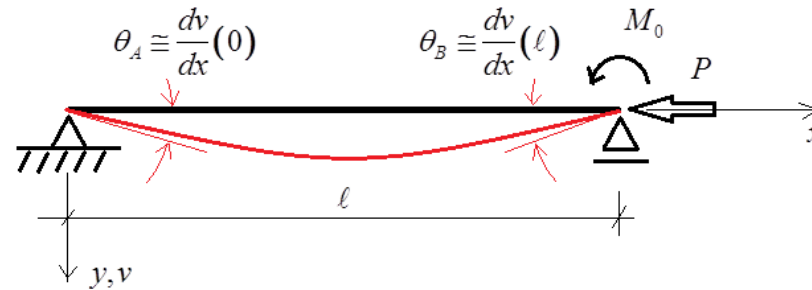


PEF-5916

Dinâmica e Estabilidade das Estruturas

Prof. Dr. Carlos Eduardo Nigro Mazzilli

Viga-coluna: teoria de segunda ordem linearizada



Problema fundamental

$$\frac{d^2v}{dx^2} = -\frac{M(x)}{EI} = -\frac{M_0}{EI} \left(\frac{x}{\ell} \right) - \frac{P}{EI} v \Rightarrow \frac{d^2v}{dx^2} + k^2 v = -\frac{M_0}{EI} \left(\frac{x}{\ell} \right)$$

$$v(x) = A \operatorname{sen} kx + B \cos kx - \frac{M_0}{P} \left(\frac{x}{\ell} \right) \quad \left. \begin{array}{l} v(0) = v(\ell) = 0 \end{array} \right\} v(x) = \frac{M_0}{P} \left(\frac{\operatorname{sen} kx}{\operatorname{sen} k\ell} - \frac{x}{\ell} \right)$$

$$\theta_A = \frac{M_0 \ell}{6EI} \Phi(u)$$

$$\theta_B = \frac{M_0 \ell}{3EI} \Psi(u)$$

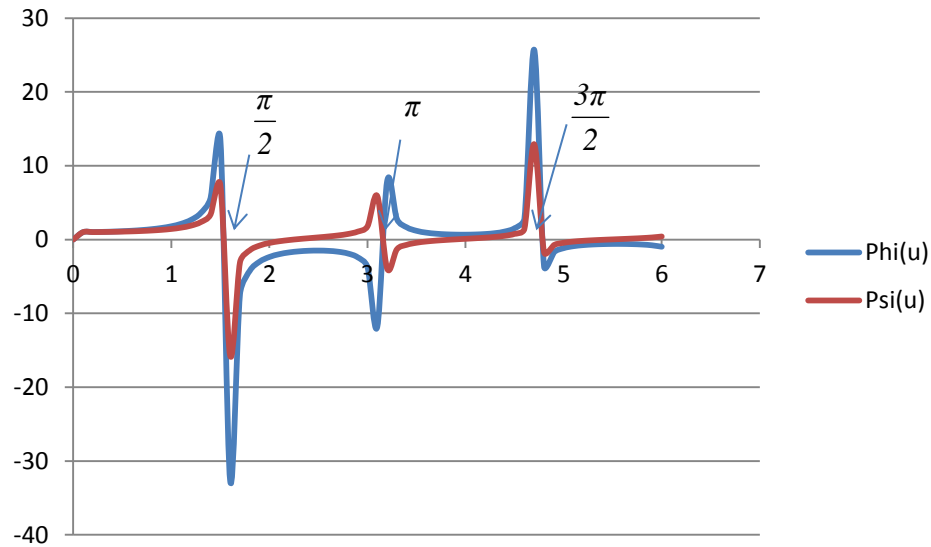
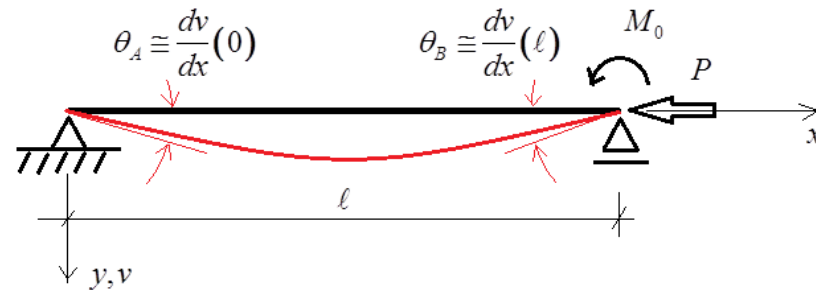
$$u = \frac{k\ell}{2}$$

com

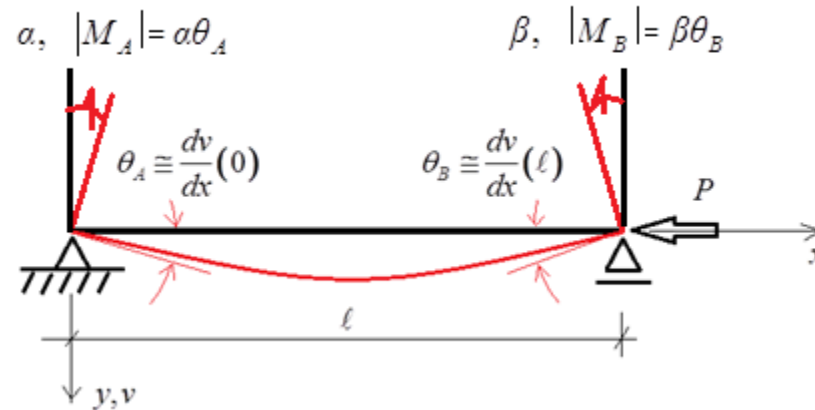
$$\Phi(u) = \frac{3}{u} \left(\frac{1}{\operatorname{sen} 2u} - \frac{1}{2u} \right)$$

$$\Psi(u) = \frac{3}{2u} \left(\frac{1}{2u} - \frac{1}{\tan 2u} \right)$$

Viga-coluna: teoria de segunda ordem linearizada



Carga crítica para barra com engastamento elástico



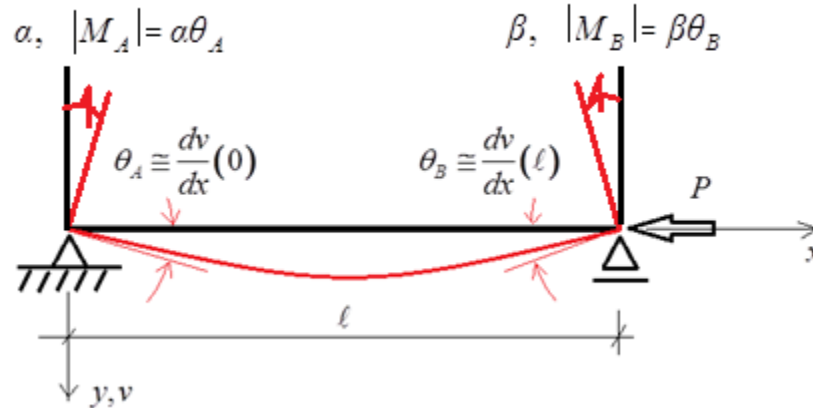
Compatibilidade:

$$\begin{aligned}
 \theta_A = -\frac{M_A}{\alpha} &= \frac{M_A \ell}{3EI} \Psi(u) + \frac{M_B \ell}{6EI} \Phi(u) \\
 \theta_B = -\frac{M_B}{\beta} &= \frac{M_A \ell}{6EI} \Phi(u) + \frac{M_B \ell}{3EI} \Psi(u)
 \end{aligned}
 \Rightarrow
 \begin{bmatrix}
 \frac{1}{\alpha} + \frac{\ell}{3EI} \Psi(u) & \frac{\ell}{6EI} \Phi(u) \\
 \frac{\ell}{6EI} \Phi(u) & \frac{1}{\beta} + \frac{\ell}{3EI} \Psi(u)
 \end{bmatrix}
 \begin{Bmatrix} M_A \\ M_B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Soluções não triviais:

$$\left[\frac{1}{\alpha} + \frac{\ell}{3EI} \Psi(u) \right] \left[\frac{1}{\beta} + \frac{\ell}{3EI} \Psi(u) \right] - \left[\frac{\ell}{6EI} \Phi(u) \right]^2 = 0 \Rightarrow u \Rightarrow P_{cr}$$

Carga crítica para barra com engastamento elástico



Exemplo: barra biengastada

$$\alpha = \beta = \infty$$

$$4[\Psi(u)]^2 - [\Phi(u)]^2 = 0 \Rightarrow [\Phi(u)] = \pm 2[\Psi(u)]$$

$$[\Phi(u)] = +2[\Psi(u)] \Rightarrow u = \tan u \Rightarrow u = 4,493 \Rightarrow P_{cr} = 80,75 \frac{EI}{\ell^2}$$

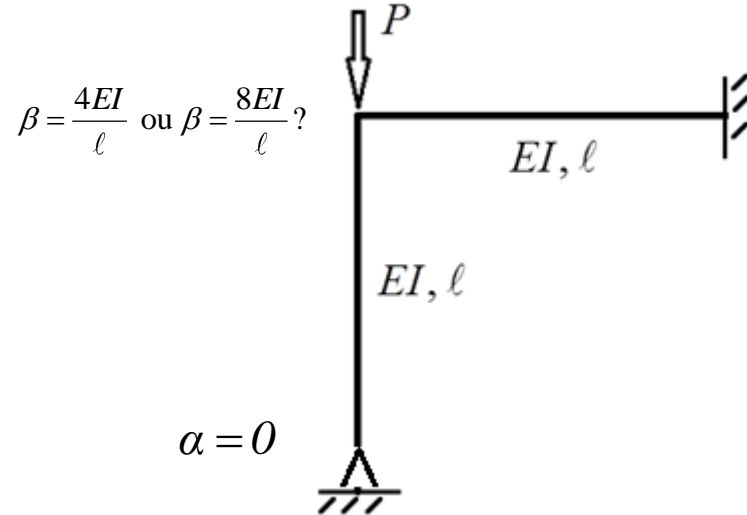
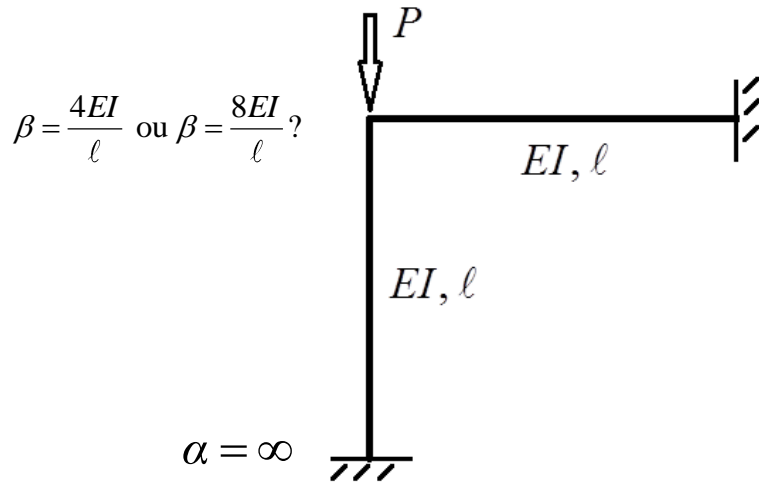


$$[\Phi(u)] = -2[\Psi(u)] \Rightarrow \cos 2u = 1 \Rightarrow u = \pi \Rightarrow P_{cr} = 4\pi^2 \frac{EI}{\ell^2}$$



Carga crítica para barra com engastamento elástico

Exemplo: pórtico simples



$$4\Psi(u) \left[\frac{3}{4} + \Psi(u) \right] - [\Phi(u)]^2 = 0 \Rightarrow u \Rightarrow P_{cr}$$



Lista 10

$$M_A = 0 \quad \text{se } \beta = \frac{4EI}{\ell} \dots$$

$$\theta_B = -\frac{M_B}{\beta} = \frac{M_B \ell}{3EI} \Psi(u)$$

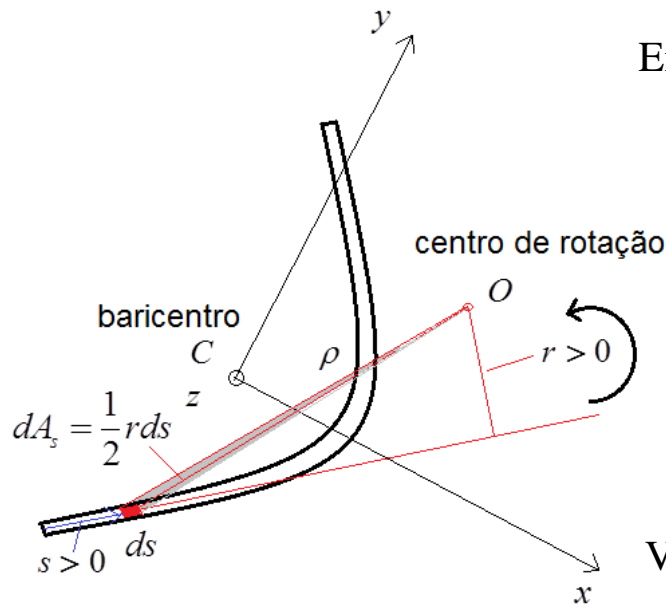


$$\Psi(u) = -\frac{3}{4} \Rightarrow u = 1,915 \Rightarrow P_{cr} = 14,7 \frac{EI}{\ell^2}$$

Torção uniforme de perfis abertos de parede delgada

Hipóteses:

- Rotação de torção por unidade de comprimento constante $\Phi' = \frac{d\Phi}{dz} = const$
- Fibras longitudinais permanecem retilíneas
- Empenamento da seção transversal livre $w_s \neq 0$



Empenamento:
$$\frac{dw_s}{ds} = -r\Phi' \Rightarrow w_s = w_0 - \Phi' \int_0^s r ds$$

Empenamento médio:
$$\bar{w}_s = w_0 - \Phi' \frac{1}{m} \int_0^m \left(\int_0^s r ds \right) ds$$

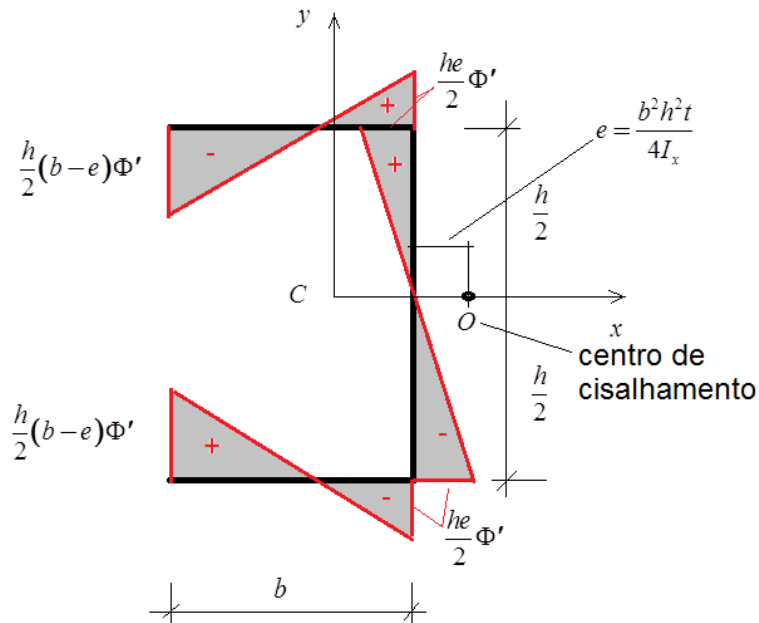
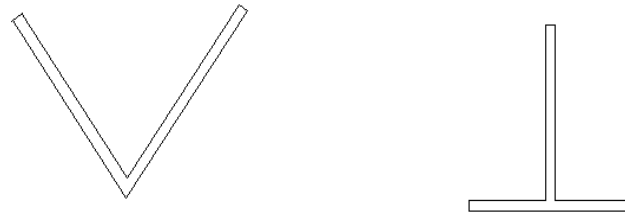
Função de empenamento:
$$\omega_s = \int_0^s r ds = 2A_s$$

Valor médio da função de empenamento:
$$\bar{\omega}_s = \frac{1}{m} \int_0^m \omega_s ds$$

Empenamento relativo:
$$w = w_s - \bar{w}_s = \Phi' (\bar{\omega}_s - \omega_s)$$

Torção uniforme de perfis abertos de parede delgada

Nota: se a seção transversal for constituída por flanges que se interceptam no centro de rotação O, a seção não empena!



Função de empenamento para perfil “C”

Torção não uniforme de perfis abertos de parede delgada

Hipóteses:

- Rotação de torção por unidade de comprimento variável $\Phi' = \frac{d\Phi}{dz}$
- Fibras longitudinais deixam de ser retilíneas, exceto a que passa pelo centro de cisalhamento O
- Empenamento da seção transversal livre ou impedido, dependendo das condições de contorno

$$M_t = \underbrace{M_{t1}}_{\text{momento de torção pura}} + \underbrace{M_{t2}}_{\text{momento de flexo-torção (flexão das flanges)}}$$

$$M_{t1} = C\Phi', \quad \text{com } C = GI_t \quad \text{e} \quad I_t = \sum \frac{m_i t_i^3}{3}$$

$$M_{t2} = -C_1\Phi''', \quad \text{com } C_1 = EC_\omega \quad \text{e} \quad C_\omega = \int_0^m (\bar{\omega}_s - \omega_s)^2 t ds$$

$$M_t = C\Phi' - C_1\Phi'''$$

Torção não uniforme de perfis abertos de parede delgada

Equilíbrio:

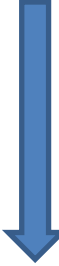
$$\frac{d\sigma_z}{dz} t dz ds + \frac{d(\tau t)}{ds} ds dz = 0 \Rightarrow \frac{d(\tau t)}{ds} = - \frac{d\sigma_z}{dz} t = - Et \underbrace{(\bar{\omega}_s - \omega_s)}_{\frac{d\epsilon_z}{dz}} \Phi'''$$



$$\tau t = -E\Phi''' \int_0^s (\bar{\omega}_s - \omega_s) t ds$$

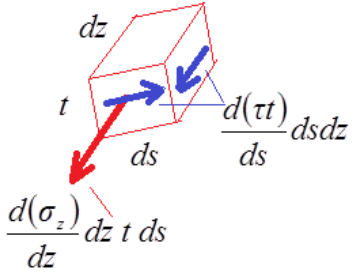
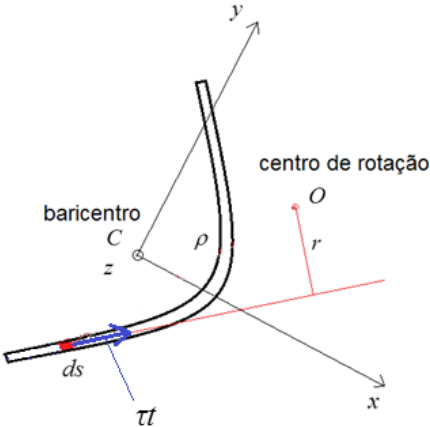


$$M_{t2} = \int_0^m \tau t r ds = -E\Phi''' \int_0^m \left[\int_0^s (\bar{\omega}_s - \omega_s) t ds \right] r ds$$

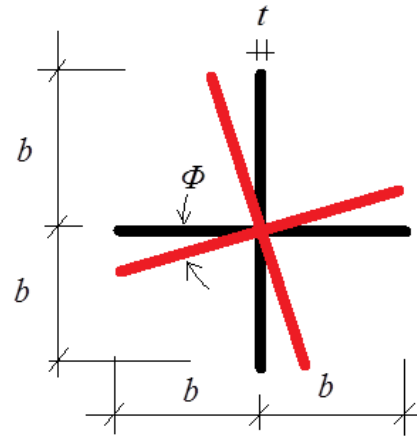
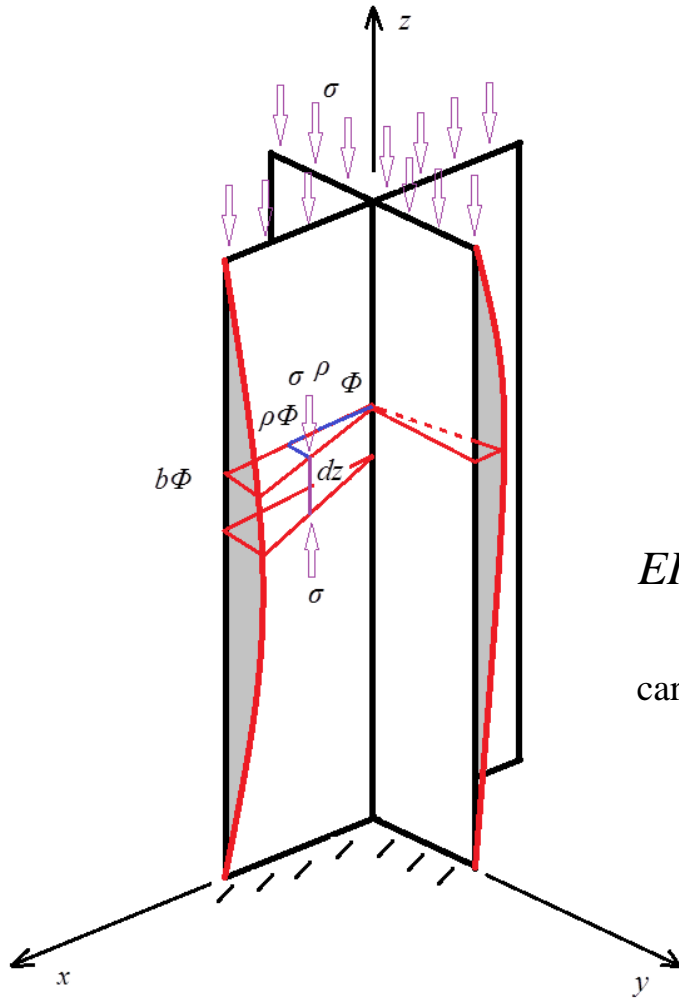


$$r ds = d\omega_s = - \frac{d(\bar{\omega}_s - \omega_s)}{ds} ds$$

$$M_{t2} = -C_1 \Phi''', \text{ com } C_1 = EC_\omega \text{ e } C_\omega = \int_0^m (\bar{\omega}_s - \omega_s)^2 t ds$$



Flambagem torcional de perfil cruciforme



$C_I = 0$
(seção não empenna)

$$C = G \frac{4bt^3}{3}$$

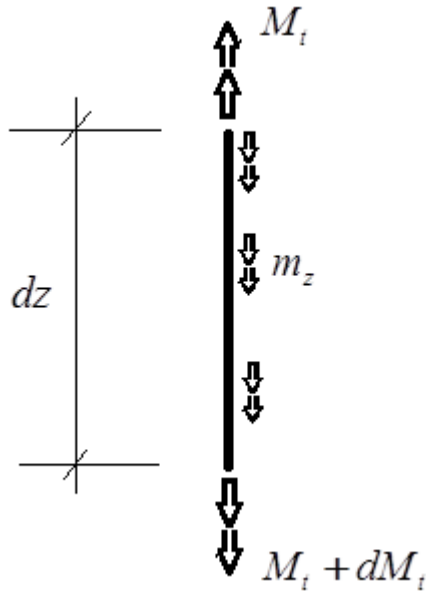
$$EI_x \frac{d^4 v}{dz^4} = -P \frac{d^2 v}{dz^2}$$

carga transversal equivalente $q = -P \frac{d^2 v}{dz^2} = -\sigma t d\rho(\rho\Phi'')$

momento torçor por unidade de comprimento

$$m_z = \int_A -\sigma t d\rho(\rho\Phi'')\rho = -\sigma\Phi'' \int_A \rho^2 dA = -\sigma\Phi'' I_o$$

Flambagem torcional de perfil cruciforme



Equilíbrio:

$$\frac{dM_t}{dz} = -m_z \Rightarrow C\Phi'' = \sigma\Phi''I_o \Rightarrow (C - \sigma I_o)\Phi'' = 0$$

Solução não trivial para:

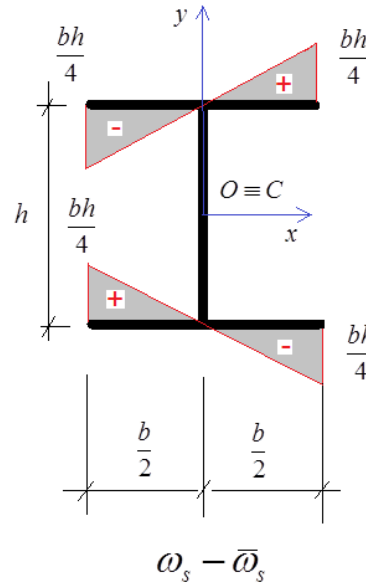
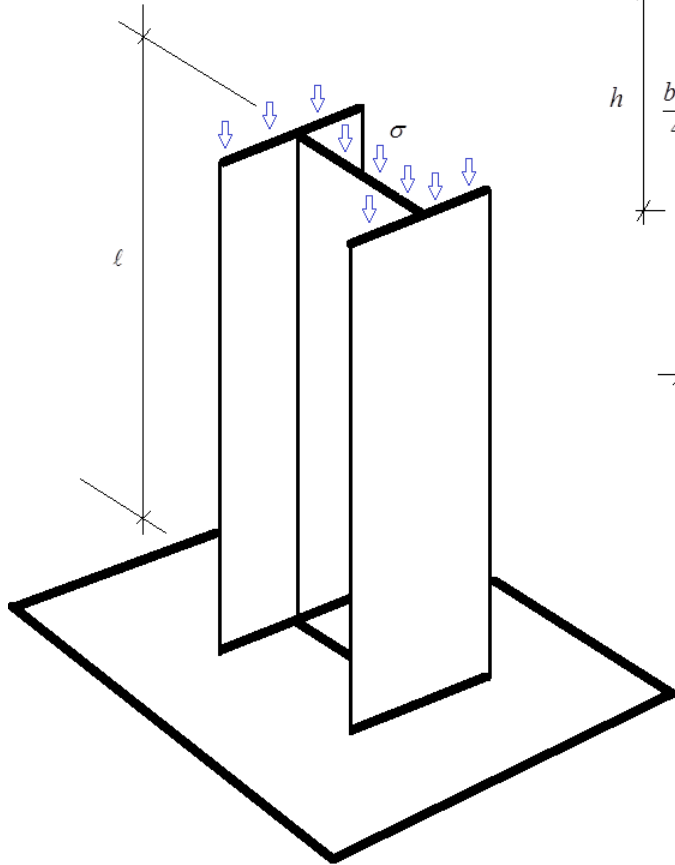
$$\sigma_{cr} = \frac{C}{I_o} = G\left(\frac{t}{b}\right)^2$$

Solução obtida para instabilidade da chapa:

$$\sigma_{cr} = \frac{C}{I_o} = \left[0,456 + \left(\frac{b}{\ell}\right)^2\right] \frac{\pi^2}{6(1-\nu)} G\left(\frac{t}{b}\right)^2$$

Para $b \ll \ell$ e $\nu = 0,25$ \longrightarrow $\sigma_{cr} \cong G\left(\frac{t}{b}\right)^2$

Flambagem torcional de perfil I



$$C = \frac{(2b+h)t^3}{3}$$

$$I_o = \frac{(6b+h)th^2}{12}$$

$$C_\omega = \int_s (\bar{\omega}_s - \omega_s)^2 t ds = 2 \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\frac{hx}{2} \right)^2 t ds = \frac{b^3 h^2 t}{24}$$

$$C_1 = EC_\omega = E \frac{b^3 h^2 t}{24} \neq 0$$

Equilíbrio:

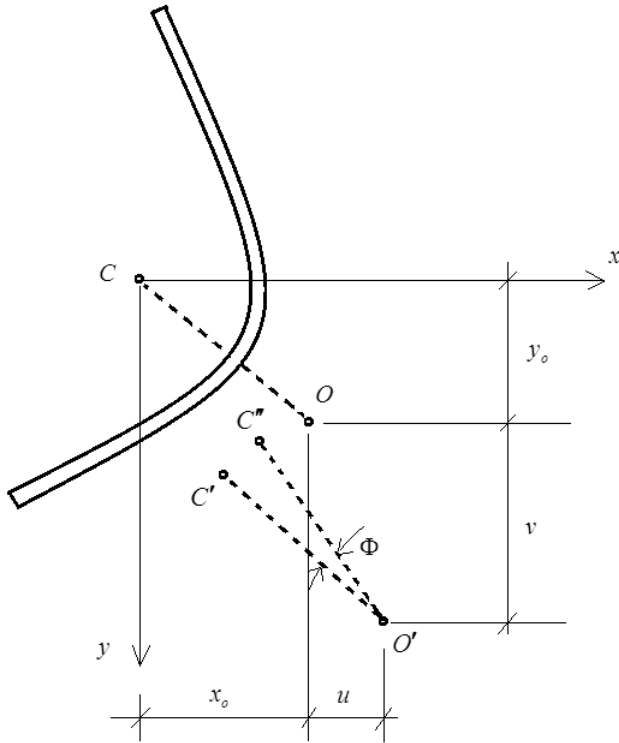
$$\frac{dM_t}{dz} = -m_z \Rightarrow C\Phi'' - C_1\Phi^{IV} = \sigma\Phi''I_o \Rightarrow \Phi^{IV} + p^2\Phi'' = 0$$

$$\text{com } p^2 = \frac{(\sigma I_o - C)}{C_1}$$



$$\Phi(z) = A_1 \text{sen}pz + A_2 \text{cosp}z + A_3 z + A_4$$

Flambagem por torção e flexão



Equilíbrio:

$$EI_x \frac{d^2 v}{dz^2} = -P(v - x_o \Phi)$$

$$EI_y \frac{d^2 u}{dz^2} = -P(u + y_o \Phi)$$

$$C_1 \Phi^{IV} + \left(\frac{I_0}{A} P - C \right) \Phi'' - P x_o v'' + P y_o u'' = 0$$

Problema desacoplado para $O \equiv C$, isto é, $x_o = y_o = 0$

$$P_{cr} = \min(P_x, P_y, P_\Phi)$$

$$P_x = \frac{\pi^2 EI_x}{\ell_x^2}, \quad P_y = \frac{\pi^2 EI_y}{\ell_y^2}, \quad P_\Phi = \left[\frac{C + p_{cr}^2 C_1}{I_o} \right] A$$

Flambagem por torção e flexão

Problema acoplado para $x_o \neq 0$ ou $y_o \neq 0$

Exemplo:

$$\left. \begin{aligned}
 u(0) = u(\ell) = 0; \quad u''(0) = u''(\ell) = 0 \\
 v(0) = v(\ell) = 0; \quad v''(0) = v''(\ell) = 0 \\
 \Phi(0) = \Phi(\ell) = 0; \quad \Phi''(0) = \Phi''(\ell) = 0
 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned}
 u(z) &= A_1 \operatorname{sen} \frac{\pi z}{\ell} \\
 v(z) &= A_2 \operatorname{sen} \frac{\pi z}{\ell} \\
 \Phi(z) &= A_3 \operatorname{sen} \frac{\pi z}{\ell}
 \end{aligned} \right.$$

$$\underbrace{\begin{bmatrix} P - P_y & 0 & P y_o \\ 0 & P - P_x & -P x_o \\ P y_o & -P x_o & \frac{I_o}{A} (P - P_\Phi) \end{bmatrix}}_{\mathbf{K}} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \Rightarrow |\mathbf{K}| = 0 \Rightarrow P_{cr}$$