different values of $\arg z$

(a) $\arg z = 0$

(b) $\arg z$ is almost $\pi/2$

Frequencies where W_z rises most steeply contribute most to the integral

Freudenberg-Looze equality-3



The bounds for $|S|$ hold provided

$$\mu \geq \left(\frac{1}{\varepsilon}\right)^{\frac{W_z(\omega_1)}{1-W_z(\omega_1)}} \cdot \left|B_{\text{poles}}^{-1}(z)\right|^{\frac{1}{1-W_z(\omega_1)}}$$

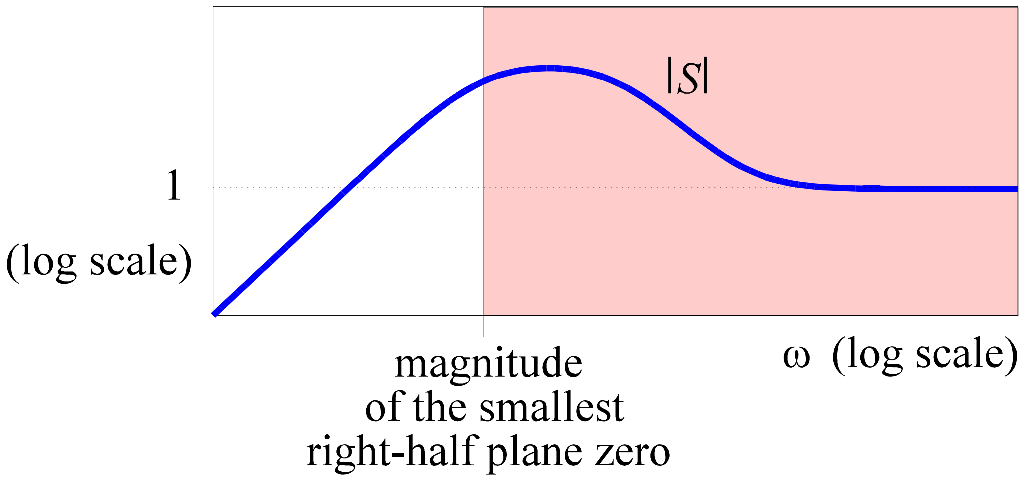
The dependence of the right-hand side on the various parameters may be analyzed

Freudenberg-Looze equality-4

Effects of right-half plane zeros on S

$|S|$ may be made small up to the frequency $\min_i |z_i|$. Attempting to make $|S|$ small beyond this frequency makes $|S|$ peak

- Right-half plane poles further impair the achievable reduction of $|S|$ (in particular, nearly-cancelling right-half plane pole-zero pairs)



Freudenberg-Looze equality-5

Rederivation of the Freudenberg-Looze equality while

- replacing L with $1/L$, so that $S = \dfrac{1}{1+L} \rightarrow \dfrac{1}{1+\dfrac{1}{L}} = \dfrac{L}{1+L} = T$
- interchanging the roles of the poles and the zeros

leads to

$$\int_0^{\infty} \log(|T(j\omega)|)\, dW_p(\omega) = \log\left|B_{\text{zeros}}^{-1}(p)\right| \geq 0$$

Freudenberg-Looze equality-6

$$\int_0^\infty \log(|T(j\omega)|)\, dW_p(\omega) = \log\left|B_{\text{zeros}}^{-1}(p)\right| \geq 0$$

p is any right-half plane pole of the loop gain, and

$$B_{\text{zeros}}(s) = \prod_i \frac{z_i - s}{\bar{z}_i + s}$$

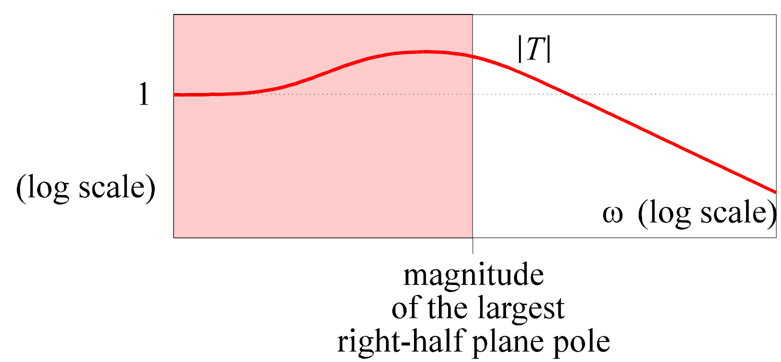
In the application of the equality, interchange the roles of *low* and *high* frequencies

Freudenberg-Looze equality-7

Effects of right-half plane poles on T

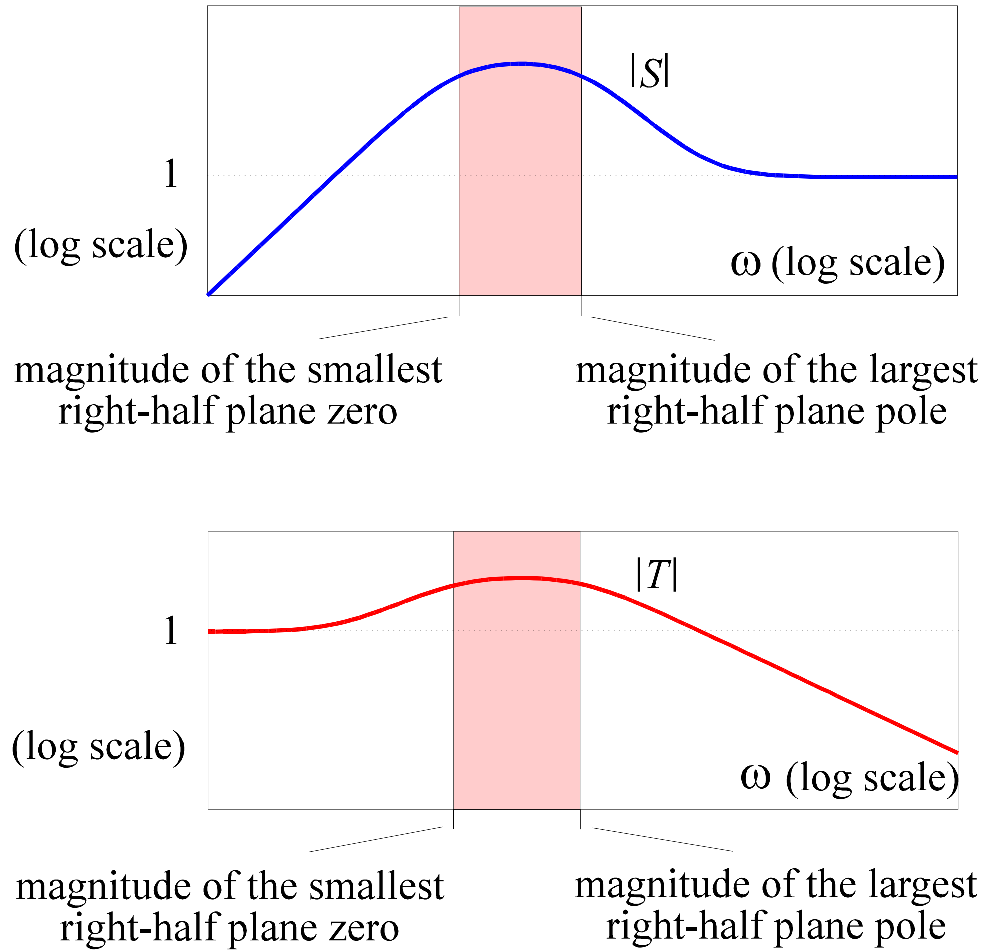
$|T|$ may be made small above the frequency $\max_i |p_i|$. Attempting to make $|T|$ small below this frequency makes $|T|$ peak

- Right-half plane zeros further impair the achievable reduction of $|T|$ (in particular, nearly-cancelling right-half plane pole-zero pairs)

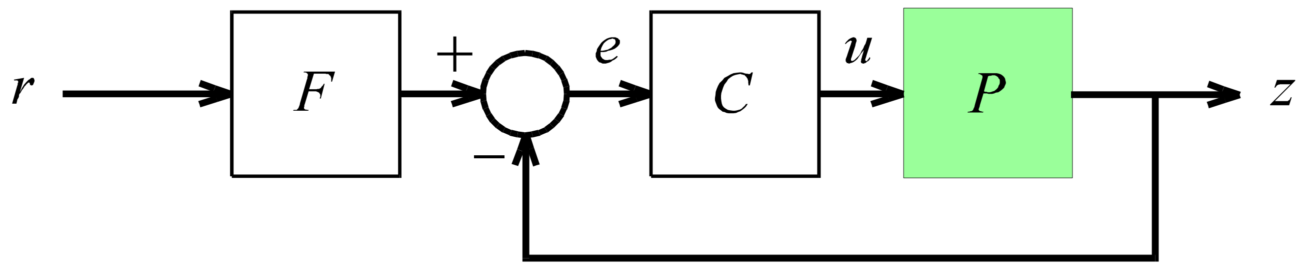


Freudenberg-Looze equality-8

Consequences for S and T



Two-degree-of-freedom systems-1



Let $P = \dfrac{N}{D}, \quad C = \dfrac{Y}{X}$

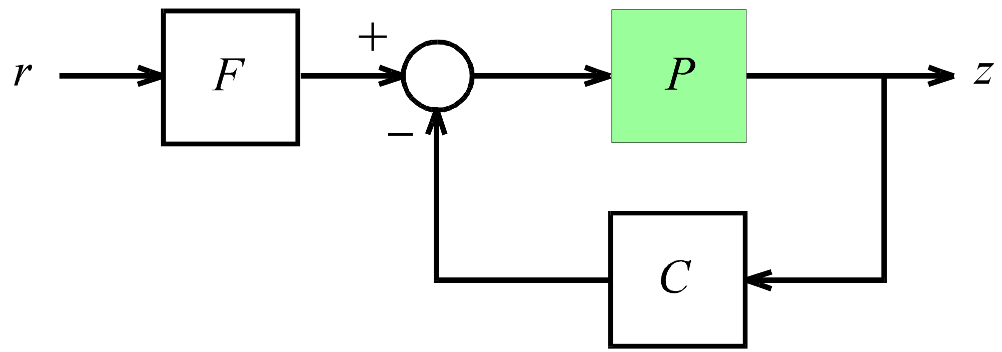
Then the closed-loop transfer function is

$$H = \frac{PC}{1+PC}F = \frac{NY}{D_{\mathrm{cl}}}F \qquad D_{\mathrm{cl}} = DX + NY$$

with D_{cl} the closed-loop characteristic polynomial

Two-degree-of-freedom systems-2

Other two-degree-of-freedom configuration:



$$H = \frac{NX}{D_{\mathrm{cl}}} F$$

has zeros at the roots of N and X

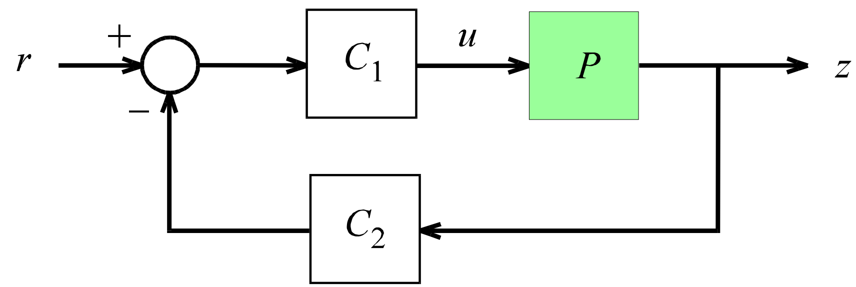
$$H = \frac{NY}{D_{\mathrm{cl}}} F$$

has zeros at the roots of N and Y

Can the zeros of H be made independent of the feedback compensator?

Two-degree-of-freedom systems-3

Further two-degree-of-freedom configuration



$$P = \frac{N}{D}, \quad C_1 = \frac{Y_1}{X_1} \quad C_2 = \frac{Y_2}{X_2}$$

Need $X_1 X_2 = X, \quad Y_1 Y_2 = Y$

to achieve the same loop gain as in the two previous cases

Have

$$H = \frac{PC_1}{1+PC} = \frac{N X_2 Y_1}{D_{\text{cl}}}$$

Two-degree-of-freedom systems-4

$$H = \frac{NX_2Y_1}{D_{\mathrm{cl}}}$$

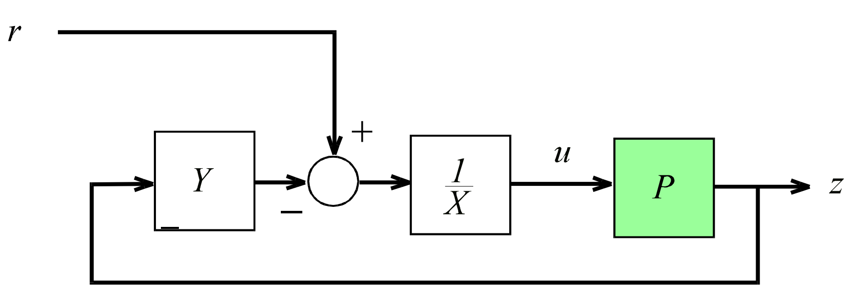
H is independent of the compensator if we let

$$Y_1 = X_2 = 1 \quad \Rightarrow \quad Y_2 = Y, \quad X_1 = X$$

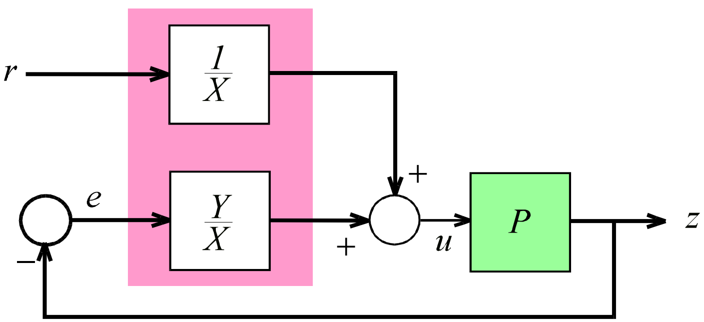
so that

$$C_1 = \frac{1}{X}, \quad C_2 = Y, \quad H = \frac{N}{D_{\mathrm{cl}}}$$

Two-degree-of-freedom systems-5



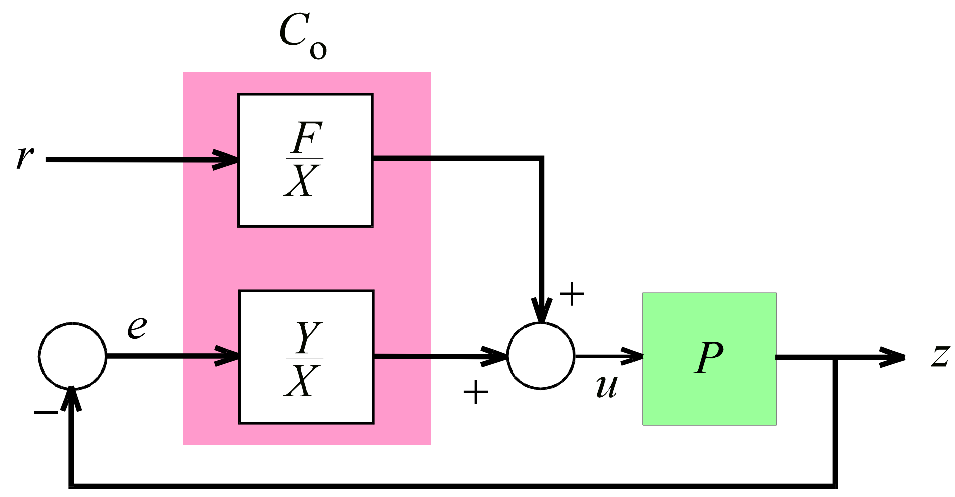
Resulting feedback system



Equivalent configuration

Two-degree-of-freedom systems-6

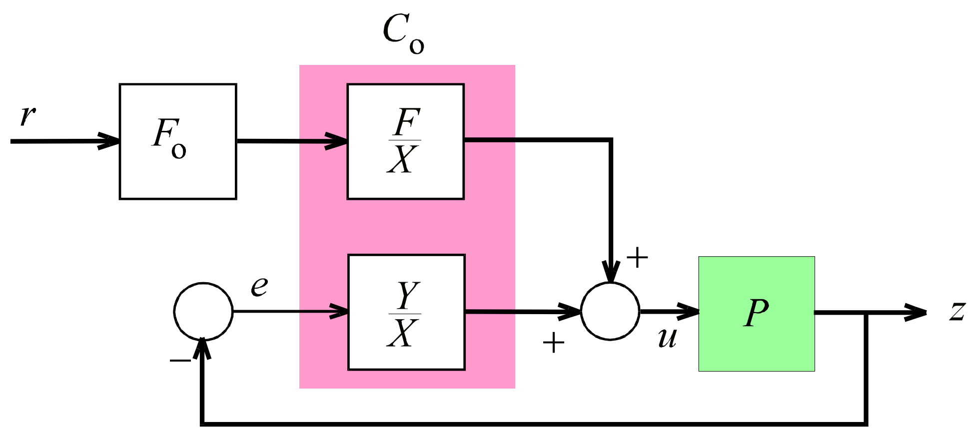
Extension



May choose F polynomial so that we obtain a "1½-degree-of-freedom" system

Two-degree-of-freedom systems-7

Further extension:



F polynomial, F_o rational:

“2½-degree-of-freedom” system

$$H = \frac{NF}{D_{cl}} F_o$$

$F = 1$, F_o rational:

2-degree-of-freedom system