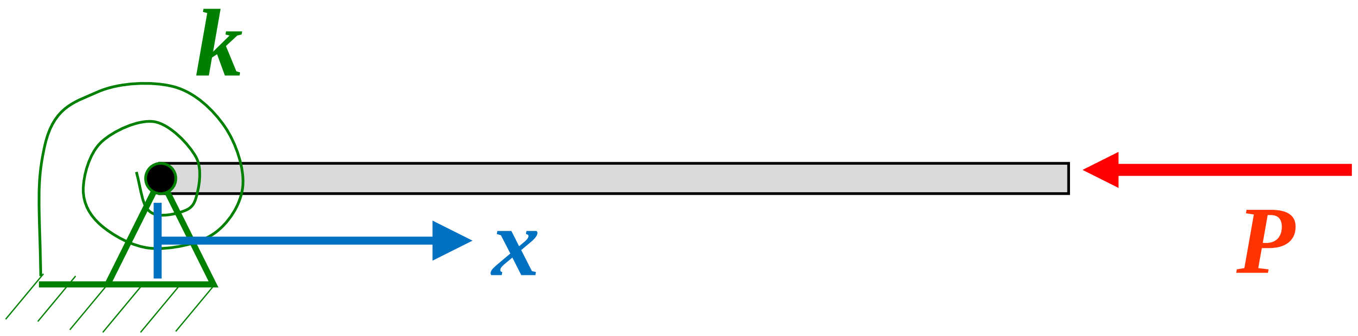
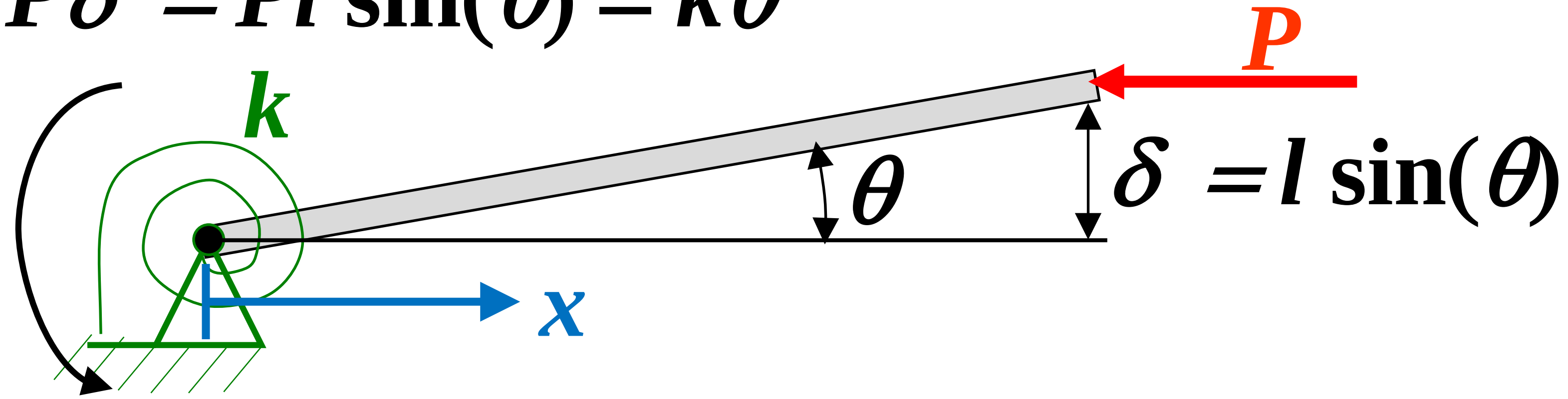
Problems

Consider a rigid rod of length l simply supported at $x = 0$ under a rotational spring of elastic constant k. The rod is under a compressive load P as depicted in the figure. Compute the buckling load for the rod.



Solution

$$M = P\delta = Pl\sin(\theta) = k\theta$$



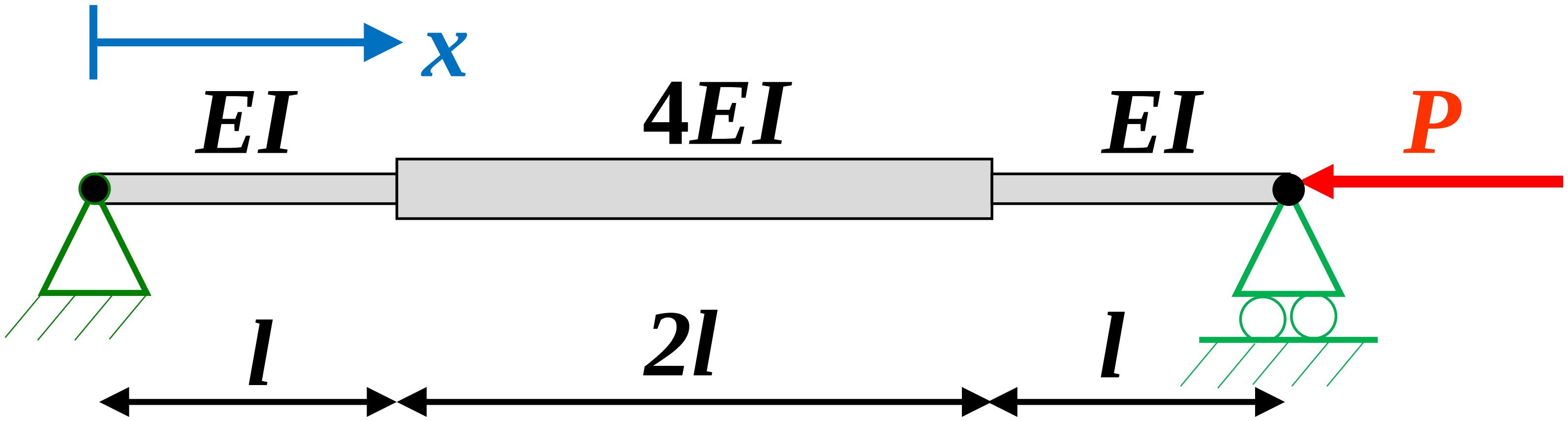
Equilibrium equation: $Pl\sin(\theta) = k\theta$

Buckling load: $Pcrit = k\theta / l\sin(\theta)$

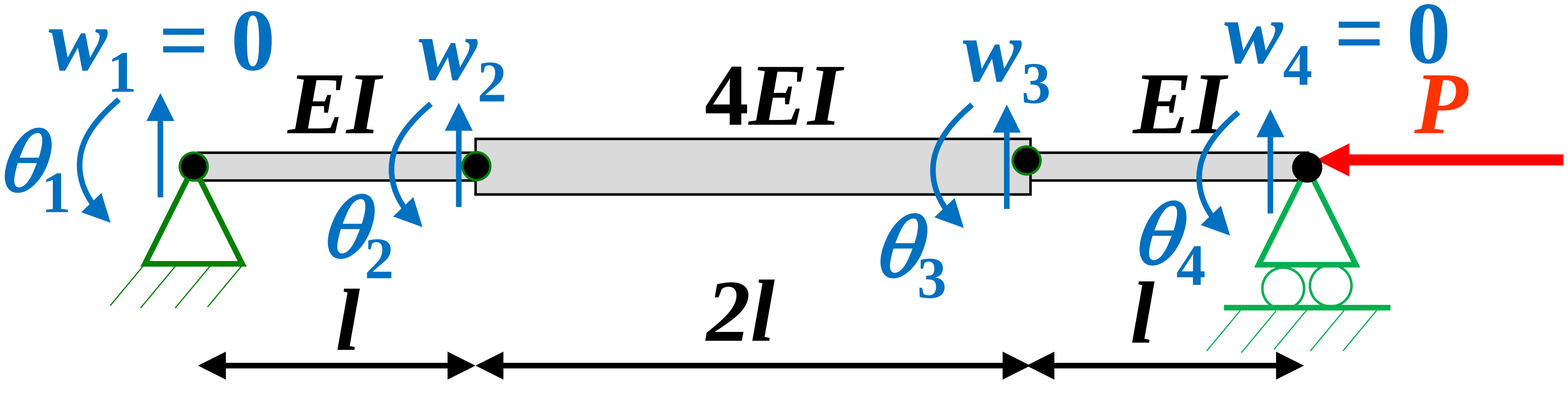
Interpretation:

Buckling occurs when the restoring momentum of the spring is lower then the applied momentum

Consider the simply supported beam with the geometry and properties depicted in the figure. The beam is under a compressive load. Assemble a finite element model for the system.



Solution 1



Vectors of nodal displacements

Global

$$\begin{Bmatrix} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ \theta_4 \\ w_1 = 0 \\ w_4 = 0 \end{Bmatrix}$$

Elem. 1

$$\begin{Bmatrix} w_1 = 0 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Elem. 2

$$\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{Bmatrix}$$

Elem. 3

$$\begin{Bmatrix} w_3 \\ \theta_3 \\ w_4 = 0 \\ \theta_4 \end{Bmatrix}$$

Stiffness matrices

Elem. 1

$$
\begin{array}{c}
\begin{array}{cccc} w_1 & \theta_1 & w_2 & \theta_2 \end{array} \\
\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}
\end{array}
\begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array}
$$

Elem. 3

$$
\begin{array}{c}
\begin{array}{cccc} w_3 & \theta_3 & w_4 & \theta_4 \end{array} \\
\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}
\end{array}
\begin{array}{c} w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{array}
$$

Elem. 2

$$
\begin{array}{c}
\begin{array}{cccc} w_2 & \theta_2 & w_3 & \theta_3 \end{array} \\
\frac{EI}{2l^3}\begin{bmatrix} 12 & 12l & -12 & 12l \\ 12l & 16l^2 & -12l & 8l^2 \\ -12 & -12l & 12 & -12l \\ 12l & 8l^2 & -12l & 16l^2 \end{bmatrix}
\end{array}
\begin{array}{c} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{array}
$$

Global

$$
\begin{array}{c}
\begin{array}{cccccc} \theta_1 & w_2 & \theta_2 & w_3 & \theta_3 & \theta_4 \end{array} \\
\frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 & 0 & 0 \\ -6l & 18 & 0 & -6 & 6l & 0 \\ 2l^2 & 0 & 12l^2 & -6l & 4l^2 & 0 \\ 0 & -6 & -6l & 18 & 0 & 6l \\ 0 & 6l & 4l^2 & 0 & 12l^2 & 2l^2 \\ 0 & 0 & 0 & 6l & 2l^2 & 4l^2 \end{bmatrix}
\end{array}
\begin{array}{c} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ \theta_4 \end{array}
$$

Geometric stiffness matrices

Pre-buckling problem: longitudinal force is equal to P

Elem. 1

$$\frac{1}{30}\frac{P}{l}\begin{array}{c} \begin{array}{cccc} w_1 & \theta_1 & w_2 & \theta_2 \end{array} \\ \left[\begin{array}{cccc} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{array}\right] \end{array} \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array}$$

Elem. 3

$$\frac{1}{30}\frac{P}{l}\begin{array}{c} \begin{array}{cccc} w_3 & \theta_3 & w_4 & \theta_4 \end{array} \\ \left[\begin{array}{cccc} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{array}\right] \end{array} \begin{array}{c} w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{array}$$

Elem. 2

$$\frac{1}{30}\frac{P}{2l}\begin{array}{c} \begin{array}{cccc} w_2 & \theta_2 & w_3 & \theta_3 \end{array} \\ \left[\begin{array}{cccc} 36 & 6l & -36 & 6l \\ 6l & 16l^2 & -6l & -4l^2 \\ -36 & -6l & 36 & -6l \\ 6l & -4l^2 & -6l & 16l^2 \end{array}\right] \end{array} \begin{array}{c} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{array}$$

Global

$$\frac{1}{30}\frac{P}{l}\begin{array}{c} \begin{array}{cccccc} \theta_1 & w_2 & \theta_2 & w_3 & \theta_3 & \theta_4 \end{array} \\ \left[\begin{array}{cccccc} 4l^2 & -3l & -l^2 & 0 & 0 & 0 \\ -3l & 54 & 0 & -18 & 3l & 0 \\ -l^2 & 0 & 12l^2 & -3l & -2l^2 & 0 \\ 0 & -18 & -3l & 54 & 0 & 3l \\ 0 & 3l & -2l^2 & 0 & 12l^2 & -l^2 \\ 0 & 0 & 0 & 3l & -l^2 & 4l^2 \end{array}\right] \end{array} \begin{array}{c} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ \theta_4 \end{array}$$

Buckling problem:

$$\left[\frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 & 0 & 0 \\ -6l & 18 & 0 & -6 & 6l & 0 \\ 2l^2 & 0 & 12l^2 & -6l & 4l^2 & 0 \\ 0 & -6 & -6l & 18 & 0 & 6l \\ 0 & 6l & 4l^2 & 0 & 12l^2 & 2l^2 \\ 0 & 0 & 0 & 6l & 2l^2 & 4l^2 \end{bmatrix} - \frac{1}{30}\frac{P}{l}\begin{bmatrix} 4l^2 & -3l & -l^2 & 0 & 0 & 0 \\ -3l & 54 & 0 & -18 & 3l & 0 \\ -l^2 & 0 & 12l^2 & -3l & -2l^2 & 0 \\ 0 & -18 & -3l & 54 & 0 & 3l \\ 0 & 3l & -2l^2 & 0 & 12l^2 & -l^2 \\ 0 & 0 & 0 & 3l & -l^2 & 4l^2 \end{bmatrix}\right]\begin{Bmatrix} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

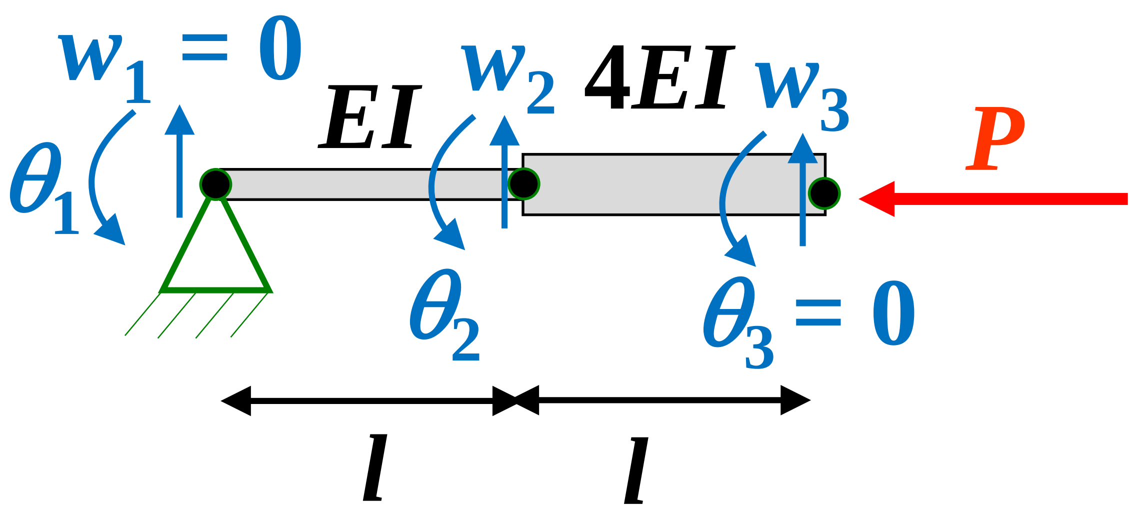
The solution of the above problem yields 6 pairs of eigenvalues / eigenvectors

Both, symmetric and anti-symmetric modes are computed

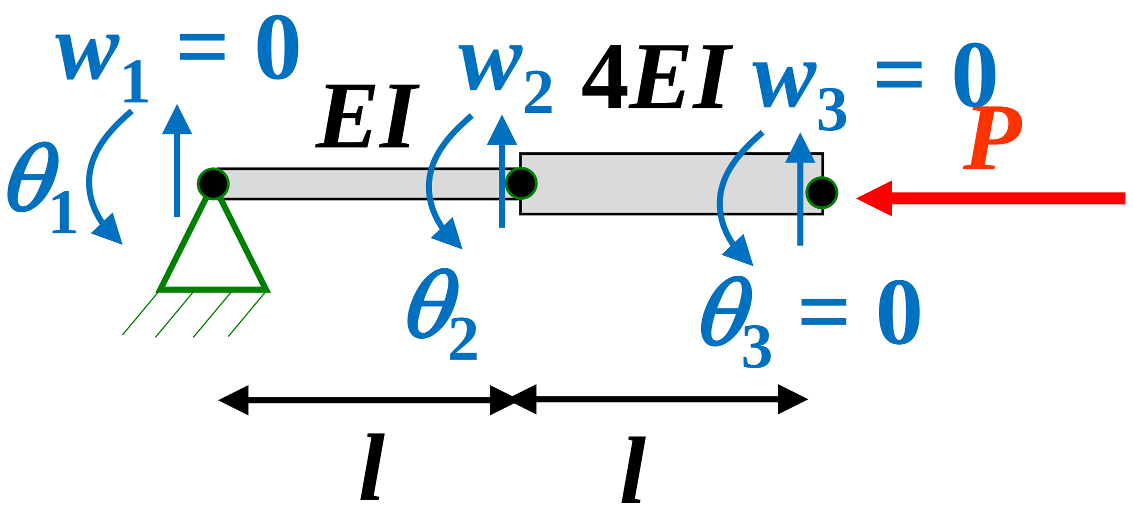
Solution 2

The problem is symmetric but has symmetric and anti-symmetric modes. One alternative is to compute them separately

Symmetric modes



Anti-symmetric modes



Symmetric modes

Vectors of nodal displacements

Global

$$\begin{Bmatrix} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ w_1 = 0 \\ \theta_3 = 0 \end{Bmatrix}$$

Elem. 1

$$\begin{Bmatrix} w_1 = 0 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Elem. 2

$$\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 = 0 \end{Bmatrix}$$

Stiffness matrices

Elem. 1

$$\frac{EI}{l^3}\begin{array}{cc} \begin{array}{cccc} w_1 & \theta_1 & w_2 & \theta_2 \end{array} & \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} & \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array} \end{array}$$

Elem. 2

$$\frac{EI}{l^3}\begin{array}{cc} \begin{array}{cccc} w_2 & \theta_2 & w_3 & \theta_3 \end{array} & \\ \begin{bmatrix} 48 & 24l & -48 & 24l \\ 24l & 16l^2 & -24l & 8l^2 \\ -48 & -24l & 48 & -24l \\ 24l & 8l^2 & -24l & 16l^2 \end{bmatrix} & \begin{array}{c} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{array} \end{array}$$

Global

$$\frac{EI}{l^3}\begin{array}{cc} \begin{array}{cccc} \theta_1 & w_2 & \theta_2 & w_3 \end{array} & \\ \begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 \\ -6l & 60 & 18l & -48 \\ 2l^2 & 18l & 20l^2 & -24l \\ 0 & -48 & -24l & 48 \end{bmatrix} & \begin{array}{c} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \end{array} \end{array}$$

Geometric stiffness matrices

Pre-buckling problem: longitudinal force is equal to P

Elem. 1

$$\begin{array}{cc} & \begin{array}{cccc} w_1 & \theta_1 & w_2 & \theta_2 \end{array} \\ \dfrac{1}{30}\dfrac{P}{l} & \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array} \end{array}$$

Elem. 2

$$\begin{array}{cc} & \begin{array}{cccc} w_2 & \theta_2 & w_3 & \theta_3 \end{array} \\ \dfrac{1}{30}\dfrac{P}{l} & \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \begin{array}{c} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{array} \end{array}$$

Global

$$\begin{array}{cc} & \begin{array}{cccc} \theta_1 & w_2 & \theta_2 & w_3 \end{array} \\ \dfrac{1}{30}\dfrac{P}{l} & \begin{bmatrix} 4l^2 & -3l & -l^2 & 0 \\ -3l & 72 & 0 & -36 \\ -l^2 & 0 & 8l^2 & -3l \\ 0 & -36 & -3l & 36 \end{bmatrix} \begin{array}{c} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \end{array} \end{array}$$

Buckling problem:

$$\left[\frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 \\ -6l & 60 & 18l & -48 \\ 2l^2 & 18l & 20l^2 & -24l \\ 0 & -48 & -24l & 48 \end{bmatrix} - \frac{1}{30}\frac{P}{l}\begin{bmatrix} 4l^2 & -3l & -l^2 & 0 \\ -3l & 72 & 0 & -36 \\ -l^2 & 0 & 8l^2 & -3l \\ 0 & -36 & -3l & 36 \end{bmatrix}\right]\begin{Bmatrix} \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The solution of the above problem yields 4 pairs of eigenvalues / eigenvectors

Only symmetric modes are computed

Since the critical buckling load is symmetric only this analysis have to be performed if the other eigenvalues are of no interest

Anti-symmetric modes

Vectors of nodal displacements

Elem. 1 **Elem. 2**

Global
$$\begin{Bmatrix} \theta_1 \\ w_2 \\ \theta_2 \\ \theta_3 \\ w_1 = 0 \\ w_3 = 0 \end{Bmatrix}$$

Elem. 1
$$\begin{Bmatrix} w_1 = 0 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Elem. 2
$$\begin{Bmatrix} w_2 \\ \theta_2 \\ w_3 = 0 \\ \theta_3 \end{Bmatrix}$$

Stiffness matrices

Elem. 1

$$
\frac{EI}{l^3}
\begin{array}{c}
\begin{array}{cccc} w_1 & \theta_1 & w_2 & \theta_2 \end{array} \\
\begin{bmatrix}
12 & 6l & -12 & 6l \\
6l & 4l^2 & -6l & 2l^2 \\
-12 & -6l & 12 & -6l \\
6l & 2l^2 & -6l & 4l^2
\end{bmatrix}
\end{array}
\begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array}
$$

Elem. 2

$$
\frac{EI}{l^3}
\begin{array}{c}
\begin{array}{cccc} w_2 & \theta_2 & w_3 & \theta_3 \end{array} \\
\begin{bmatrix}
48 & 24l & -48 & 24l \\
24l & 16l^2 & -24l & 8l^2 \\
-48 & -24l & 48 & -24l \\
24l & 8l^2 & -24l & 16l^2
\end{bmatrix}
\end{array}
\begin{array}{c} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{array}
$$

Global

$$
\frac{EI}{l^3}
\begin{array}{c}
\begin{array}{cccc} \theta_1 & w_2 & \theta_2 & \theta_3 \end{array} \\
\begin{bmatrix}
4l^2 & -6l & 2l^2 & 0 \\
-6l & 60 & 18l & 24l \\
2l^2 & 18l & 20l^2 & 8l^2 \\
0 & 24l & 8l^2 & 16l^2
\end{bmatrix}
\end{array}
\begin{array}{c} \theta_1 \\ w_2 \\ \theta_2 \\ \theta_3 \end{array}
$$

Geometric stiffness matrices

Pre-buckling problem: longitudinal force is equal to P

Elem. 1

$$
\begin{array}{cc}
 & \begin{array}{cccc} w_1 & \theta_1 & w_2 & \theta_2 \end{array} \\
\dfrac{1}{30}\dfrac{P}{l} & \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \begin{array}{c} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{array}
\end{array}
$$

Elem. 2

$$
\begin{array}{cc}
 & \begin{array}{cccc} w_2 & \theta_2 & w_3 & \theta_3 \end{array} \\
\dfrac{1}{30}\dfrac{P}{l} & \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \begin{array}{c} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{array}
\end{array}
$$

Global

$$
\begin{array}{cc}
 & \begin{array}{cccc} \theta_1 & w_2 & \theta_2 & \theta_3 \end{array} \\
\dfrac{1}{30}\dfrac{P}{l} & \begin{bmatrix} 4l^2 & -3l & -l^2 & 0 \\ -3l & 72 & 0 & 3l \\ -l^2 & 0 & 8l^2 & -l^2 \\ 0 & 3l & -l^2 & 4l^2 \end{bmatrix} \begin{array}{c} \theta_1 \\ w_2 \\ \theta_2 \\ \theta_3 \end{array}
\end{array}
$$

Buckling problem:

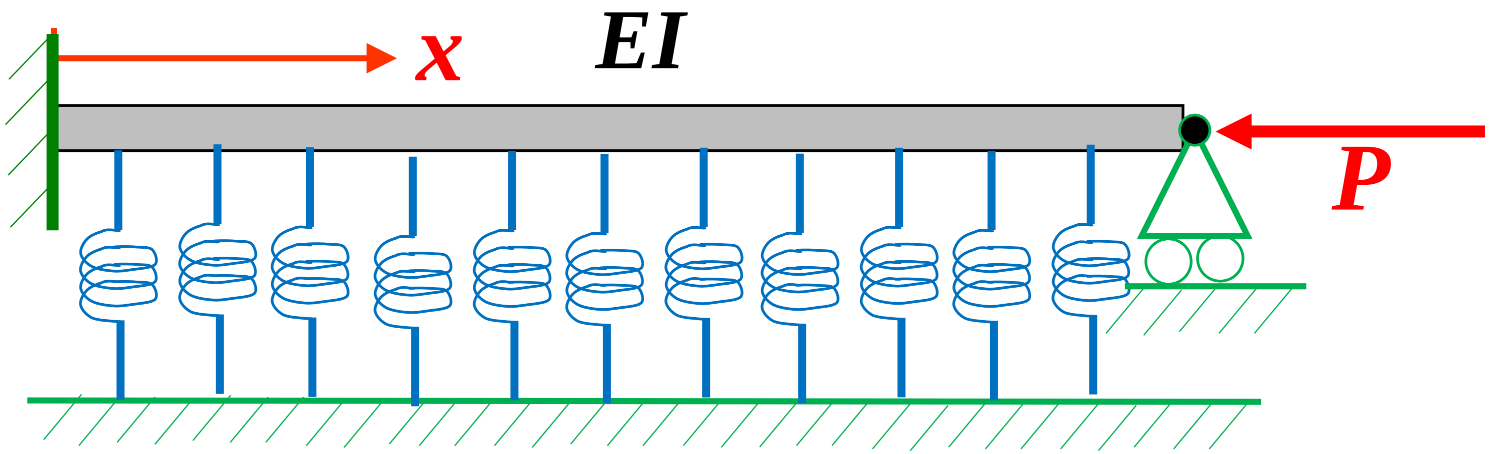
$$\left[\frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 & 0 \\ -6l & 60 & 18l & 24l \\ 2l^2 & 18l & 20l^2 & 8l^2 \\ 0 & 24l & 8l^2 & 16l^2 \end{bmatrix} - \frac{1}{30}\frac{P}{l}\begin{bmatrix} 4l^2 & -3l & -l^2 & 0 \\ -3l & 72 & 0 & 3l \\ -l^2 & 0 & 8l^2 & -l^2 \\ 0 & 3l & -l^2 & 4l^2 \end{bmatrix}\right]\begin{Bmatrix} \theta_1 \\ w_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The solution of the above problem yields 4 pairs of eigenvalues / eigenvectors

Only anti-symmetric modes are computed

Since the critical buckling load is symmetric this analysis does not have to be performed if the other eigenvalues are of no interest

A uniform straight beam of length l and flexural stiffness EI is clamped at $x = 0$ and simply supported at $x = l$. The beam is also attached to an elastic foundation of constant stiffness k per unit length. Derive the finite element formulation for the elastic foundation and estimate the buckling load using a single element.



Solution

The first step it to obtain the stiffness matrix of the elastic foundation.

The infinitesimal stored energy is:

$$dV = \frac{1}{2} k\left(x\right) dx \; w^2\left(x\right)$$

The transverse displacements $w(x)$ are given by: $w\left(\xi\right) = \left[N\left(\xi\right)\right]\left\{w_\theta\right\}$

Therefore: $w^2\left(\xi\right) = \left\{w_\theta\right\}^T \left[N\left(\xi\right)\right]^T \left[N\left(\xi\right)\right]\left\{w_\theta\right\}$

where $[N(\xi)]$ are the interpolation functions and $\{w_\theta\}$ are nodal displacements of the beam:

$$\left\{w_\theta\right\}^T = \left[w_1 \quad l\theta_1 \quad w_2 \quad l\theta_2\right]$$

The total stored energy is:

$$V = \frac{1}{2}\{w_\theta\}^T \left[k\int_0^l [N(x)]^T [N(x)]\,dx \right]\{w_\theta\} = \frac{1}{2}\{w_\theta\}^T \left[k\int_0^1 [N(\xi)]^T [N(\xi)]\,l\,d\xi \right]\{w_\theta\}$$

In the problem, only θ_2 is different from 0, therefore:

$$V = \frac{1}{2}\{l\theta_2\}^T \left[kl\int_0^1 [N_4(\xi)]^T [N_4(\xi)]\,d\xi \right]\{l\theta_2\} = \frac{1}{2}\left[kl\int_0^1 N_4^2(\xi)\,d\xi \right] l^2\theta_2^2$$

$N_4(\xi)$ is: $N_4(\xi) = -\xi^2 + \xi^3 \Rightarrow \int_0^1 N_4^2(\xi)\,d\xi = \frac{1}{105}$

Therefore: $V = \frac{1}{2}\{l\theta_2\}^T \left[\frac{kl}{105} \right]\{l\theta_2\}$

Stiffness matrices:

$$\frac{EI}{l^3}\begin{array}{c} \begin{array}{cccc} w_1 & l\theta_1 & w_2 & l\theta_2 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{array} \qquad \begin{array}{c} l\theta_2 \\ \begin{bmatrix} \dfrac{kl}{105} \end{bmatrix} \end{array} \begin{array}{c} \\ l\theta_2 \end{array}$$

The pre-buckling problem is not affected by the elastic foundation

Geometric stiffness matrices:

$$\frac{1}{30}\frac{P}{l}\begin{array}{c} \begin{array}{cccc} w_1 & l\theta_1 & w_2 & l\theta_2 \end{array} \\ \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{array}$$

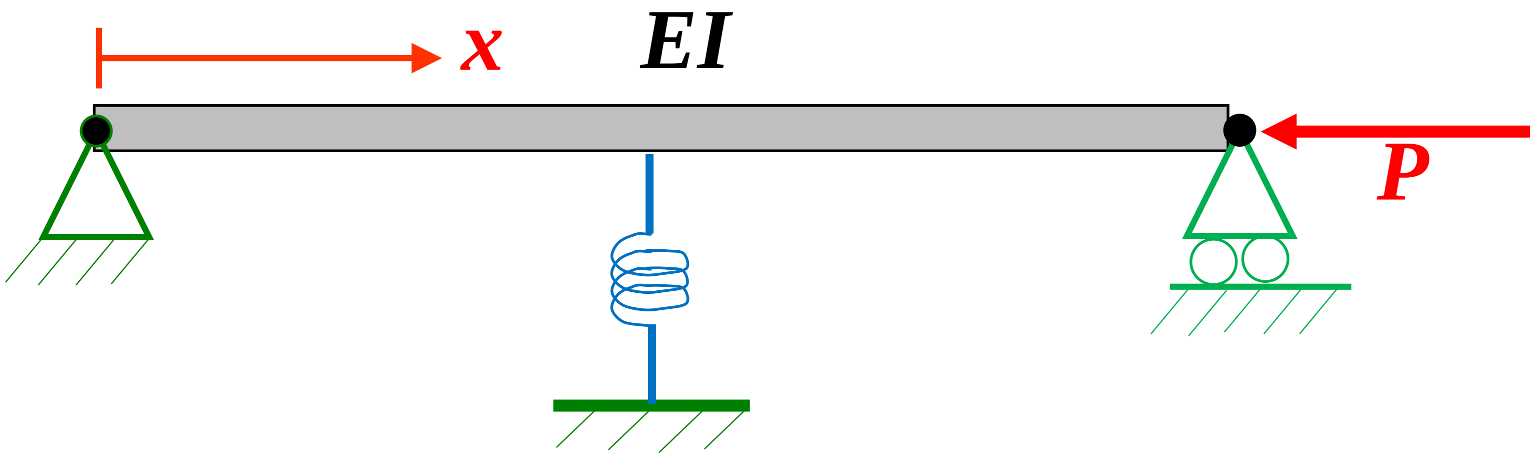
Keeping only the values for θ_2:

$$\left(\frac{4EI}{l^3}+\frac{kl}{105}-\frac{4}{30}\frac{P}{l}\right)(l\theta_2)=0$$

$$\frac{4}{30}\frac{P_{crit}}{l}=\frac{4EI}{l^3}+\frac{kl}{105}$$

$$\boxed{P_{crit}=\frac{30EI}{l^2}+\frac{kl^2}{14}}$$

A uniform straight beam of length l and flexural stiffness EI is simply supported at both ends and is compressed by a load P. The beam is attached to a spring of stiffness K at its center. Estimate the buckling load for the first symmetric mode using a single element.



Solution

The energy stored at the spring is:

$$V = \frac{1}{2} K w^2 \left(\frac{l}{2} \right)$$ **but:** $$x = \frac{l}{2} \Rightarrow \xi = \frac{1}{2}$$

$$w_1 = w_2 = 0 \Rightarrow w\left(\frac{1}{2}\right) = N_2\left(\frac{1}{2}\right) l\theta_1 + N_4\left(\frac{1}{2}\right) l\theta_2 = \frac{1}{8} l\theta_1 - \frac{1}{8} l\theta_2$$

$$\Rightarrow w^2\left(\frac{l}{2}\right) = \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix}^T \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} \end{bmatrix}^T \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} \end{bmatrix} \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix} = \frac{1}{64} \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix}$$

$$\Rightarrow V = \frac{1}{2} \frac{K}{64} \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} l\theta_1 \\ l\theta_2 \end{Bmatrix}$$

Stiffness matrices:

$$\frac{EI}{l^3}\begin{array}{c} \\ \end{array}\begin{array}{cccc|c} w_1 & l\theta_1 & w_2 & l\theta_2 & \\ \end{array}$$

$$\frac{EI}{l^3}\left[\begin{array}{cccc} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{array}\right]\begin{array}{c} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{array} \qquad \begin{array}{cc} l\theta_1 & l\theta_2 \end{array}$$

$$\frac{K}{64}\left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}\right]\begin{array}{c} l\theta_1 \\ l\theta_2 \end{array}$$

The pre-buckling problem is not affected by the spring

Geometric stiffness matrices:

$$\begin{array}{cccc} w_1 & l\theta_1 & w_2 & l\theta_2 \end{array}$$

$$\frac{1}{30}\frac{P}{l}\left[\begin{array}{cccc} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{array}\right]\begin{array}{c} w_1 \\ l\theta_1 \\ w_2 \\ l\theta_2 \end{array}$$

Keeping only the values for θ_1 and θ_2:

$$\left[\frac{EI}{l^3}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}+\frac{K}{64}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}-\frac{1}{30}\frac{P}{l}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}\right]\begin{Bmatrix}l\theta_1\\ l\theta_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Therefore:

$$\begin{bmatrix}4+\alpha-4\lambda & 2-\alpha+\lambda\\ 2-\alpha+\lambda & 4+\alpha-4\lambda\end{bmatrix}\begin{Bmatrix}l\theta_1\\ l\theta_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

where:

$$\alpha=\frac{Kl^3}{64EI}$$

$$\lambda=\frac{Pl^2}{30EI}$$

$$\Rightarrow \det\begin{bmatrix}4+\alpha-4\lambda & 2-\alpha+\lambda\\ 2-\alpha+\lambda & 4+\alpha-4\lambda\end{bmatrix}=0 \Rightarrow 15\lambda^2-(36+6\alpha)\lambda+12(1+\alpha)=0$$

$$\lambda_{crit}=\frac{36+6\alpha-\sqrt{(36+6\alpha)^2-4(15)12(1+\alpha)}}{30}=\frac{36+6\alpha-\sqrt{576-288\alpha+36\alpha^2}}{30}$$

$$\lambda_{crit} = \frac{36 + 6\alpha - \sqrt{576 - 288\alpha + 36\alpha^2}}{30}$$

$$P_{crit} = 30\frac{EI\lambda_{crit}}{l^2}$$

The value of the eigenvalue is bounded by: $0 \leq \lambda_{crit} \leq 2$

Consider a uniform, simply supported square plate with length l and thickness t. The modulus of elasticity is E and the Poisson ratio is ν. The plate is compressed by a uniaxial load N_x. Estimate the thickness for which the critical buckling load coincides with the load that causes yielding. The yielding stress is σ_y. Note that, for the intended application, wing panels should not buckle or yield.

Solution

Stress in the plate: $\boxed{\sigma_x = \dfrac{N_x}{t} \le \sigma_y}$

Critical buckling load:

$$N_{x,crit} = \frac{k_c \pi^2 D}{b^2}$$

where: *a* = *b* = *l* and *n* = 1

$$D = \frac{Et^3}{12\left(1-\nu^2\right)} \qquad k_c = \left(\frac{mb}{a} + \frac{n^2 a}{mb}\right)^2 = \left(m + \frac{1}{m}\right)^2$$

The minimum value for k_c is for *m* = 1: $k_c = 4$

$$\Rightarrow \quad N_{x,crit} = \frac{4\pi^2 D}{l^2} = \frac{4\pi^2 Et^3}{12\left(1-\nu^2\right) l^2}$$

The minimum value for k_c is for m = 1:

$$\boxed{\sigma_{x,crit} = \frac{N_{x,crit}}{t} = \frac{4\pi^2 E t^2}{12\left(1-\nu^2\right)l^2} \le \sigma_y}$$

Critical buckling stress = yield stress:

$$\frac{3\pi^2 E}{\left(1-\nu^2\right)}\frac{t^2}{l^2} = \sigma_y \Rightarrow \boxed{t = l\sqrt{\frac{3\left(1-\nu^2\right)}{\pi^2}\frac{\sigma_y}{E}}}$$