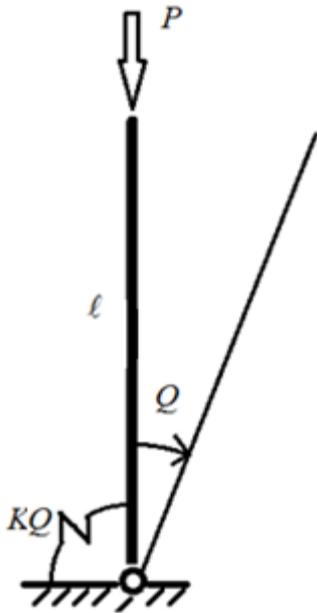


**PEF-5916**

# **Dinâmica e Estabilidade das Estruturas**

Prof. Dr. Carlos Eduardo Nigro Mazzilli

# Sistema sem Imperfeição: Bifurcação Simétrica Estável

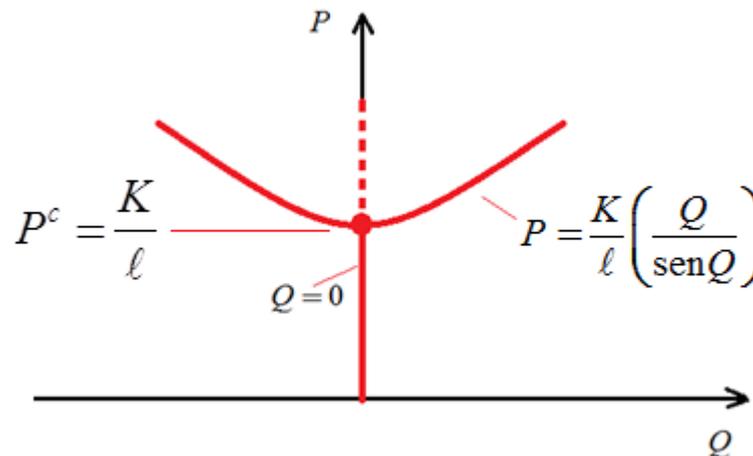


$$V = \frac{1}{2}KQ^2 - Pl(1 - \cos Q)$$

Equilíbrio

$$V_{,Q} = \frac{dV}{dQ} = KQ - Pl\text{sen}Q = 0$$

$$\Rightarrow Q = 0 \text{ ou } Q \neq 0 \text{ e } P = \frac{K}{\ell} \left( \frac{Q}{\text{sen}Q} \right) \cong \frac{K}{\ell} \left( 1 + \frac{Q^2}{6} \right)$$



# Sistema sem Imperfeição: Bifurcação Simétrica Estável

Estabilidade da trajetória primária (0)

$$\delta V = \frac{1}{2!} V_{,QQ}^0 \delta Q^2 + \frac{1}{3!} V_{,QQQ}^0 \delta Q^3 + \frac{1}{4!} V_{,QQQQ}^0 \delta Q^4 + \dots$$

$$V_{,QQ}^0 = K - P\ell; \quad V_{,QQQ}^0 = 0; \quad V_{,QQQQ}^0 = P\ell$$

Discussão:

$$P < \frac{K}{\ell} \Rightarrow V_{,QQ}^0 > 0 \Rightarrow Q = 0 \text{ é estável!}$$

$$P > \frac{K}{\ell} \Rightarrow V_{,QQ}^0 < 0 \Rightarrow Q = 0 \text{ é instável!}$$

$$P = \frac{K}{\ell} \Rightarrow V_{,QQ}^0 = 0; V_{,QQQ}^0 = 0; V_{,QQQQ}^0 > 0; \Rightarrow Q = 0 \text{ é estável!}$$

# Sistema sem Imperfeição: Bifurcação Simétrica Estável

Estabilidade da trajetória secundária (\*)

$$\delta V = \frac{1}{2!} V_{,QQ}^* \delta Q^2 + \frac{1}{3!} V_{,QQQ}^* \delta Q^3 + \frac{1}{4!} V_{,QQQQ}^* \delta Q^4 + \dots$$

$$V_{,QQ}^* = K - Pl \cos Q = K \left[ 1 - \frac{Q}{\tan Q} \right] > 0, \quad Q \neq 0$$

Portanto, a trajetória secundária (\*) é estável

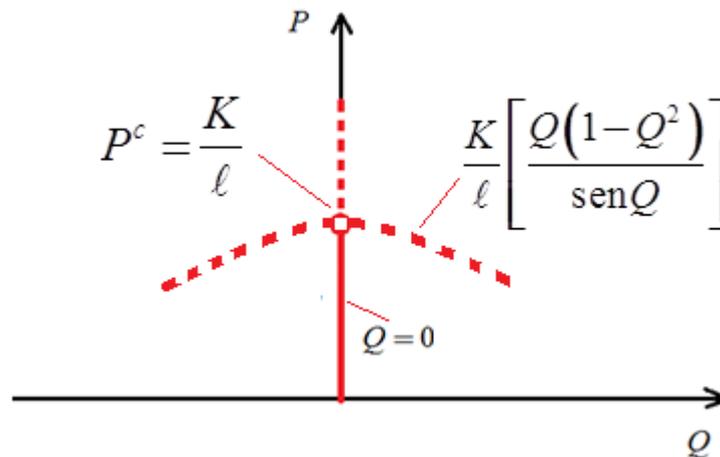
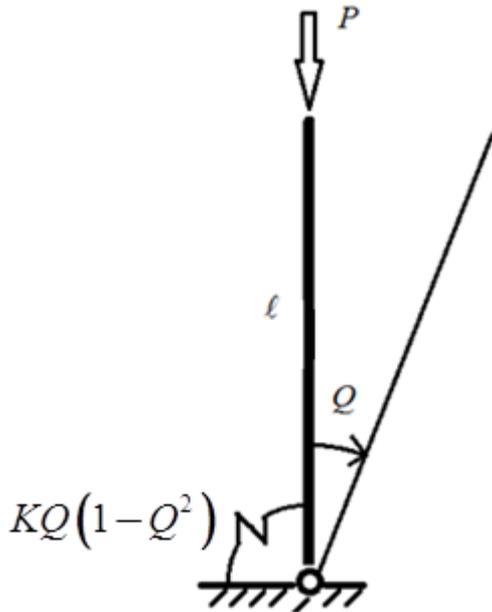
# Sistema sem Imperfeição: Bifurcação Simétrica Instável

$$V = K \left( \frac{Q^2}{2} - \frac{Q^4}{4} \right) - P\ell(1 - \cos Q)$$

Equilíbrio

$$V_{,Q} = \frac{dV}{dQ} = KQ(1 - Q^2) - P\ell \operatorname{sen} Q = 0$$

$$\Rightarrow Q = 0 \text{ ou } Q \neq 0 \text{ e } P = \frac{K}{\ell} \left[ \frac{Q(1 - Q^2)}{\operatorname{sen} Q} \right] \cong \frac{K}{\ell} (1 - Q^2)$$



# Sistema sem Imperfeição: Bifurcação Simétrica Instável

Estabilidade da trajetória primária (0)

$$\delta V = \frac{1}{2!} V_{,QQ}^0 \delta Q^2 + \frac{1}{3!} V_{,QQQ}^0 \delta Q^3 + \frac{1}{4!} V_{,QQQQ}^0 \delta Q^4 + \dots$$

$$V_{,QQ}^0 = K - P\ell; \quad V_{,QQQ}^0 = 0; \quad V_{,QQQQ}^0 = -6K + P\ell$$

Discussão:

$$P < \frac{K}{\ell} \Rightarrow V_{,QQ}^0 > 0 \Rightarrow Q = 0 \text{ é estável!}$$

$$P > \frac{K}{\ell} \Rightarrow V_{,QQ}^0 < 0 \Rightarrow Q = 0 \text{ é instável!}$$

$$P = \frac{K}{\ell} \Rightarrow V_{,QQ}^0 = 0; V_{,QQQ}^0 = 0; V_{,QQQQ}^0 < 0; \Rightarrow Q = 0 \text{ é instável!}$$

# Sistema sem Imperfeição: Bifurcação Simétrica Instável

Estabilidade da trajetória secundária (\*)

$$Q \neq 0 \text{ e } P = \frac{K}{\ell} \left[ \frac{Q(1-Q^2)}{\text{sen}Q} \right] \cong \frac{K}{\ell} (1-Q^2)$$

$$\delta V = \frac{1}{2!} V_{,QQ}^* \delta Q^2 + \frac{1}{3!} V_{,QQQ}^* \delta Q^3 + \frac{1}{4!} V_{,QQQQ}^* \delta Q^4 + \dots$$

$$V_{,QQ}^* = K(1-3Q^2) - P\ell \cos Q \cong -\frac{3}{2} KQ^2 < 0$$

Portanto, a trajetória secundária (\*) é instável

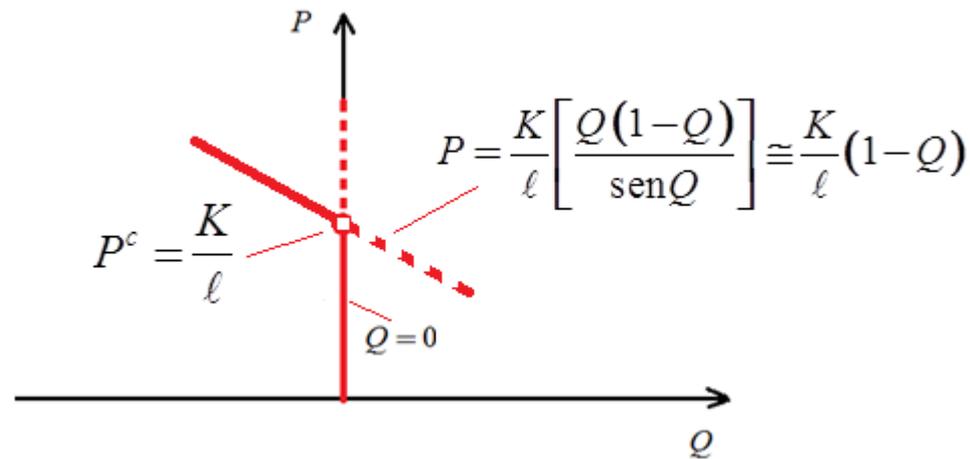
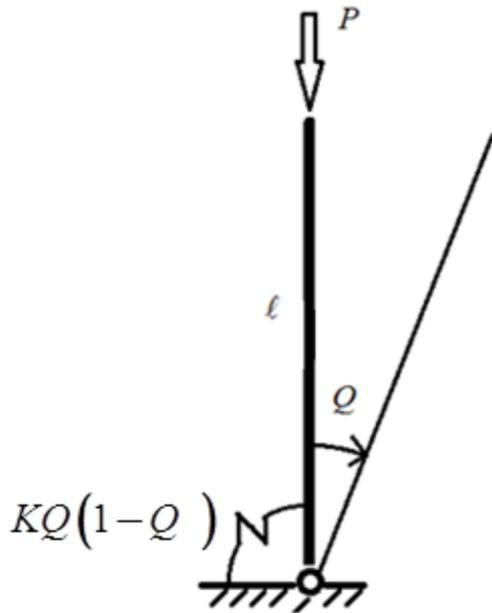
# Sistema sem Imperfeição: Bifurcação Assimétrica (Instável)

$$V = K \left( \frac{Q^2}{2} - \frac{Q^3}{3} \right) - P\ell(1 - \cos Q)$$

Equilíbrio

$$V_{,Q} = \frac{dV}{dQ} = KQ(1 - Q) - P\ell \operatorname{sen} Q = 0$$

$$\Rightarrow Q = 0 \text{ ou } Q \neq 0 \text{ e } P = \frac{K}{\ell} \left[ \frac{Q(1 - Q)}{\operatorname{sen} Q} \right] \cong \frac{K}{\ell} (1 - Q)$$



# Sistema sem Imperfeição: Bifurcação Assimétrica (Instável)

Estabilidade da trajetória primária (0)

$$\delta V = \frac{1}{2!} V_{,QQ}^0 \delta Q^2 + \frac{1}{3!} V_{,QQQ}^0 \delta Q^3 + \frac{1}{4!} V_{,QQQQ}^0 \delta Q^4 + \dots$$

$$V_{,QQ}^0 = K - P\ell; \quad V_{,QQQ}^0 = -2K; \quad V_{,QQQQ}^0 = P\ell$$

Discussão:

$$P < \frac{K}{\ell} \Rightarrow V_{,QQ}^0 > 0 \Rightarrow Q = 0 \text{ é estável!}$$

$$P > \frac{K}{\ell} \Rightarrow V_{,QQ}^0 < 0 \Rightarrow Q = 0 \text{ é instável!}$$

$$P = \frac{K}{\ell} \Rightarrow V_{,QQ}^0 = 0; V_{,QQQ}^0 \neq 0; V_{,QQQQ}^0 > 0; \Rightarrow Q = 0 \text{ é instável!}$$

# Sistema sem Imperfeição: Bifurcação Assimétrica (Instável)

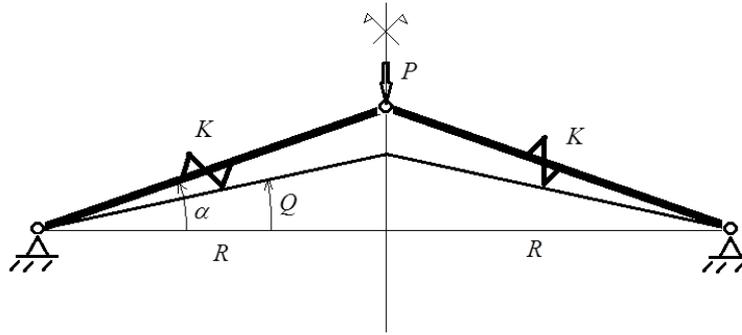
Estabilidade da trajetória secundária (\*)

$$Q \neq 0 \text{ e } P = \frac{K}{\ell} \left[ \frac{Q(1-Q)}{\text{sen}Q} \right] \cong \frac{K}{\ell} (1-Q)$$

$$\delta V = \frac{1}{2!} V_{,QQ}^* \delta Q^2 + \frac{1}{3!} V_{,QQQ}^* \delta Q^3 + \frac{1}{4!} V_{,QQQQ}^* \delta Q^4 + \dots$$

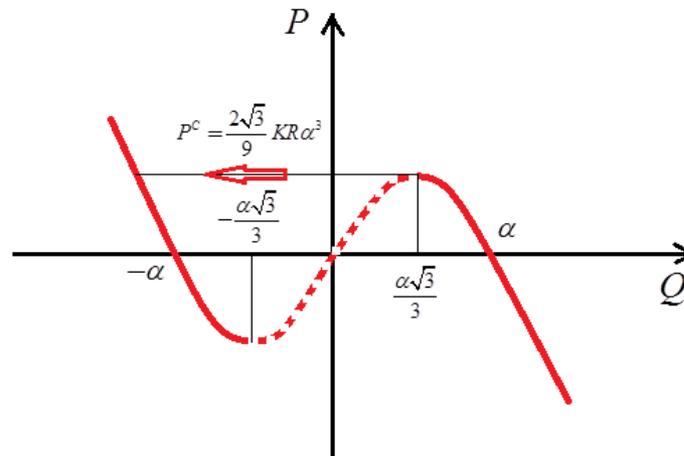
$$V_{,QQ}^* = K(1-2Q) - P\ell \cos Q \cong -KQ \Rightarrow \begin{cases} V_{,QQ}^* < 0 \text{ para } Q > 0 \Rightarrow \text{instável} \\ V_{,QQ}^* > 0 \text{ para } Q < 0 \Rightarrow \text{estável} \end{cases}$$

# Sistema sem Imperfeição: Ponto Limite (Instável)



$$V = 2 \frac{1}{2} KR^2 \left[ \frac{1}{\cos \alpha} - \frac{1}{\cos Q} \right]^2 - PR(\tan \alpha - \tan Q) \cong \frac{KR^2}{4} (\alpha^2 - Q^2)^2 - PR(\alpha - Q)$$

Equilíbrio  $V_{,Q} = \frac{dV}{dQ} \cong KR^2 Q(Q^2 - \alpha^2) + PR \cong 0 \Rightarrow P \cong KRQ(\alpha^2 - Q^2)$



# Sistema sem Imperfeição: Ponto Limite (Instável)

Estabilidade da trajetória (\*)

$$\delta V = \frac{1}{2!} V_{,QQ}^0 \delta Q^2 + \frac{1}{3!} V_{,QQQ}^0 \delta Q^3 + \frac{1}{4!} V_{,QQQQ}^0 \delta Q^4 + \dots$$

$$V_{,QQ}^* = KR^2 (3Q^2 - \alpha^2) + \frac{\partial P}{\partial Q} R = 2KR^2 (3Q^2 - \alpha^2);$$

$$V_{,QQQ}^* = 12KR^2 Q; \quad V_{,QQQQ}^0 = 12KR^2$$

Discussão:

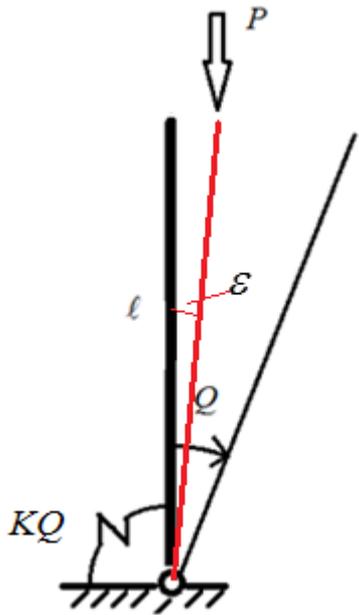
$$Q < -\frac{\alpha\sqrt{3}}{3} \quad \vee \quad Q > \frac{\alpha\sqrt{3}}{3} \Rightarrow V_{,QQ}^0 > 0 \Rightarrow (*) \text{ é estável!}$$

$$-\frac{\alpha\sqrt{3}}{3} < Q < \frac{\alpha\sqrt{3}}{3} \Rightarrow V_{,QQ}^0 < 0 \Rightarrow (*) \text{ é instável!}$$

$$Q = \pm \frac{\alpha\sqrt{3}}{3} \Rightarrow V_{,QQ}^* = 0 \quad \text{e} \quad V_{,QQQ}^* \neq 0 \Rightarrow (*) \text{ é instável!}$$

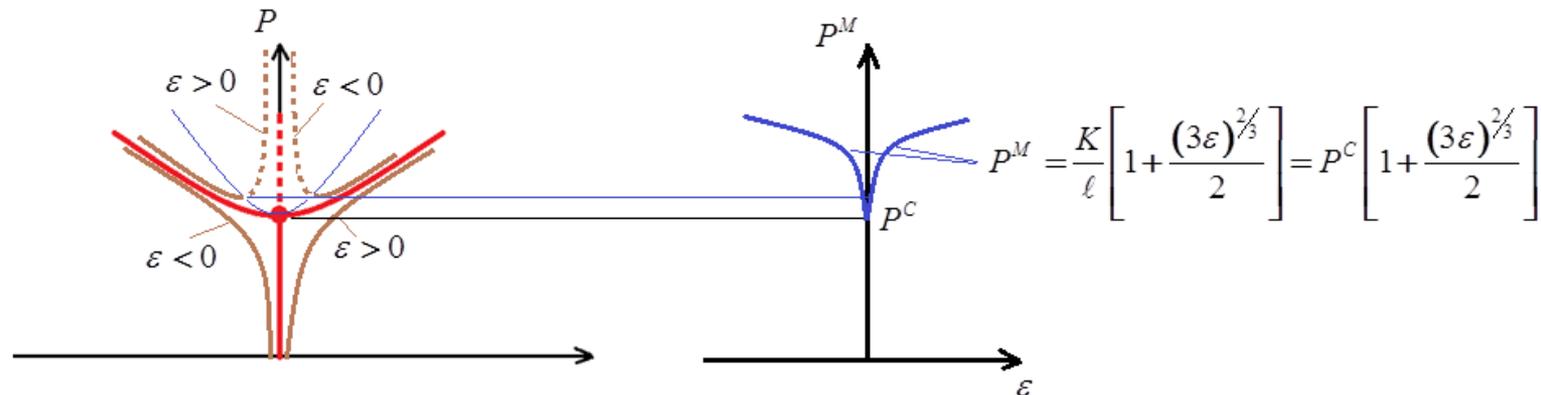
# Sistema com Imperfeição: Bifurcação Simétrica Estável

$$V = \frac{1}{2}K(Q - \varepsilon)^2 - Pl(\cos \varepsilon - \cos Q) \cong K\left(\frac{Q^2}{2} - Q\varepsilon + \frac{\varepsilon^2}{2}\right) - Pl\left(\frac{Q^2}{2} - \frac{Q^4}{24} - \frac{\varepsilon^2}{2}\right)$$



Equilíbrio

$$V_{,Q} = \frac{dV}{dQ} = K(Q - \varepsilon) - Pl\left(Q - \frac{Q^3}{6}\right) = 0 \Rightarrow P \cong \frac{K}{l}\left(1 + \frac{Q^2}{6} - \frac{\varepsilon}{Q}\right)$$

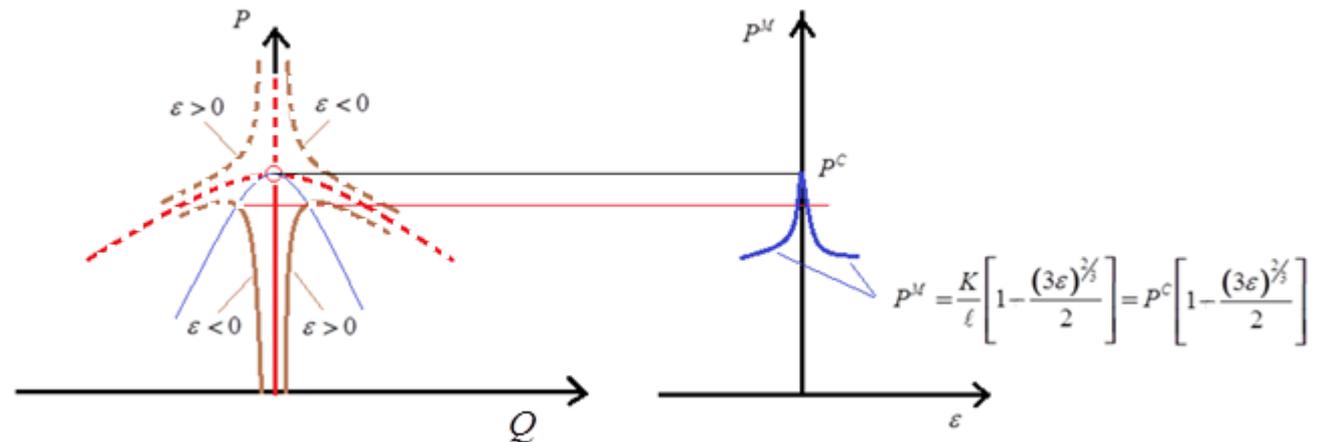
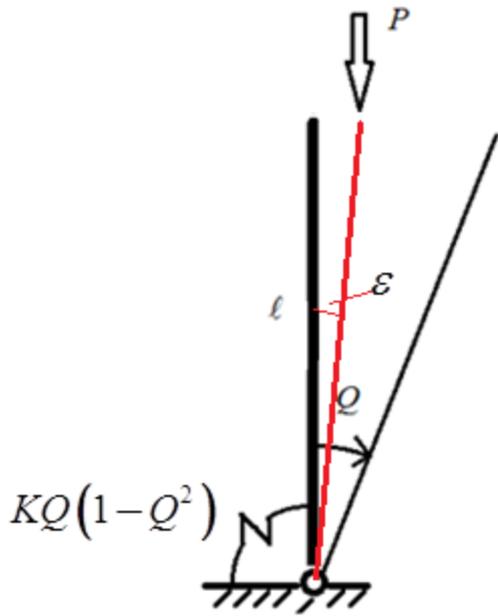


# Sistema com Imperfeição: Bifurcação Simétrica Instável

$$V = K \left[ \frac{(Q - \varepsilon)^2}{2} - \frac{(Q - \varepsilon)^4}{4} \right] - Pl(\cos \varepsilon - \cos Q) \cong K \left( \frac{Q^2}{2} - Q\varepsilon + \frac{\varepsilon^2}{2} - \frac{Q^4}{4} + \frac{\varepsilon^4}{4} \right) - Pl \left( \frac{Q^2}{2} - \frac{\varepsilon^2}{2} \right)$$

Equilíbrio

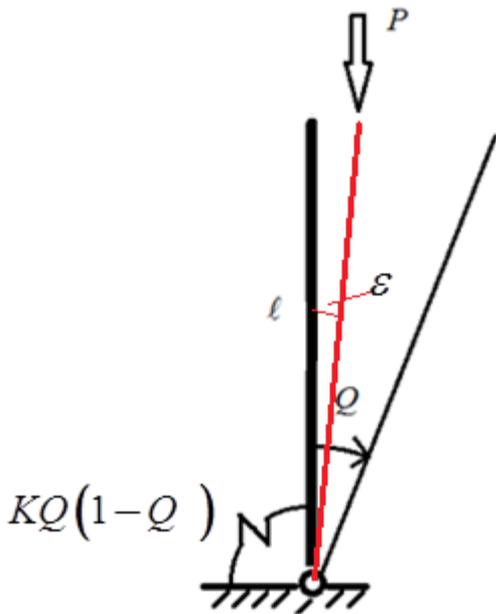
$$V_{,Q} = \frac{dV}{dQ} = K(Q - \varepsilon - Q^3) - PlQ = 0 \Rightarrow P \cong \frac{K}{\ell} \left( 1 - Q^2 - \frac{\varepsilon}{Q} \right)$$



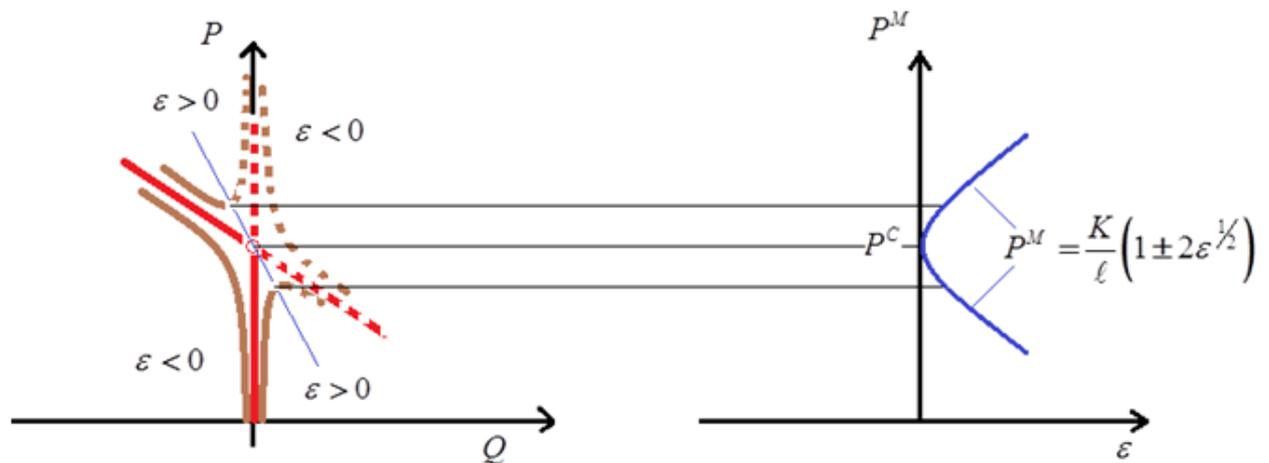
# Sistema com Imperfeição: Bifurcação Assimétrica (Instável)

$$V = K \left[ \frac{(Q - \varepsilon)^2}{2} - \frac{(Q - \varepsilon)^3}{3} \right] - Pl(\cos \varepsilon - \cos Q) \cong K \left[ \frac{Q^2}{2} - \frac{Q^3}{3} - Q\varepsilon + \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} \right] - Pl \left( \frac{Q^2}{2} - \frac{\varepsilon^2}{2} \right)$$

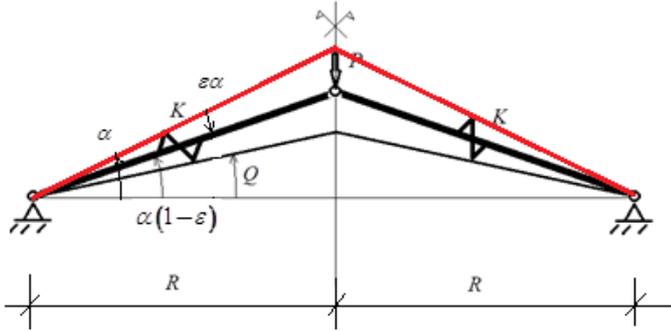
Equilíbrio



$$V_{,Q} = \frac{dV}{dQ} = K(Q - Q^2 - \varepsilon) - PlQ = 0 \Rightarrow P = \frac{K}{l} \left( 1 - Q - \frac{\varepsilon}{Q} \right)$$



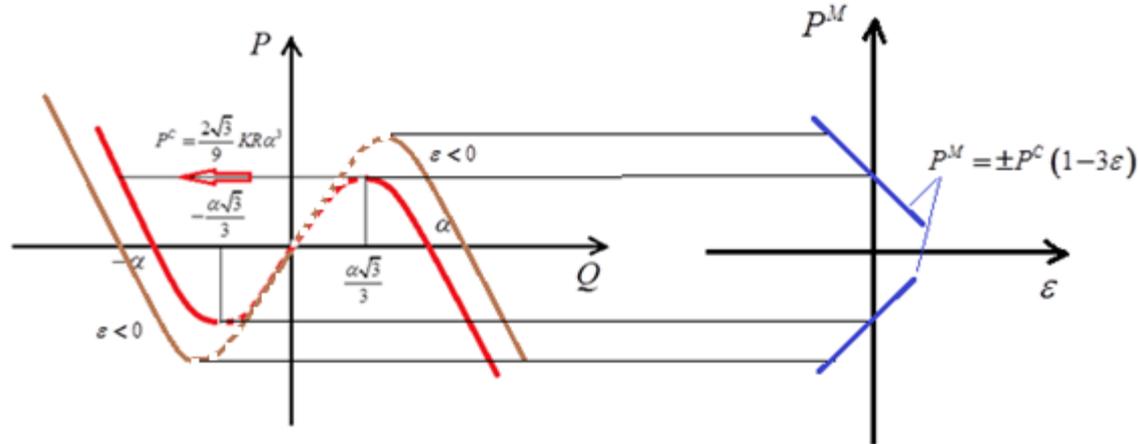
# Sistema com Imperfeição: Ponto Limite (Instável)



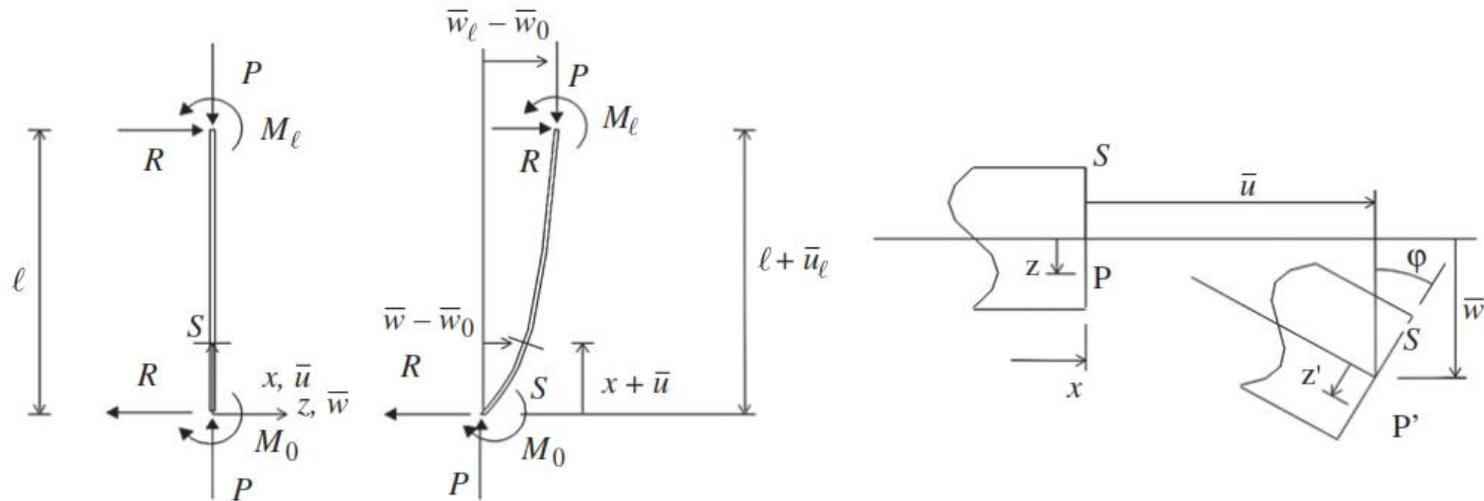
$$V = 2 \frac{1}{2} KR^2 \left[ \frac{1}{\cos \alpha(1-\varepsilon)} - \frac{1}{\cos Q} \right]^2 - PR [\tan \alpha(1-\varepsilon) - \tan Q]$$

$$\cong \frac{KR^2}{4} [\alpha^4(1-\varepsilon)^4 - 2\alpha^2(1-\varepsilon)^2 Q^2 + Q^4] - PR [\alpha(1-\varepsilon) - Q]$$

Equilíbrio  $V_{,Q} = \frac{dV}{dQ} \cong KR^2 Q [Q^2 - \alpha^2(1-\varepsilon)^2] + PR \cong 0 \Rightarrow P \cong KRQ [\alpha^2(1-\varepsilon)^2 - Q^2]$



# Flambagem de Euler: Barra Prismática sem Imperfeição



$$u = \bar{u} - z \sin \varphi,$$

$$w = \bar{w} - z(1 - \cos \varphi),$$

$$\sin \varphi = \frac{\bar{w}'}{\bar{\lambda}} \Rightarrow \bar{w}' = \bar{\lambda} \sin \varphi,$$

$$\cos \varphi = \frac{1 + \bar{u}'}{\bar{\lambda}} \Rightarrow 1 + \bar{u}' = \bar{\lambda} \cos \varphi, \quad \bar{\lambda} = \sqrt{(1 + \bar{u}')^2 + (\bar{w}')^2}.$$

# Flambagem de Euler: Barra Prismática sem Imperfeição

Lei de Hooke

$$\sigma = E(\lambda - 1)$$

$$\lambda = \sqrt{(1 + u'^2) + w'^2} = \bar{\lambda} - z\varphi' \Rightarrow \sigma = E(\bar{\lambda} - 1) - Ez\varphi' \quad \Rightarrow$$

Esforços resistentes

$$N = EA(\bar{\lambda} - 1),$$

$$M = -EI\varphi'.$$

$$N = -P \cos \varphi + R \sin \varphi,$$

Esforços solicitantes

$$M = M_\ell - R[(\ell + \bar{u}_\ell) - (x + \bar{u})] - P(\bar{w}_\ell - \bar{w})$$

$$= M_0 + R(x + \bar{u}) + P\bar{w},$$

Portanto

$$EA(\bar{\lambda} - 1) = -P \cos \varphi + R \sin \varphi, \quad \Rightarrow \quad \bar{\lambda} = 1 - \left( \frac{P}{EA} \cos \varphi - \frac{R}{EA} \sin \varphi \right).$$

$$-EI\varphi' = M_\ell - R[(\ell + \bar{u}_\ell) - (x + \bar{u})] - P(\bar{w}_\ell - \bar{w})$$

$$= M_0 + R(x + \bar{u}) + P\bar{w}.$$



$$-EI\varphi'' = R(1 + \bar{u}') + P\bar{w}' = \bar{\lambda}(R \cos \varphi + P \sin \varphi).$$

# Flambagem de Euler: Barra Prismática sem Imperfeição

Equação geral

$$EI\varphi'' + \left[ 1 - \left( \frac{P}{EA} \cos \varphi - \frac{R}{EA} \sin \varphi \right) \right] (R \cos \varphi + P \sin \varphi) = 0.$$

Normalizando

Exata

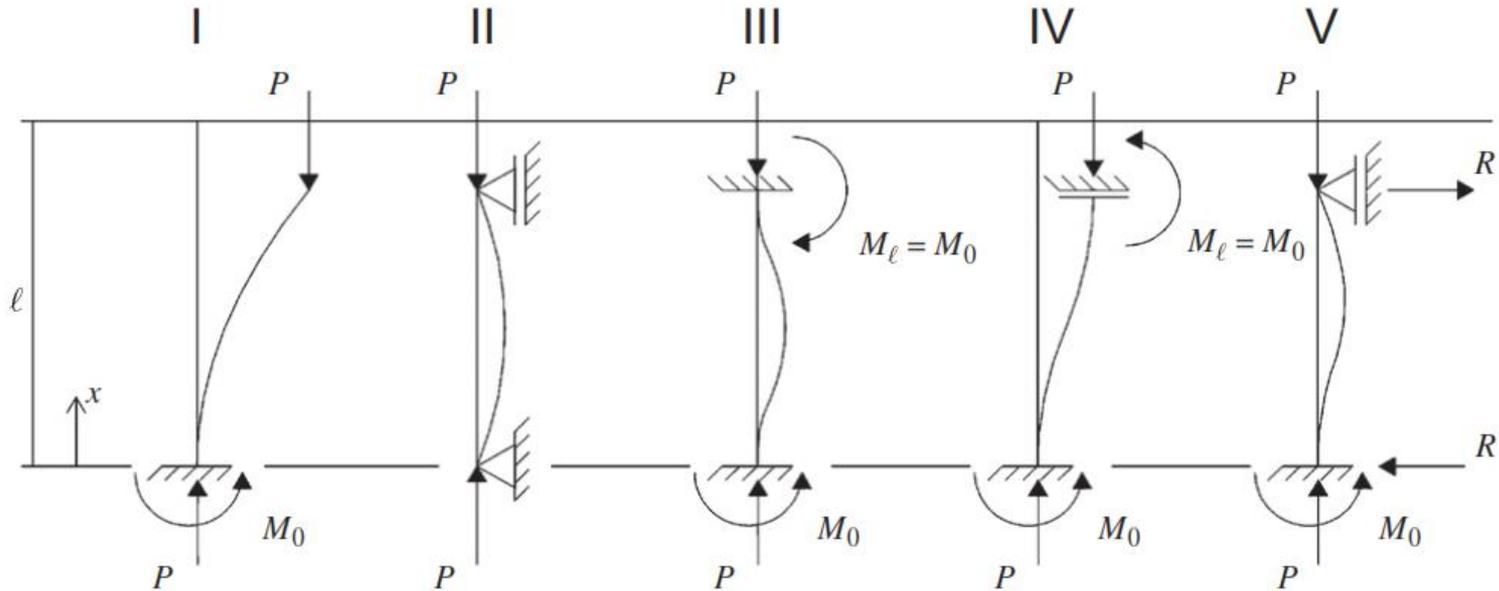
$$\left\{ \begin{array}{l} \frac{d^2 \varphi}{d\xi^2} + \frac{p}{\eta} [1 - p(\cos \varphi - \alpha \sin \varphi)] (\alpha \cos \varphi + \sin \varphi) = 0, \\ p = \frac{P}{EA}, \quad \alpha = \frac{R}{P}, \quad \xi = \frac{x}{\ell}, \quad \eta = \frac{I}{A\ell^2}. \end{array} \right.$$

Aproximada

$$\left\{ \begin{array}{l} \frac{d^2 \varphi}{d\xi^2} + \alpha_1 \varphi + \varepsilon \alpha_2 \varphi^2 + \alpha_3 \varphi^3 = \varepsilon \alpha_0, \\ \varepsilon \alpha_0 = -\frac{\alpha p(1-p)}{\eta}, \quad \varepsilon \alpha_2 = -\frac{\alpha p(1-4p)}{2\eta}, \\ \alpha_1 = \frac{p(1-p+\alpha^2 p)}{\eta}, \quad \alpha_3 = -\frac{p(1-4p+4\alpha^2 p)}{6\eta}. \end{array} \right.$$

$0 < \varepsilon \ll 1$

# Flambagem de Euler: Barra Prismática sem Imperfeição

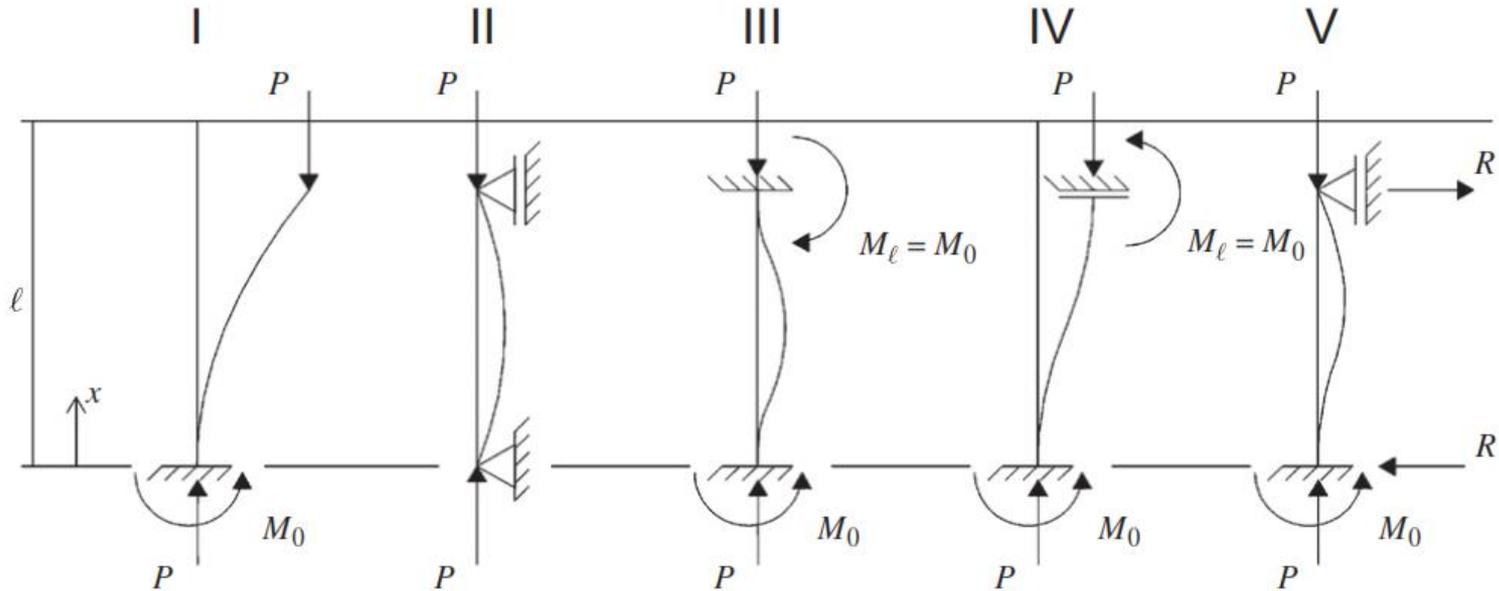


Equação linearizada

$$\frac{d^2 \varphi}{d\xi^2} + \alpha_1 \varphi = \varepsilon \alpha_0.$$

Casos I a IV  $\alpha = 0 \Rightarrow \varepsilon \alpha_0 = 0,$   
 Caso V  $\alpha = \frac{R}{P} = \frac{EI \varphi'(0)}{Pl \left(1 + \frac{\bar{u}_l}{l}\right)} \approx \frac{\frac{d\varphi}{d\xi}(0)}{\frac{p(1-p)}{\eta}} \Rightarrow \frac{d\varphi}{d\xi}(0) \approx \frac{\alpha p(1-p)}{\eta}$   
 $\Rightarrow \varepsilon \alpha_0 \approx -\frac{d\varphi}{d\xi}(0).$

# Flambagem de Euler: Barra Prismática sem Imperfeição



Equação linearizada da barra extensível

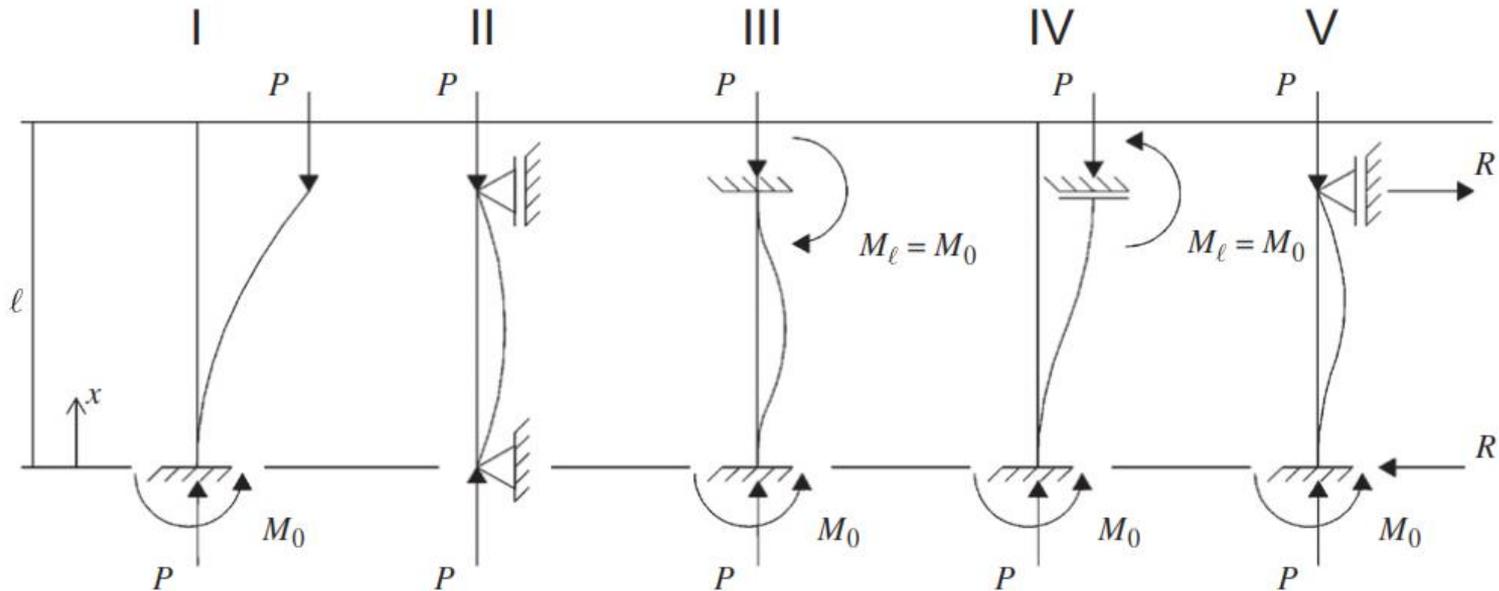
$$\frac{d^2 \varphi}{d\xi^2} + \alpha_1 \varphi = \varepsilon \alpha_0.$$

X

Equação linearizada da barra inextensível

$$\frac{d^2 \varphi}{d\xi^2} + (kl)^2 \varphi = \varepsilon \alpha_0, \quad kl = \sqrt{\frac{p}{\eta}}$$

# Flambagem de Euler: Barra Prismática sem Imperfeição



Solução para barra extensível

$$\varphi = \frac{\varepsilon \alpha_0}{\alpha_1} + C_1 \sin(\sqrt{\alpha_1} \xi) + C_2 \cos(\sqrt{\alpha_1} \xi)$$

$$\text{Case I} \Rightarrow \varphi(0) = 0 \quad \text{and} \quad \frac{d\varphi}{d\xi}(1) = 0,$$

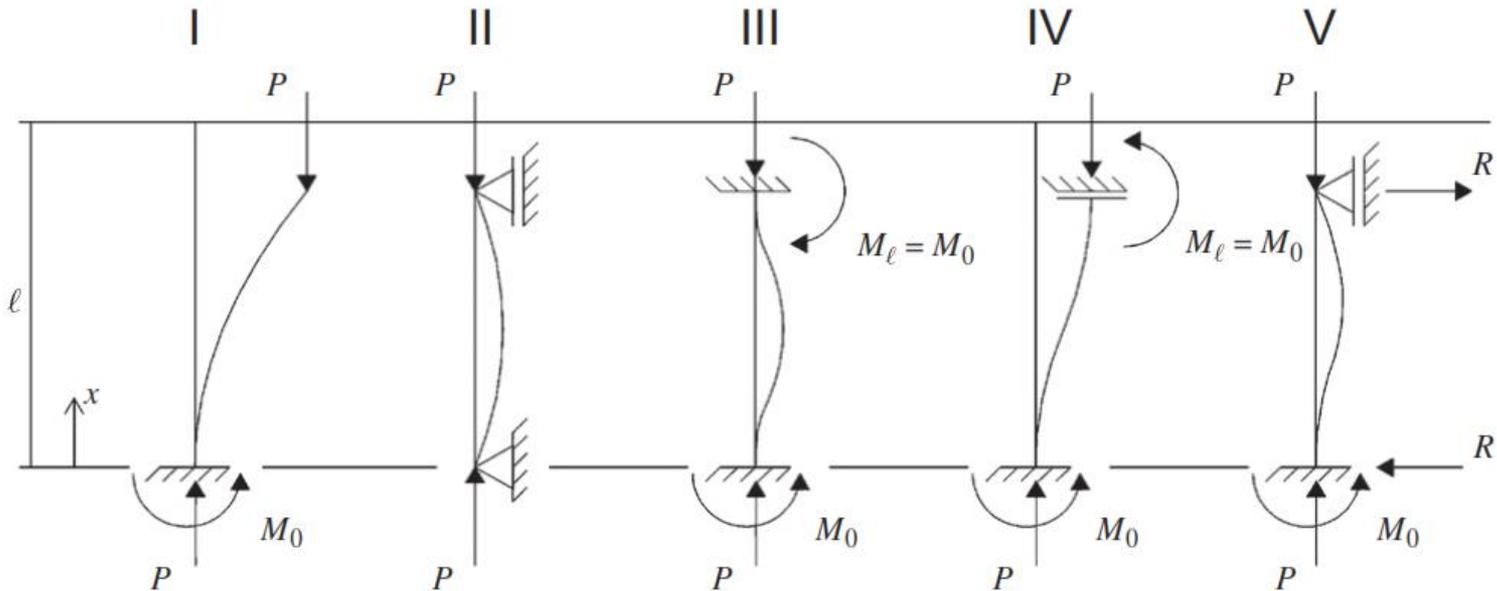
$$\text{Case II} \Rightarrow \frac{d\varphi}{d\xi}(0) = 0 \quad \text{and} \quad \frac{d\varphi}{d\xi}(1) = 0,$$

$$\text{Case III} \Rightarrow \varphi(0) = 0 \quad \text{and} \quad \varphi\left(\frac{1}{2}\right) = 0,$$

$$\text{Case IV} \Rightarrow \varphi(0) = 0 \quad \text{and} \quad \varphi(1) = 0,$$

$$\text{Case V} \Rightarrow \varphi(0) = 0 \quad \text{and} \quad \frac{d\varphi}{d\xi}(1) = 0.$$

# Flambagem de Euler: Barra Prismática sem Imperfeição



Barra inextensível

$$\text{Case I } k\ell = \frac{\pi}{2} \Rightarrow p_E = \frac{\pi^2 \eta}{4},$$

$$\text{Case II } k\ell = \pi \Rightarrow p_E = \pi^2 \eta,$$

$$\text{Case III } \frac{k\ell}{2} = \pi \Rightarrow p_E = 4\pi^2 \eta,$$

$$\text{Case IV } k\ell = \pi \Rightarrow p_E = \pi^2 \eta,$$

$$\text{Case V } k\ell = 4.493 \Rightarrow p_E = 20.19\eta.$$

Barra extensível

$$\text{Case I } \sqrt{\alpha_1} = \frac{\pi}{2} \Rightarrow p_{cr} = \frac{1 - \sqrt{1 - \pi^2 \eta}}{2},$$

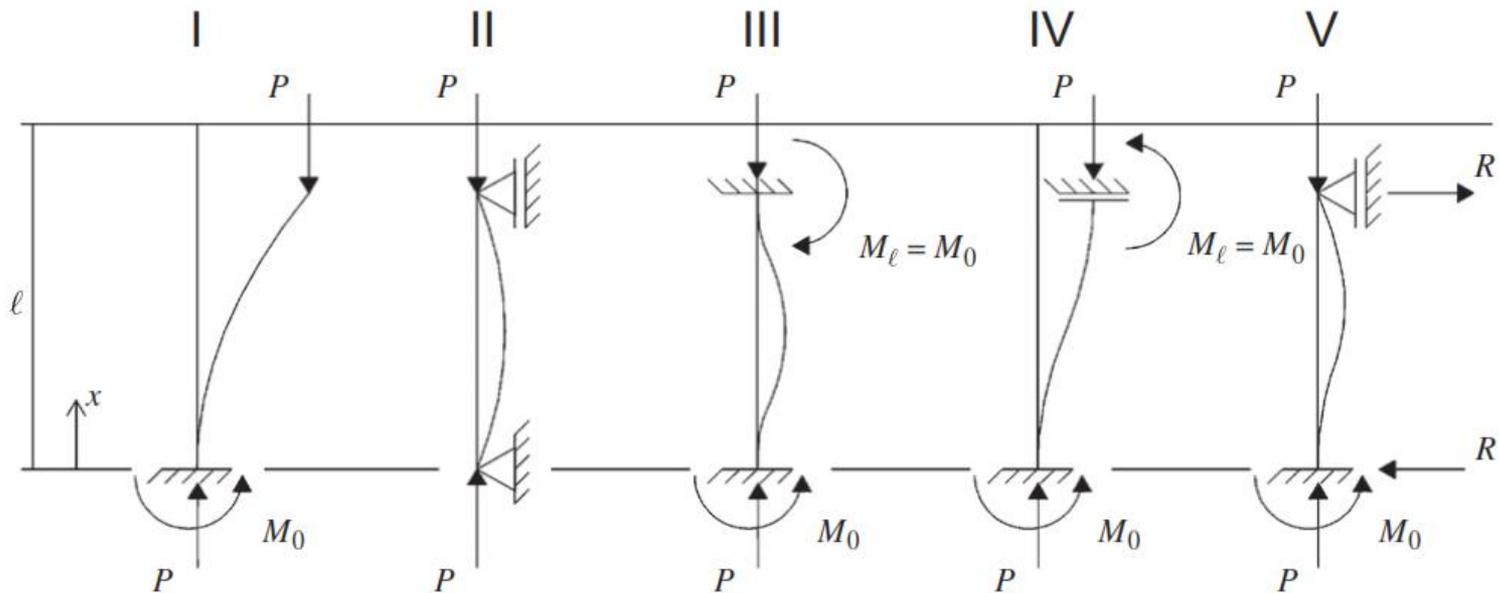
$$\text{Case II } \sqrt{\alpha_1} = \pi \Rightarrow p_{cr} = \frac{1 - \sqrt{1 - 4\pi^2 \eta}}{2},$$

$$\text{Case III } \sqrt{\alpha_1} = 2\pi \Rightarrow p_{cr} = \frac{1 - \sqrt{1 - 16\pi^2 \eta}}{2},$$

$$\text{Case IV } \sqrt{\alpha_1} = \pi \Rightarrow p_{cr} = \frac{1 - \sqrt{1 - 4\pi^2 \eta}}{2},$$

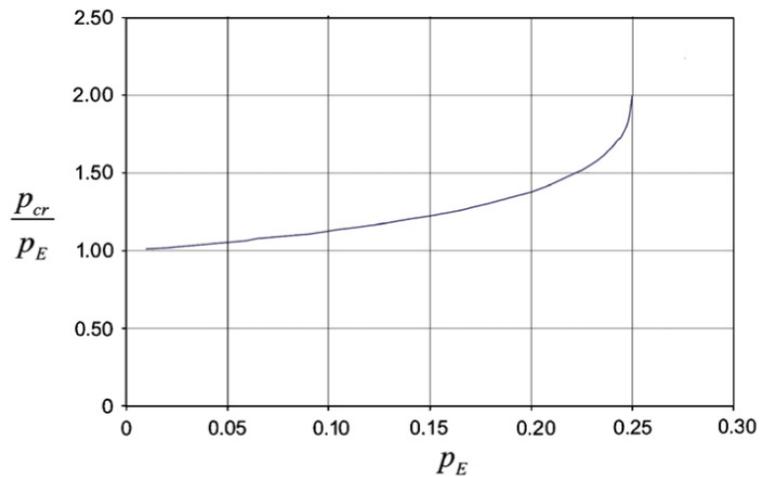
$$\text{Case V } \sqrt{\alpha_1} = 4.493 \Rightarrow p_{cr} = \frac{1 - \sqrt{1 - 80.76\eta}}{2}.$$

# Flambagem de Euler: Barra Prismática sem Imperfeição

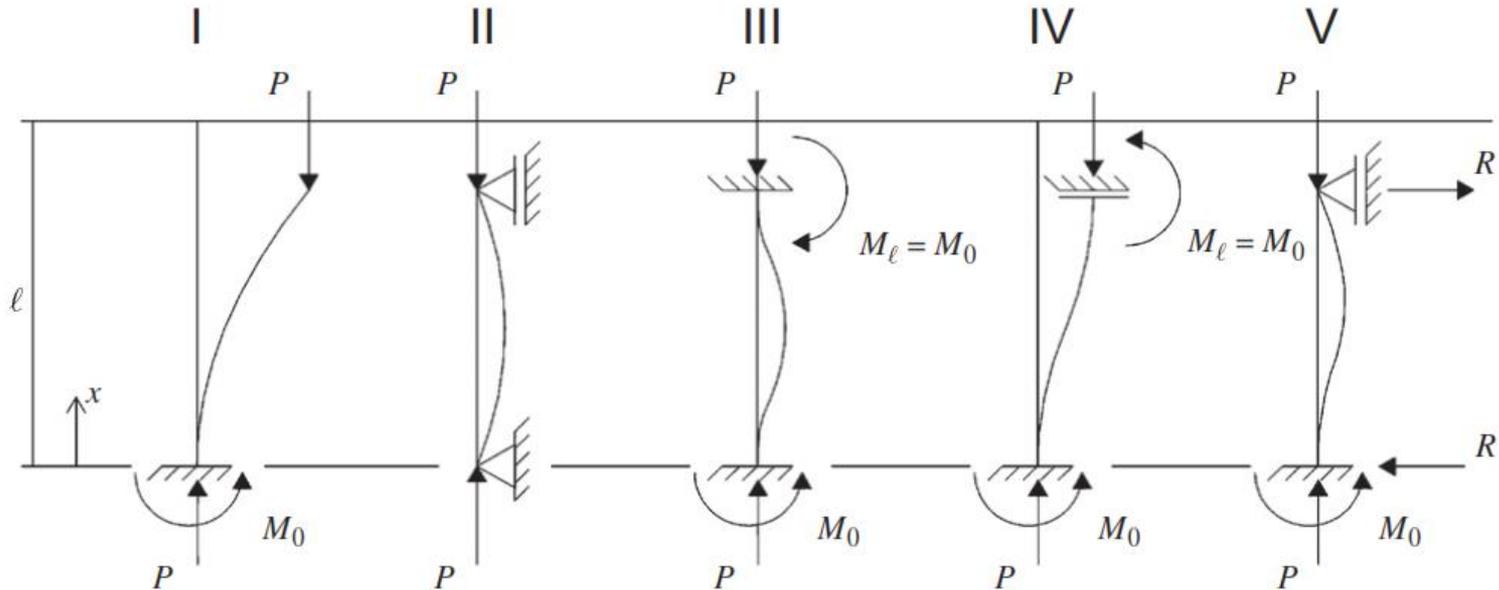


Barra extensível

$$p_{cr} = \frac{1 - \sqrt{1 - 4p_E}}{2}$$



# Flambagem de Euler: Barra Prismática sem Imperfeição



Nota (Caso II): não há carga crítica para  $l < 2\pi r_1$  !

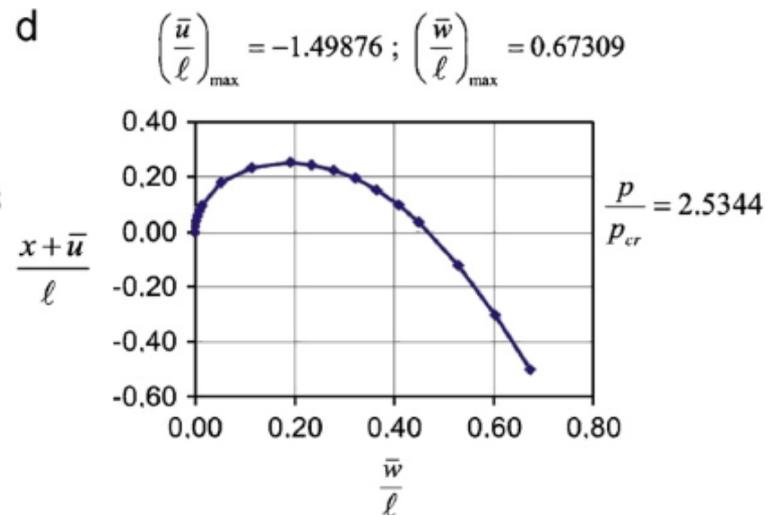
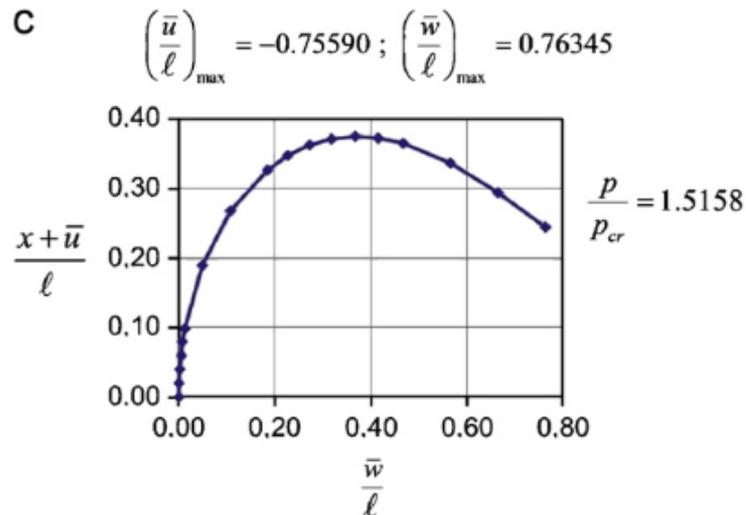
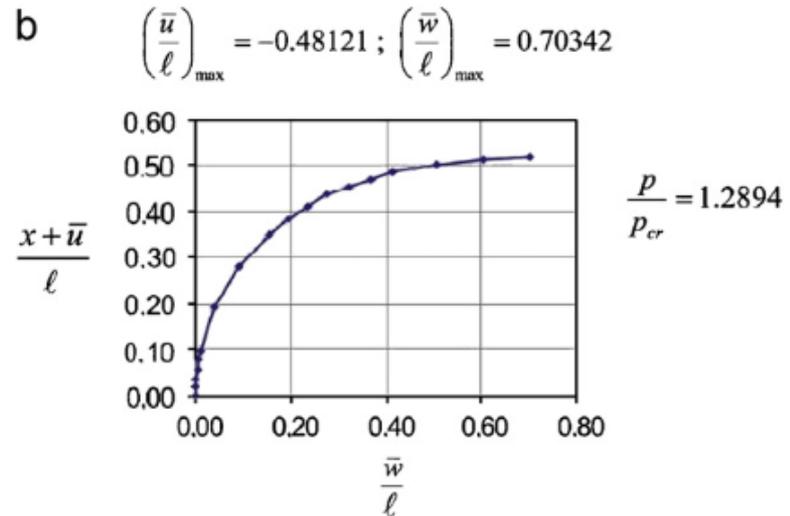
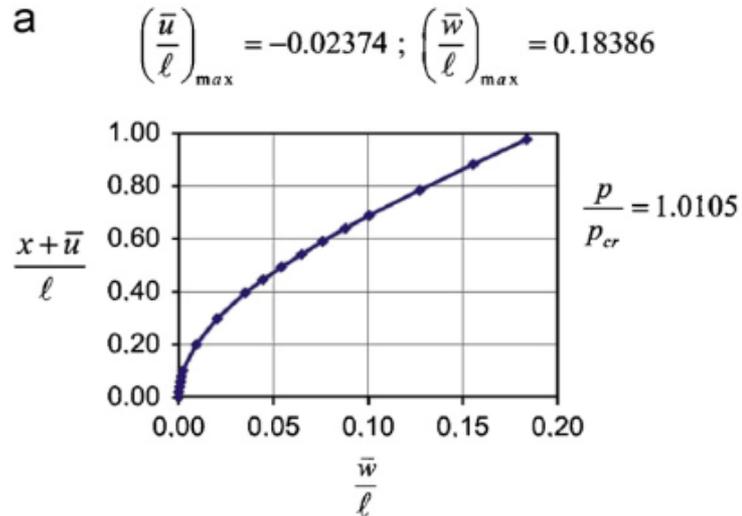
número de cargas críticas: max inteiro  $\leq (l/2\pi r_1)$  ou max inteiro  $\leq \frac{1}{(2\pi\sqrt{\eta})}$

Exemplo: para  $\eta = 0.001$  há somente cinco cargas críticas!

Nota (geral): não há carga crítica para distância entre pontos de inflexão  $< 2\pi r_1$  !

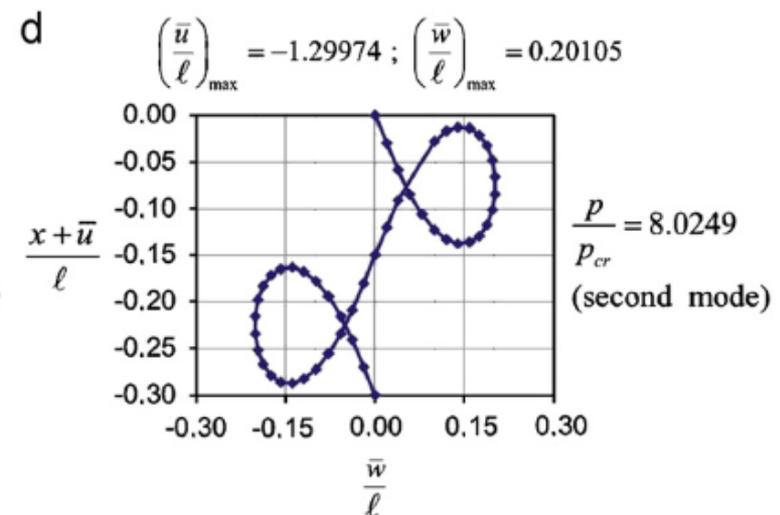
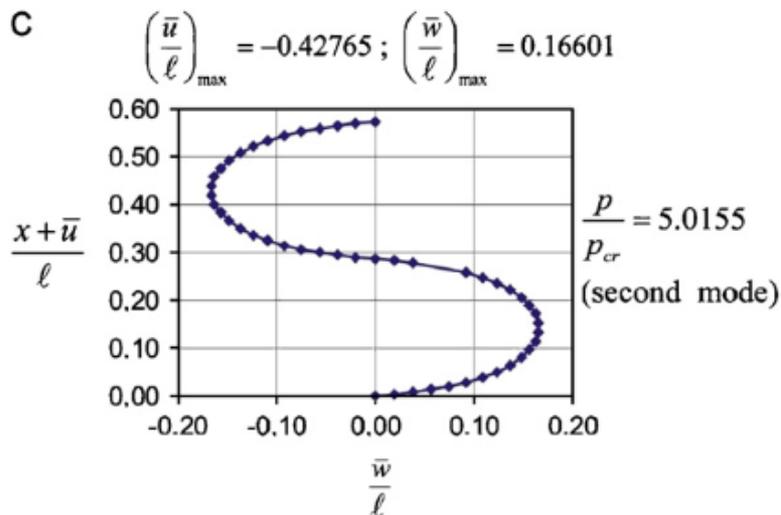
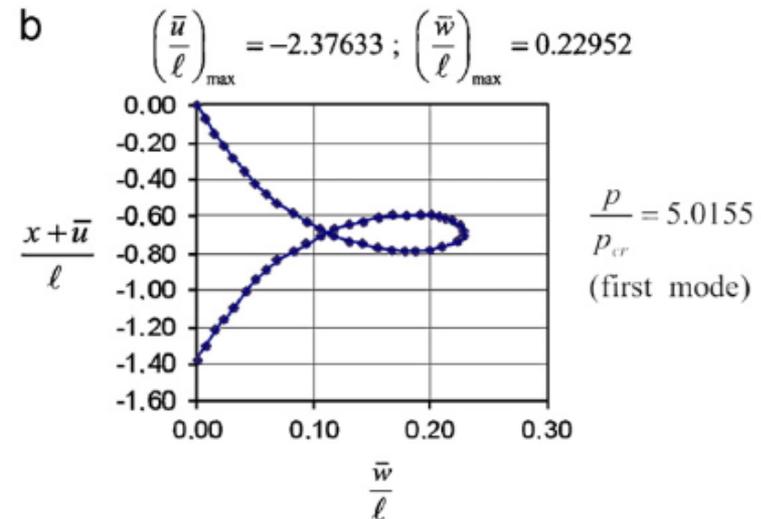
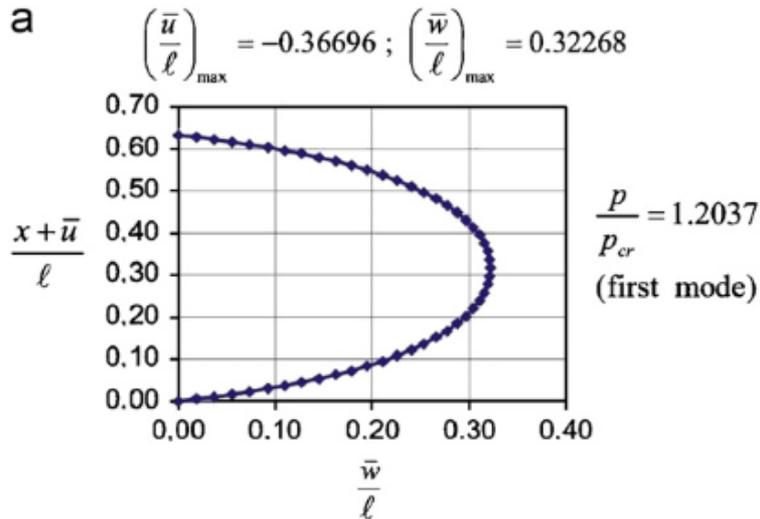
# Pós-Flambagem de Euler: Caso I

## Barra Prismática sem Imperfeição



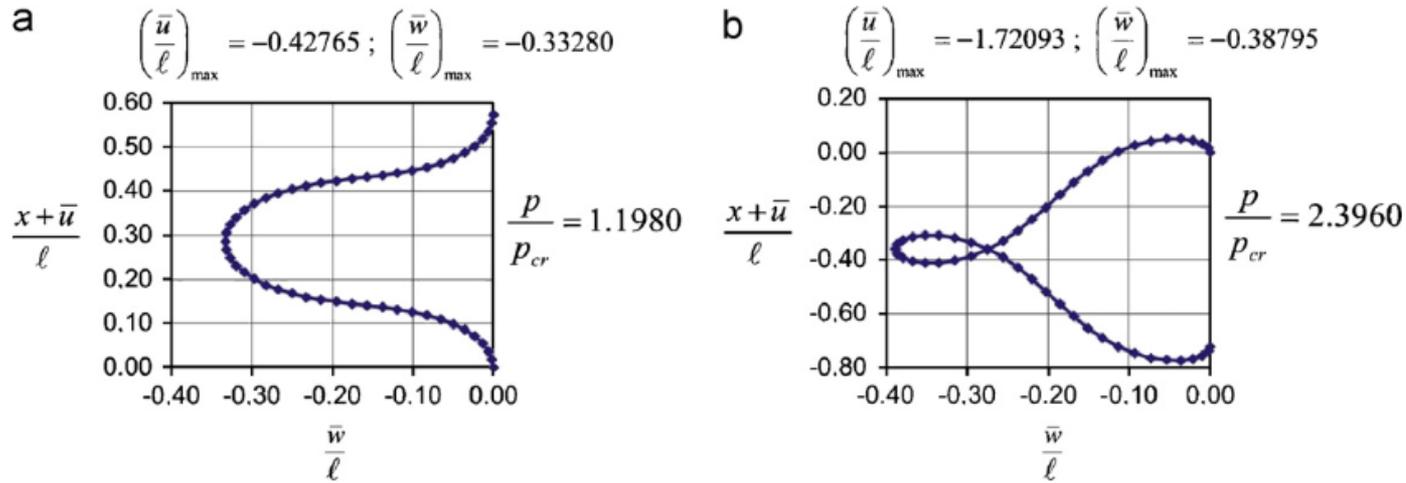
# Pós-Flambagem de Euler: Caso II

## Barra Prismática sem Imperfeição

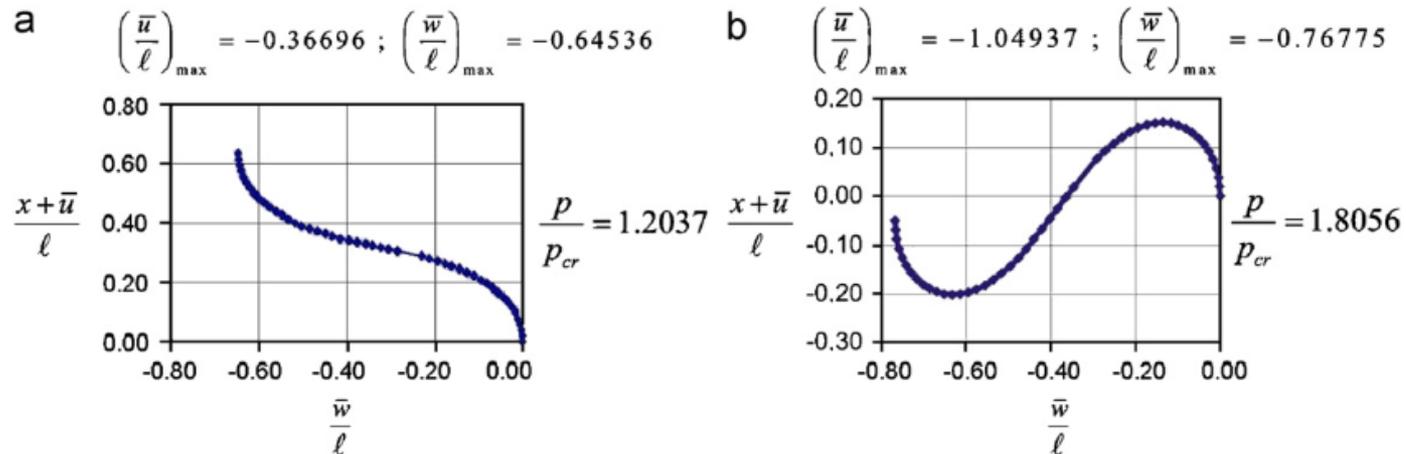


# Pós-Flambagem de Euler: Casos III e IV

## Barra Prismática sem Imperfeição



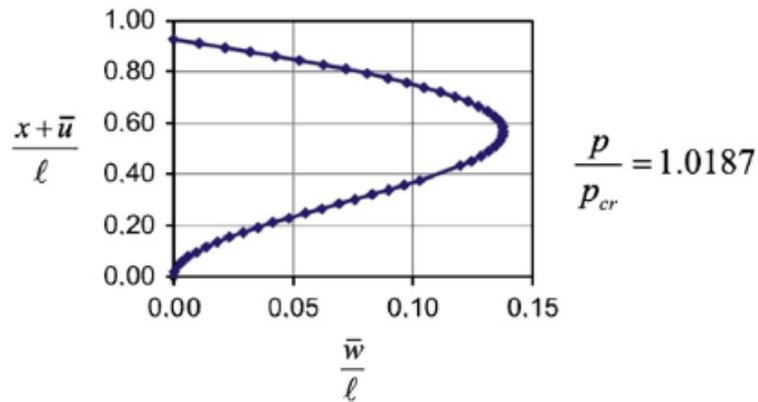
**Fig. 6.** Post-buckled configurations and maximum displacements for (a)  $p/p_{cr} = 1.1980$ ; (b)  $p/p_{cr} = 2.3960$ .



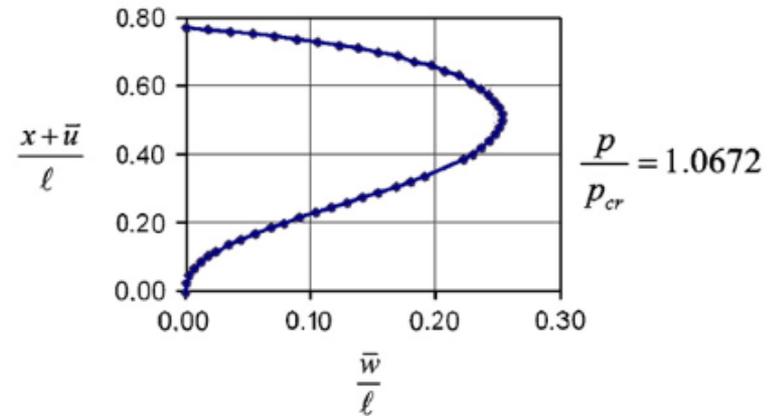
# Pós-Flambagem de Euler: Caso V

## Barra Prismática sem Imperfeição

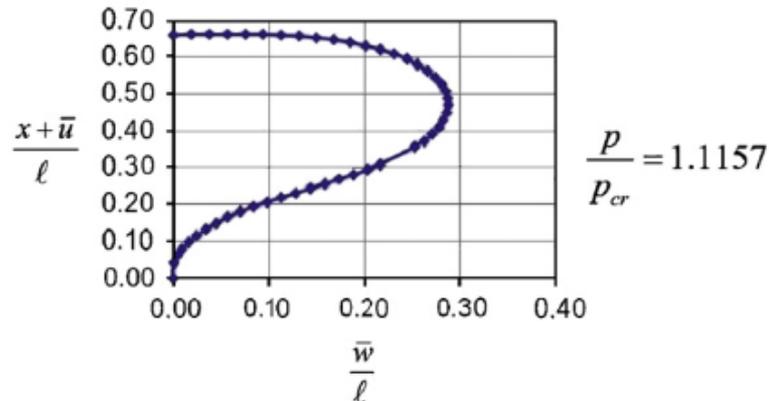
**a**  $\left(\frac{\bar{u}}{\ell}\right)_{\max} = -0.07271$  ;  $\left(\frac{\bar{w}}{\ell}\right)_{\max} = 0.13788$



**b**  $\left(\frac{\bar{u}}{\ell}\right)_{\max} = -0.22988$  ;  $\left(\frac{\bar{w}}{\ell}\right)_{\max} = 0.25313$



**c**  $\left(\frac{\bar{u}}{\ell}\right)_{\max} = -0.33984$  ;  $\left(\frac{\bar{w}}{\ell}\right)_{\max} = 0.28787$



**d**  $\left(\frac{\bar{u}}{\ell}\right)_{\max} = -1.58426$  ;  $\left(\frac{\bar{w}}{\ell}\right)_{\max} = 0.26387$

