Monography on

Hamiltonian formulation of EM

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Abstract

The study of constraints will be of great importance in developing a quantum theory for electrodynamics since the canonical prescription for quantization requires the correct Hamiltonian formalism of the classical theory. We start by reviewing the Lagrangian formalism and then we proceed carefully through the Hamiltonian formalism with constraints, as constructed by Dirac. Finally, we use such formalism to perform the canonical quantization of the electromagnetic free field on Coulomb's gauge.

Content

1

Constrained Hamiltonian systems

1.1 Lagrangian field theory

Let Ψ be a generic field in a D-dimensional space-time parametrized by coordinates $x = \{x^\mu\}_{\mu=0}^{D-1} = (t, x^1, x^2, x^3)$. It can be a scalar field $\phi(x)$, a vector field $A_\mu(x)$, the metric tensor field $g_{\mu\nu}(x)$ or a fermionic spin-$1/2$ field $\psi(x)$. The equations of motion for the field Ψ are determined by a Lagrangian (density) $\mathcal{L}$ which is a function of the field itself and is derivatives $\partial_\mu \Psi(x)$, noticing that $\partial_\mu \equiv \partial / \partial x^\mu = (\partial_0, \boldsymbol{\nabla})$ and is assumed to contain every relevant information about the physical system. Of course, different space-time regions may be described by different Lagrangians. Lets denote the space-time region where the Lagrangian is valid by $\mathcal{V}$. The action is defined as the space-time integral of the Lagrangian over $\mathcal{V}$, which means

$$S = \int\limits_{\mathcal{V}} d^D x \, \mathcal{L}\left(\Psi, \partial_\mu \Psi; x\right)$$

Hamilton's principle states that true field Ψ is the one for which $\delta S\left(\Psi\right) = 0$, which implies

$$\begin{aligned}
\delta S\left(\Psi\right) &= \int\limits_{\mathcal{V}} d^D x \left(\frac{\partial \mathcal{L}}{\partial \Psi} \, \delta \Psi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi\right)} \, \delta\left(\partial_\mu \Psi\right)\right) \\
&= \int\limits_{\mathcal{V}} d^D x \left(\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi\right)}\right) \delta \Psi + \int\limits_{\mathcal{V}} d^D x \, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi\right)} \, \delta \Psi\right) \\
&= \int\limits_{\mathcal{V}} d^D x \left(\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi\right)}\right) \delta \Psi + \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi\right)} \, \delta \Psi\right) \Bigg|_{\partial \mathcal{V}} = 0
\end{aligned} \tag{1.1}$$

If we now make the further assumption that $\delta\Psi = 0$ at $\partial\mathcal{V}$, then for (1.1) to be satisfied for all allowed $\delta\Psi$ it suffices that

$$\frac{\partial\mathcal{L}}{\partial\Psi} - \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi\right)} \right) = 0$$

which is the so called *Euler-Lagrange equation*.

1.2 Hamiltonian field theory

Lets define the conjugate momentum of Ψ, denoted as π, as

$$\pi = \frac{\partial\mathcal{L}}{\partial\left(\partial_0\Psi\right)}$$

then the *Hamiltonian density* $\mathcal{H}$ is defined as

$$\mathcal{H}\left(\Psi, \boldsymbol{\nabla}\Psi, \pi, \boldsymbol{\nabla}\pi; x\right) = \pi \cdot \partial_0\Psi - \mathcal{L}$$

and must always be expressed in terms of the field Ψ with its spatial derivatives $\boldsymbol{\nabla}\Psi$, and its conjugate momentum π with its spatial derivative $\boldsymbol{\nabla}\pi$. The full Hamiltonian H, defined as the spatial integral of $\mathcal{H}$, is then

$$H = \int\limits_{\mathcal{V}} d^{D-1}x\, \mathcal{H}$$

Lets write the action in terms of $\mathcal{H}$

$$S = \int\limits_{\mathcal{V}} d^D x \left(\pi \cdot \partial_0\Psi - \mathcal{H}\right)$$

Its important to notice that now the action is a functional of both Ψ and π, not only of Ψ as before, so we must vary Ψ and π *independently*. Treating Ψ and π independently, the equations of motion are obtained through the following: lets start computing δS by computing its parts separately:

$$\begin{aligned}
\delta\left(\pi \cdot \partial_0\Psi\right) &= \pi \cdot \delta\left(\partial_0\Psi\right) + \delta\pi \cdot \partial_0\Psi \\
&= \pi \cdot \partial_0\left(\delta\Psi\right) + \delta\pi \cdot \partial_0\Psi \\
&= \partial_0\left(\pi \cdot \delta\Psi\right) - \partial_0\pi \cdot \delta\Psi + \delta\pi \cdot \partial_0\Psi
\end{aligned}$$

$$\delta\mathcal{H} = \frac{\partial\mathcal{H}}{\partial\Psi}\,\delta\Psi + \frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\,\delta\left(\boldsymbol{\nabla}\Psi\right) + \frac{\partial\mathcal{H}}{\partial\pi}\,\delta\pi + \frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\pi\right)}\,\delta\left(\boldsymbol{\nabla}\pi\right)$$
$$= \frac{\partial\mathcal{H}}{\partial\Psi}\,\delta\Psi + \frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\boldsymbol{\nabla}\left(\delta\Psi\right) + \frac{\partial\mathcal{H}}{\partial\pi}\,\delta\pi + \frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\pi\right)}\boldsymbol{\nabla}\left(\delta\pi\right)$$
$$\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\boldsymbol{\nabla}\left(\delta\Psi\right) = \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\,\delta\Psi\right) - \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\right)\delta\Psi$$
$$\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\pi\right)}\boldsymbol{\nabla}\left(\delta\pi\right) = \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\pi\right)}\,\delta\pi\right) - \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\pi\right)}\right)\delta\pi$$

Now we can write δS (after collecting the $\delta\Psi$ and $\delta\pi$ terms) as

$$\delta S = \int_{\mathcal{V}} d^D x \left\{\left[-\partial_0\pi - \frac{\partial\mathcal{H}}{\partial\Psi} + \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\right)\right]\delta\Psi + \right.$$
$$\left.\left[\partial_0\Psi - \frac{\partial\mathcal{H}}{\partial\pi} + \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\right)\right]\delta\pi\right\} +$$
$$-\left[\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\,\delta\Psi\right]\Bigg|_{\partial\mathcal{V}} - \left[\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\pi\right)}\,\delta\pi\right]\Bigg|_{\partial\mathcal{V}} = 0 \quad (1.2)$$

Assuming that $\delta\Psi = 0$ and $\delta\pi = 0$ at $\partial\mathcal{V}$, then for (1.2) to be satisfied for all allowed $\delta\Psi$ and $\delta\pi$ is sufficies that

$$\partial_0\Psi = \frac{\partial\mathcal{H}}{\partial\pi} - \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\pi\right)}\right) = \frac{\delta H}{\delta\pi} \quad (1.3)$$

and

$$\partial_0\pi = -\frac{\partial\mathcal{H}}{\partial\Psi} + \boldsymbol{\nabla}\cdot\left(\frac{\partial\mathcal{H}}{\partial\left(\boldsymbol{\nabla}\Psi\right)}\right) = -\frac{\delta H}{\delta\Psi} \quad (1.4)$$

which are the so called *Hamilton's equations*.

Let F and G be any functionals of Ψ and π. Then the *Poisson bracket* between F and G is defined as

$$\{F,\ G\}_{\mathrm{PB}} = \int d^3x \left(\frac{\delta F}{\delta\Psi}\frac{\delta G}{\delta\pi} - \frac{\delta F}{\delta\pi}\frac{\delta G}{\delta\Psi}\right)$$

and has the following properties:

$$\{F,\ G\}_{\mathrm{PB}} = -\{G,\ F\}_{\mathrm{PB}} \quad \text{antisymmetry}$$

$$\{F_1 + F_2,\ G\}_{\mathrm{PB}} = \{F_1,\ G\}_{\mathrm{PB}} + \{F_2,\ G\}_{\mathrm{PB}} \quad \text{linear}$$

$$\{F_1F_2,\ G\}_{\text{PB}} = F_1 \{F_2,\ G\}_{\text{PB}} + \{F_1,\ G\}_{\text{PB}} F_2 \quad \text{product law}$$

and finally the *Jacobi Identity*

$$\{F,\ \{G,\ H\}_{\text{PB}}\}_{\text{PB}} + \{G,\ \{H,\ F\}_{\text{PB}}\}_{\text{PB}} + \{H,\ \{F,\ G\}_{\text{PB}}\}_{\text{PB}} = 0$$

With the above definition we can now take $\{F,\ H\}_{\text{PB}}$ to find

$$\begin{aligned}\{F,\ H\}_{\text{PB}} &= \int d^3x \left(\frac{\delta F}{\delta \Psi}\frac{\delta H}{\delta \pi} - \frac{\delta F}{\delta \pi}\frac{\delta H}{\delta \Psi}\right) \\ &= \int d^3x \left(\frac{\delta F}{\delta \Psi}\partial_0 \Psi - \frac{\delta F}{\delta \pi}\left(-\partial_0 \pi\right)\right) \\ &= \int d^3x \left(\frac{\delta F}{\delta \Psi}\partial_0 \Psi + \frac{\delta F}{\delta \pi}\partial_0 \pi\right)\end{aligned}$$

also, lets determine the variation of a functional F with respect to time, meaning $\delta\Psi = \partial_0 \Psi\, \delta t$ and $\delta\pi = \partial_0 \pi\, \delta t$:

$$\delta F = \frac{\partial F}{\partial t}\,\delta t + \int d^3x \left(\frac{\delta F}{\delta \Psi}\partial_0 \Psi\, \delta t + \frac{\delta F}{\delta \pi}\partial_0 \pi\, \delta t\right)$$

and since time is a scalar parameter it follows that variations with respect to it are the same as derivatives with respect to it, which implies that

$$\frac{\delta F}{\delta t} \equiv \frac{dF}{dt} = \{F,\ H\}_{\text{PB}} + \frac{\partial F}{\partial t}$$

and now, noticing that

$$\frac{\delta f_i(x)}{\delta f_j(y)} = \delta_{ij}\delta(x-y)$$

it follows that Hamilton's equations can be stated as

$$\partial_0 \Psi = \{\Psi,\ H\}_{\text{PB}} \quad \text{and} \quad \partial_0 \pi = \{\pi,\ H\}_{\text{PB}}$$

Also, we notice that

$$\{\Psi(t,\mathbf{x}),\ \pi(t,\mathbf{y})\}_{\text{PB}} = \delta(\mathbf{x}-\mathbf{y})$$

$$\{\Psi(t,\mathbf{x}),\ \Psi(t,\mathbf{y})\}_{\text{PB}} = \{\pi(t,\mathbf{x}),\ \pi(t,\mathbf{y})\}_{\text{PB}} = 0$$

However, in doing so we imposed the independence between Ψ and π, and such independence its not original from the Lagrangian, so we don't want to do that. This statement is very important since it shows that, indeed, the Lagrangian must hold every relevant information about the physical system, including the contraints. There are different kinds of constraints but all of them are represented as a function ϕ of the field Ψ and its conjugate momentum π such that

$$\phi(\Psi, \pi) = 0$$

The constraints originated exclusively from the *form* of the Lagrangian are called *primary constraints*, and, of course, they must not vary with time. In the presence of primary constraints we don't have a uniquely determined Hamiltonian since we take the ordinary Hamiltonian H and add to it any combination of the M primary constraints, which means

$$H^* = H + c_m \phi_m, \quad m = 1, ..., M$$

where the coefficients c_m can be *any functions of* Ψ *and* π.

Its weird to write $H^* = H + c_m \phi_m$ since $\phi_m = 0$, isn't this simply $H^* = H$? The answer is *yes* and *no*. A crucial matter in developing this theory is *when* to impose the constraints. To make this more clear, lets define the so called *weak equality* or *weak equation*, which will be denoted by a $\approx$ instead of a $=$. The use of weak equations is this: the left hand side of an weak equality will only achieve the right hand side when after you 'massage' the left hand side and impose the constraint. An example: let the constraint be $\phi = y - x = 0$ and lets look at the functions $f = x$ and $g = y$, we can then write $f \approx g$ since if we impose the constraint on f, we achieve g. We write the primary constraints themselves as weak equations:

$$\phi_m \approx 0$$

And of course, we can write

$$H^* \approx H$$

so this *modified Hamiltonian* is as good as the ordinary one.

Now, starting from the modified Hamiltonian H^*, lets obtain the equations of motion. It's crucial to notice that we can, now, vary the field and its conjugate momentum independently, since any constraint between them has been taken into account through the lagrange multipliers c_m. The only new thing we must consider is the variation of the constraints, which is

$$\begin{aligned}\delta\left(c_m\phi_m\right) &= \delta\left(c_m\right)\phi_m + c_m\,\delta\left(\phi_m\right)\\ &= \left(\frac{\partial c_m}{\partial\Psi}\,\delta\Psi + \frac{\partial c_m}{\partial\pi}\,\delta\pi\right)\phi_m + c_m\left(\frac{\partial\phi_m}{\partial\Psi}\,\delta\Psi + \frac{\partial\phi_m}{\partial\pi}\,\delta\pi\right)\\ &= \left(\phi_m\frac{\partial c_m}{\partial\Psi} + c_m\frac{\partial\phi_m}{\partial\Psi}\right)\delta\Psi + \left(\phi_m\frac{\partial c_m}{\partial\pi} + c_m\frac{\partial\phi_m}{\partial\pi}\right)\delta\pi\end{aligned}$$

Now comes another interesting thing: irrespective of ϕ_m being 0, its partial derivative is not. For instance, if we take the previous example $\phi = y - x = 0$, we find that $\frac{\partial\phi}{\partial x} = 1$. So we can impose the constraints $\phi_m = 0$ *now*, and obtain the weak equality

$$\delta\left(c_m\phi_m\right) \approx \left(c_m\frac{\partial\phi_m}{\partial\Psi}\right)\delta\Psi + \left(c_m\frac{\partial\phi_m}{\partial\pi}\right)\delta\pi$$

and such variation results in the following Hamilton's equations

$$\partial_0\Psi = \frac{\delta H}{\delta\pi} + c_m\frac{\partial\phi_m}{\partial\pi}$$

and

$$\partial_0\pi = -\frac{\delta H}{\delta\Psi} - c_m\frac{\partial\phi_m}{\partial\Psi}$$

which enable us to write

$$\partial_0 F(\Psi,\pi) = \{F,\ H\}_{\text{PB}} + c_m\,\{F,\ \phi_m\}_{\text{PB}} \tag{1.5}$$

Lets compute the following

$$\begin{aligned}\{F,\ H^*\}_{\text{PB}} &= \{F,\ H + c_m\phi_m\}_{\text{PB}} = \{F,\ H\}_{\text{PB}} + \{F,\ c_m\phi_m\}_{\text{PB}}\\ &= \{F,\ H\}_{\text{PB}} - \{c_m\phi_m,\ F\}_{\text{PB}}\\ &= \{F,\ H\}_{\text{PB}} - c_m\,\{\phi_m,\ F\}_{\text{PB}} - \{c_m,\ F\}_{\text{PB}}\,\phi_m\\ &= \{F,\ H\}_{\text{PB}} + c_m\,\{F,\ \phi_m\}_{\text{PB}} + \{F,\ c_m\}_{\text{PB}}\,\phi_m\\ &\approx \{F,\ H\}_{\text{PB}} + c_m\,\{F,\ \phi_m\}_{\text{PB}}\end{aligned}$$

so we can write

$$\partial_0 F \approx \{F,\ H^*\}_{\text{PB}}$$

and notice that $\{F,\ \phi_m\}_{\text{PB}}$ didn't vanish even with the constraints imposed since the Poisson bracket consists of some partial derivatives (that don't necessarly vanish, as we saw on the example).

We can then write $\partial_0\phi_n$ as

$$\partial_0\phi_n \approx \{\phi_n,\ H^*\}_{\text{PB}} \approx \{\phi_n,\ H\}_{\text{PB}} + c_m \{\phi_n,\ \phi_m\}_{\text{PB}}$$

and since the constraints must not vary with time, we find the following

$$\{\phi_n,\ H\}_{\text{PB}} + c_m \{\phi_n,\ \phi_m\}_{\text{PB}} \approx 0 \tag{1.6}$$

Several things can now happen, the first being an inconsistency: we could find $1 \approx 0$, and this will only happen if the Lagrangian itself has inconsistent equations of motion, so lets assume this is not the case. We could find an identity like $0 \approx 0$, which is fine but doesn't add anything new and the modified hamiltonian is enough to account for every possible constraint. Now, if we notice that $\{\phi_n,\ \phi_m\}_{\text{PB}}$ can be the components ξ_{nm} of a matrix ξ, the c_m the components of a vector c, and $\{\phi_n,\ H\}_{\text{PB}}$ the components of a vector $\mathcal{F}$, (1.6) becomes the equation

$$\xi \cdot c = -\mathcal{F}$$

and if ξ is non-singular ($\det\{\xi\} \neq 0$), then it has an inverse and the coefficients c_m are univocally determined as

$$c \approx -\xi^{-1}\mathcal{F}$$

$$c_m \approx -\xi^{-1}_{mn} \{\phi_n,\ H\}_{\text{PB}}$$

and (1.5) becomes

$$\partial_0 F(\Psi, \pi) = \{F,\ H\}_{\text{PB}} - \{F,\ \phi_m\}_{\text{PB}}\, \xi^{-1}_{mn} \{\phi_n,\ H\}_{\text{PB}}$$

which we can write concisely by defining the *Dirac's bracket*

$$\{F,\ H\}_{\text{D}} \equiv \{F,\ H\}_{\text{PB}} - \{F,\ \phi_m\}_{\text{PB}}\, \xi^{-1}_{mn} \{\phi_n,\ H\}_{\text{PB}}$$

or explicitly writing the implicit sum

$$\{F,\ H\}_{\text{D}} \equiv \{F,\ H\}_{\text{PB}} - \sum_{n,m=1}^{M} \{F,\ \phi_m\}_{\text{PB}}\, \xi^{-1}_{mn} \{\phi_n,\ H\}_{\text{PB}}$$

However, as we can see this only takes the M primary constraints into account, but there may be more. The (1.6) may lead to more constraints, which we'll call *secondary constraints* denoted by

$$\chi_s(\Psi, \pi) = 0, \quad s = 1, ..., S$$

Again, such constraints must not vary with time, so they must satisfy

$$\{\chi_s,\ H\}_{\text{PB}} + c_m \{\chi_s,\ \phi_m\}_{\text{PB}} \approx 0$$

We can readily notice that we'll again have analyse the possibilities and may end up with some more secondary constraints and have to repeat the whole process. This process is called the *Dirac-Bergmann algorithm*. What matter is that if we don't find an inconsistency or a triviality, then we'll end up with M primary constraints and K secondary constraints with the final requirement that the K secondary constraints do not vary with time, and having ended the Dirac-Bergmann algorithm, we are sure to finally find the c_m. Denoting the secondary constraints as ϕ_k with $k = M+1, ..., M+K$, and letting $M + K = J$ be the total number of constraints, then the requirement that they all do not vary with time is

$$\{\phi_j,\ H\}_{\text{PB}} + \sum_{m=1}^{M} c_m \{\phi_j,\ \phi_m\}_{\text{PB}} \approx 0 \quad \text{with} \quad j = 1, ..., J$$

Notice that we have J equations and $M \leq J$ unknowns, and, of course, this must have a solution otherwise we have an inconsistency, which means the original Lagrangian is inconsistent, but since we are not considering inconsistent Lagrangians, we're fine. The general solution is

$$c_m = U_m + \sum_{a=1}^{A} v_a V_m^a$$

where U_m is the particular solution and V_m^a are the A linearly independet solutions to the homogenous equation. This means

$$\sum_{m=1}^{M} \{\phi_j,\ \phi_m\}_{\text{PB}} V_m^a \approx 0$$

We can now state the *total Hamiltonian* H_T

$$H_T = H + \sum_{m=1}^{M} U_m \phi_m + \sum_{m=1}^{M} \sum_{a=1}^{A} v_a V_m^a \phi_m \equiv H' + \sum_{a=1}^{A} v_a \Phi_a$$

where

$$H' = H + \sum_{m=1}^{M} U_m \phi_m$$

and

$$\Phi_a = \sum_{m=1}^{M} V_m^a \phi_m$$

and notice that the v's are completely arbitrary functions of time. The equations of motion are, for some arbitraty function F of Ψ and π, simply

$$\frac{dF}{dt} = \{F,\ H_T\}_{\text{PB}} + \frac{\partial F}{\partial t} = \{F,\ H\}_{\text{D}} + \frac{\partial F}{\partial t}$$

Procedure-wise, all we have to do is find every constraint, compute the constraint matrix ξ, find its inverse, then compute the Poisson bracket between every constraint with the field and its conjugate momenta and we're done: we have the fundamental Dirac brackets on which we can build our quantum theory.

2
Source free electromagnetic fields

Let A^μ be the four-potential defined as

$$A^\mu = (\phi, \mathbf{A})$$

Let

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

be the so called *field strength*. We'll be using Minkowski metric with signature $(-+++)$, which means that we can raise and lower latin lettered indices as we wish, but must always change sign for each zero-th index raised or lowered. Also, of course, we'll be using natural units ($c = \hbar = 1$). Then the Lagrangian which gives rise to the source free Maxwell's equations is

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

The field $\mathbf{E}$ is, then

$$E_j = -\partial_0 A_j + \partial_j A_0 \quad \text{or} \quad E^j = \partial^0 A^j - \partial^j A^0$$

while the field $\mathbf{B}$ is

$$B_j = -\epsilon_{jk\ell} \partial_k A_\ell$$

Lets find the equations of motion. Lets start by computing

$$\begin{aligned}
\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial\left(\partial_\nu A_\mu\right)} &= -\frac{1}{4}\left(\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)}F^{\alpha\beta} + F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)}\right)\\
&= -\frac{1}{4}\left(\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)}F^{\alpha\beta} + \eta_{a\alpha}\eta_{b\beta}F^{ab}\frac{\partial F^{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)}\right)\\
&= -\frac{1}{4}\left(\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)}F^{\alpha\beta} + F^{ab}\frac{\partial\left(\eta_{a\alpha}\eta_{b\beta}F^{\alpha\beta}\right)}{\partial\left(\partial_\nu A_\mu\right)}\right)\\
&= -\frac{1}{4}\left(\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)}F^{\alpha\beta} + F^{ab}\frac{\partial F_{ab}}{\partial\left(\partial_\nu A_\mu\right)}\right)\\
&= -\frac{1}{2}\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)}F^{\alpha\beta}
\end{aligned}$$

and by the definition of $F_{\mu\nu}$ we find that

$$\frac{\partial F_{\alpha\beta}}{\partial\left(\partial_\nu A_\mu\right)} = \delta^\nu_\alpha\delta^\mu_\beta - \delta^\nu_\beta\delta^\mu_\alpha$$

which means that

$$\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial\left(\partial_\nu A_\mu\right)} = -\frac{1}{2}\left(\delta^\nu_\alpha\delta^\mu_\beta - \delta^\nu_\beta\delta^\mu_\alpha\right)F^{\alpha\beta} = -\frac{1}{2}\left(F^{\nu\mu} - F^{\mu\nu}\right)$$

and since $F_{\mu\nu}$ is antisymmetric

$$\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial\left(\partial_\nu A_\mu\right)} = F^{\mu\nu} \tag{2.1}$$

Now, since $\partial\mathcal{L}_{\mathrm{EM}}/\partial A_\mu \equiv 0$, the equations of motion are simply

$$\partial_\mu\left(\frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial\left(\partial_\mu A_\nu\right)}\right) = -\partial_\mu F^{\mu\nu} = 0 \tag{2.2}$$

To obtain the conjugate momenta π^μ we simply set $\nu = 0$ on (2.1) and find

$$\pi^\mu = \frac{\partial \mathcal{L}_{\mathrm{EM}}}{\partial\left(\partial_0 A_\mu\right)} = F^{\mu 0} = \partial^\mu A^0 - \partial^0 A^\mu \tag{2.3}$$

Now comes the interesting part: since $F^{00} = 0$, it follows that

$$\pi^0 = 0 := \phi_1$$

which is a primary constrain, since it came directly from the form of the Lagrangian. The other momenta are

$$\pi^i = F^{i0} = \partial^i A^0 - \partial^0 A^i = E^i$$

Now we must impose $\partial_0 \phi_1 \approx 0$ using (1.6)

$$\partial_0 \phi_1 = \{\phi_1,\ H\}_{\text{PB}} + c_1 \{\phi_1,\ \phi_1\}_{\text{PB}} \equiv \{\phi_1,\ H\}_{\text{PB}} = 0$$

where $\{\phi_1,\ \phi_1\}_{\text{PB}} \equiv 0$ since the Poisson bracket is antisymmetric. Notice that this fact allows us to simply use the ordinary equations of motion for the momenta

$$\partial_0 \pi^\mu = -\frac{\partial \mathcal{H}}{\partial A^\mu} + \partial^i \left(\frac{\partial \mathcal{H}}{\partial \left(\partial^i A^\mu\right)} \right)$$

But to do so we must first find the ordinary Hamiltonian through

$$\mathcal{H} = \pi^\mu \partial_0 A_\mu - \mathcal{L}_{\text{EM}}$$

which using (2.3) we can write as

$$\mathcal{H} = \pi^\mu \pi_\mu + \pi^\mu \partial_\mu A_0 - \mathcal{L}_{\text{EM}} \approx \pi^i \pi_i + \pi^i \left(\partial_i A_0\right) - \mathcal{L}_{\text{EM}} \tag{2.4}$$

for which it follows that $\partial \mathcal{H} / \partial A^\mu \equiv 0$, meaning that we have to compute only

$$\partial^i \left(\frac{\partial \mathcal{H}}{\partial \left(\partial^i A^\mu\right)} \right) = \delta^0_\mu\, \partial^i \pi^i - \partial^i \left(\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \left(\partial^i A^\mu\right)} \right) = \delta^0_\mu\, \partial^i \pi^i - \partial^\nu \left(\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \left(\partial^\nu A^\mu\right)} \right) + \partial^0 \left(\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \left(\partial^0 A^\mu\right)} \right)$$

and since the second term of the right hand side is proportional to equation of motion (2.2) we find

$$\partial_0 \pi^\mu = \delta^0_\mu\, \partial^i \pi^i + \partial^0 \left(\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \left(\partial^0 A^\mu\right)} \right) = 0$$

which for $\mu = 0$ reads

$$\partial_0 \pi^0 = \partial^i \pi^i + \partial^0 \left(\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \left(\partial^0 A^0\right)} \right) = \partial^i \pi^i - \partial_0 \left(\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \left(\partial_0 A_0\right)} \right) \equiv \partial^i \pi^i - \partial_0 \pi^0 = 0$$

meaning

$$2 \partial_0 \pi^0 = \partial_i \pi^i \approx 0 = \boldsymbol{\nabla} \cdot \mathbf{E}$$

which in terms of the four-potential is

$$\boldsymbol{\nabla} \cdot \mathbf{E} \equiv \partial_i E^i = \partial_i \left(\partial^0 A^i - \partial^i A^0\right) = \partial^0 \left(\partial_i A^i\right) - \partial_i \partial^i \left(A^0\right) = 0 := \phi_2$$

where ϕ_2 is a secondary constraint! Thus we must now impose $\partial_0 \phi_2 \approx 0$ as

$$\partial_0 \phi_2 = \{\phi_2,\ H\}_{\text{PB}} + c_1 \{\phi_2,\ \phi_1\}_{\text{PB}} \approx 0$$

Using

$$\left\{\partial_i \pi^i,\ F\right\}_{\text{PB}} = \partial_i \left\{\pi^i,\ F\right\}_{\text{PB}}$$

we find that

$$\left\{\partial_i \pi^i,\ H\right\}_{\text{PB}} = \partial_i \left\{\pi^i,\ H\right\}_{\text{PB}} \equiv 0$$

due to (2.4), and also

$$\left\{\partial_i \pi^i,\ \pi^0\right\}_{\text{PB}} = \partial_i \left\{\pi^i,\ \pi^0\right\}_{\text{PB}} \equiv 0$$

Allowing us to conclude that there are no more constraints and thus c_1 is arbitrary.

However, we can do something interesting: we can use the primary constraint to impose a gauge, we just have to change the Lagrangian:

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \longrightarrow -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial_0 A_0 \left(G\right)$$

where G is some function of A^μ such that $G \approx 0$. It's easy to see that now we find

$$\pi^0 = -G \approx 0$$

so we retained our previous primary constraint, but now we can express it simply as $G \approx 0$. Recall that the secondary constraint that arises from $\pi^0 \approx 0$ is $\boldsymbol{\nabla} \cdot \mathbf{E} \approx 0$, which we can write in terms of A^μ as

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \partial_0 \left(\boldsymbol{\nabla} \cdot \mathbf{A}\right) - \boldsymbol{\nabla}^2 A^0 = 0$$

and solve it for A^0 as

$$A_0 = \int d^3x' \, \frac{\partial_0 \left(\boldsymbol{\nabla} \cdot \mathbf{A}(\mathbf{x}')\right)}{4\pi |\mathbf{x} - \mathbf{x}'|}$$

which means that instead of 4 degrees of freedom we have only 3.

The thing now is that if we say that $G = \boldsymbol{\nabla} \cdot \mathbf{A} \approx 0$, which is Coulomb's gauge, then $A_0 \approx 0$, and although π^0 is equivalent to $\boldsymbol{\nabla} \cdot \mathbf{A}$ ($\phi_1 = \pi^0 = \boldsymbol{\nabla} \cdot \mathbf{A}$), the poisson bracket

$$\{\phi_1,\ \phi_2\}_{\text{PB}}$$

is different, lets compute it

$$\{\phi_1,\ \phi_2\}_{\text{PB}} = \left\{\partial_i A^i,\ \partial_i \pi^i\right\}_{\text{PB}} = \partial_i \partial_i \left\{A^i,\ \pi^i\right\}_{\text{PB}} = \boldsymbol{\nabla}^2 \delta(\mathbf{x} - \mathbf{y})$$

lets also compute $\{\phi_1,\ H\}_{\text{PB}}$

$$\left\{\partial_i A^i,\ H\right\}_{\text{PB}} = \partial_i \left\{A^i,\ H\right\}_{\text{PB}}$$

which using (1.3) is

$$\partial_i \left[\frac{\partial \mathcal{H}}{\partial \pi^i} - \partial_i \left(\frac{\partial \mathcal{H}}{\partial \left(\partial_i \pi^i\right)}\right)\right] = \partial_i \left(\pi^i + \partial_i A_0\right) = \partial_i \pi^i + \partial_i \partial^i A_0 \approx 0$$

So lets find everything about the constraints, recalling that they are

$$\phi_1 = \partial_i A_i \quad \text{and} \quad \phi_2 = \partial_i \pi_i$$

and the canonical fundamental Poisson bracket

$$\{A_i(\mathbf{x}),\ \pi_j(\mathbf{y})\}_{\text{PB}} = \delta_{ij} \delta(\mathbf{x} - \mathbf{y})$$

We already have the following

$$\{\phi_1,\ \phi_1\}_{\text{PB}} = \{\phi_2,\ \phi_2\}_{\text{PB}} = 0$$

$$\{\phi_1,\ \phi_2\}_{\text{PB}} = \boldsymbol{\nabla}^2 \delta(\mathbf{x} - \mathbf{y})$$

thus we still have to find the Poisson brackets of the constraints with A_i and π_i:

$$\{\phi_1,\ A_i\}_{\text{PB}} = \partial_i \{A_i,\ A_i\}_{\text{PB}} = 0$$

$$\{\phi_2,\ A_i\}_{\text{PB}} = \partial_i \{\pi_i,\ A_i\}_{\text{PB}} = -\partial_i \delta(\mathbf{x} - \mathbf{y})$$

$$\{\phi_1,\ \pi_i\}_{\text{PB}} = \partial_i \{A_i,\ \pi_i\}_{\text{PB}} = \partial_i \delta(\mathbf{x} - \mathbf{y})$$

$$\{\phi_2,\ \pi_i\}_{\text{PB}} = \partial_i \{\pi_i,\ \pi_i\}_{\text{PB}} = 0$$

This means that the Dirac bracket between A_i and π_j is, finally

$$\begin{aligned}\{A_i,\ \pi_j\}_{\mathrm{D}} &= \{A_i,\ \pi_j\}_{\mathrm{PB}} - \{A_i,\ \phi_2\}_{\mathrm{PB}}\,\{\phi_1,\ \pi_j\}_{\mathrm{PB}}\,\xi_{21}^{-1} \\ &= \delta_{ij}\delta(\mathbf{x}-\mathbf{y}) - (\partial_i\delta(\mathbf{x}-\mathbf{y}))\,(\partial_j\delta(\mathbf{x}-\mathbf{z}))\,\xi_{12}^{-1} \\ &= \delta(\mathbf{x}-\mathbf{y})\left(\delta_{ij} - \partial_i\partial_j\ \xi_{12}^{-1}\right)\end{aligned}$$

and it's common to write

$$\xi_{12}^{-1} = \frac{1}{\boldsymbol{\nabla}^2}$$

as the Laplacian green's function, so that the final Dirac bracket is

$$\{A_i,\ \pi_j\}_{\mathrm{D}} = \left(\delta_{ij} - \frac{\partial_i\partial_j}{\boldsymbol{\nabla}^2}\right)\delta(\mathbf{x}-\mathbf{y})$$

or in momentum space

$$\{A_i,\ \pi_j\}_{\mathrm{D}} = \int \frac{d^3p}{(2\pi)^3}\left(\delta_{ij} - \frac{p_ip_j}{|\mathbf{p}|^2}\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

and now we can turn to quantization by imposing

$$\{\cdot,\cdot\}_{\mathrm{D}} \longrightarrow -i[\cdot,\cdot]$$

which means that the true commutator between A_i and π_j is

$$[A_i, \pi_j] = i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\boldsymbol{\nabla}^2}\right)\delta(\mathbf{x}-\mathbf{y})$$

while the others remain 0.

Yet in the classical realm, lets see the final equation of motion (using the Lagrangian, but then apllying the constraints we found or imposed):

$$\partial_\mu F^{\mu\nu} = \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = 0$$

imposing the constraints $A^0 \approx 0$ and $\partial_i A^i \approx 0$

$$\partial_\mu\partial^\mu A^i - \partial_\nu\left(\partial_i A^i\right) \approx \partial_\mu\partial^\mu A^i = 0$$

thus in Coulomb gauge the equation of motion for $\mathbf{A}$ is

$$\partial_\mu\partial^\mu\mathbf{A} = 0$$

which is just the wave equation for $\mathbf{A}$, meaning that $\mathbf{A}$ is a linear combination of the well known plane waves of the form

$$\mathbf{A}(\mathbf{x}) = \mathbf{A}e^{ip^{\mu}x_{\mu}} = \mathbf{A}e^{i\left(p^0 x_0 + \mathbf{p}\cdot\mathbf{x}\right)}$$

Notice that the constraint $\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{x}) = 0$ implies

$$\mathbf{A}\cdot\mathbf{p} = 0$$

which means that given some $\mathbf{p}$, our field has only 2 degrees of freedom $\boldsymbol{\epsilon}_r(\mathbf{p})$, $r = 1, 2$, such that

$$\boldsymbol{\epsilon}_r(\mathbf{p})\cdot\mathbf{p} = 0 \quad \text{and} \quad \boldsymbol{\epsilon}_r(\mathbf{p})\cdot\boldsymbol{\epsilon}_s(\mathbf{p}) = \delta_{rs}$$

this are known as the *polarization vectors*. As we now, the general solution to the wave equation can be writen as a linear combination of plane waves, and in our case, we have 2 plane waves for each polarization vector, and this is true for every $\mathbf{p}$, which means we can write $\mathbf{A}$ as

$$\mathbf{A}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} F(\mathbf{p}) \sum_{r=1}^{2} \boldsymbol{\epsilon}_r(\mathbf{p}) \left[a^r_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a^{r\dagger}_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right]$$

where the coefficients $a^r_{\mathbf{p}}$ and $a^{r\dagger}_{\mathbf{p}}$ are the amplitudes of each plane wave and are constrained since, in the end, $\mathbf{A}$ has to be real. The function $F(\mathbf{p})$ is there for future normalization purposes.

This was all classic, but what about quantum? Well, if we consider Heisenberg's picture, everything holds equally, the field and its momenta are promoted to operators, and so are the coefficients a for which we can still show the following commutation relations

$$\left[a^r_{\mathbf{p}}, a^s_{\mathbf{q}}\right] = \left[a^{r\dagger}_{\mathbf{p}}, a^{w\dagger}_{\mathbf{q}}\right] = 0$$

$$\left[a^r_{\mathbf{p}}, a^{w\dagger}_{\mathbf{q}}\right] = (2\pi)^3 \delta_{rs}\delta(\mathbf{p} - \mathbf{q})$$

which shows that these coefficients are the creation and annihilation operators.

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