PEF-5916
Dinâmica e Estabilidade das Estruturas

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Análise Modal

Sistemas sem amortecimento

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0} \qquad \text{com} \quad \mathbf{U}(0) = \mathbf{U_0} \quad \text{e} \quad \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_\mathbf{0}$$

solução geral

$$\mathbf{U} = \sum_i \hat{\mathbf{U}}_i \cos(\omega_i t - \theta_i)$$

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right)\hat{\mathbf{U}}_i = \mathbf{0} \Rightarrow \left|\mathbf{K} - \omega_i^2 \mathbf{M}\right| = 0 \quad \text{para soluções não triviais}$$

$$\Downarrow$$

$$\omega_1 \le \omega_2 \le \ldots \le \omega_n$$

$$\left(\mathbf{K} - \omega_i^2 \mathbf{M}\right)\hat{\mathbf{U}}_i = \mathbf{0} \Rightarrow \mathbf{\Phi_i} = \hat{\mathbf{U}}_\mathbf{i}$$

Análise Modal

Ortogonalidade dos modos de vibração não amortecidos

$$\mathbf{K\Phi_i} = \omega_i^2 \mathbf{M\Phi_i}$$ problema quase-estático equivalente

Teorema de Betti

$$\mathbf{\Phi}_i^T \left(\omega_j^2 \mathbf{M\Phi}_j \right) = \mathbf{\Phi}_j^T \left(\omega_i^2 \mathbf{M\Phi}_i \right) \Rightarrow \left(\omega_j^2 - \omega_i^2 \right) \mathbf{\Phi}_i^T \mathbf{M\Phi}_j = 0$$

$\omega_j^2 \neq \omega_i^2 \Rightarrow \mathbf{\Phi}_i^T \mathbf{M\Phi}_j = 0$ e também $\omega_j^2 \neq \omega_i^2 \Rightarrow \mathbf{\Phi}_i^T \mathbf{K\Phi}_j = 0$

Generalização: $\mathbf{\Phi}_i^T \mathbf{M}\left(\mathbf{M}^{-1}\mathbf{K}\right)^b \mathbf{\Phi}_j = 0, \quad i \neq j \text{ e } b \in \mathbb{Z}$

Matrizes modais: $\mathbf{M}^* = \mathbf{\Phi}^\mathbf{T}\mathbf{M\Phi}$ e $\mathbf{K}^* = \mathbf{\Phi}^\mathbf{T}\mathbf{K\Phi}$

$$M_{ij}^* = \mathbf{\Phi_i^T M \Phi_j} = 0, \quad i \neq j$$

$$K_{ij}^* = \mathbf{\Phi_i^T K \Phi_j} = 0, \quad i \neq j$$

$$K_i^* = \omega_i^2 M_i^*$$

Análise Modal

Sistemas com amortecimento "proporcional"

$$\mathbf{C} = \sum_b a_b \mathbf{M}\left(\mathbf{M}^{-1}\mathbf{K}\right)^b, \quad b \in \mathbb{Z}, \quad a_b \in \mathbb{R}$$

$$\sum_b a_b \omega_i^{2b-1} = 2\xi_i$$

Caso particular: amortecimento de Rayleigh:

$$\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K} = \alpha\mathbf{M} + \beta\mathbf{K}$$

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{0} \quad \text{com} \quad \mathbf{U}(0) = \mathbf{U_0} \quad \text{e} \quad \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_\mathbf{0}$$

solução geral
amortecimento subcrítico

$$\mathbf{U} = e^{-\xi\omega t}\hat{\mathbf{U}}\cos\left(\omega_D t - \theta\right)$$

Análise Modal

Sistemas com amortecimento de Rayleigh

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

$$\begin{bmatrix} \dfrac{1}{\omega_1} & \omega_1 \\ \dfrac{1}{\omega_2} & \omega_2 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{Bmatrix} 2\xi_1 \\ 2\xi_2 \end{Bmatrix} \Rightarrow \alpha, \beta$$

Método das Superposição Modal

Sistemas com amortecimento proporcional

$$\mathbf{M\ddot{U}}+\mathbf{C\dot{U}}+\mathbf{KU}=\mathbf{R}(t) \quad \text{com} \quad \mathbf{U}(0)=\mathbf{U_0} \qquad \mathbf{\dot{U}}(0)=\mathbf{\dot{U}_0}$$

mudança de variáveis: $\mathbf{U}=\mathbf{\Phi Y}$

$$\mathbf{\Phi^T M \Phi \ddot{Y}}+\mathbf{\Phi^T C \Phi \dot{Y}}+\mathbf{\Phi^T K \Phi Y}=\mathbf{\Phi^T R}(t)$$

$$\Downarrow$$

$$\mathbf{M}^*\mathbf{\ddot{Y}}+\mathbf{C}^*\mathbf{\dot{Y}}+\mathbf{K}^*\mathbf{Y}=\mathbf{R}^*(t)$$

$$\Downarrow$$

$$M_i^*\ddot{Y}+C_i^*\dot{Y}+K_i^*Y=R_i^*(t) \quad \Rightarrow \quad \ddot{Y}_i+2\xi_i\omega_i\dot{Y}_i+\omega_i^2Y_i=\frac{R_i^*(t)}{M_i}$$

Método das Superposição Modal

Resposta do oscilador modal

$$\ddot{Y}_i + 2\xi_i\omega_i\dot{Y}_i + \omega_i^2 Y_i = \frac{R_i^*(t)}{M_i} \Rightarrow Y_i(t)$$

Resposta nas coordenadas originais

$$\mathbf{U} = \mathbf{\Phi}\mathbf{Y} = \sum_i Y_i(t)\mathbf{\Phi_i}$$

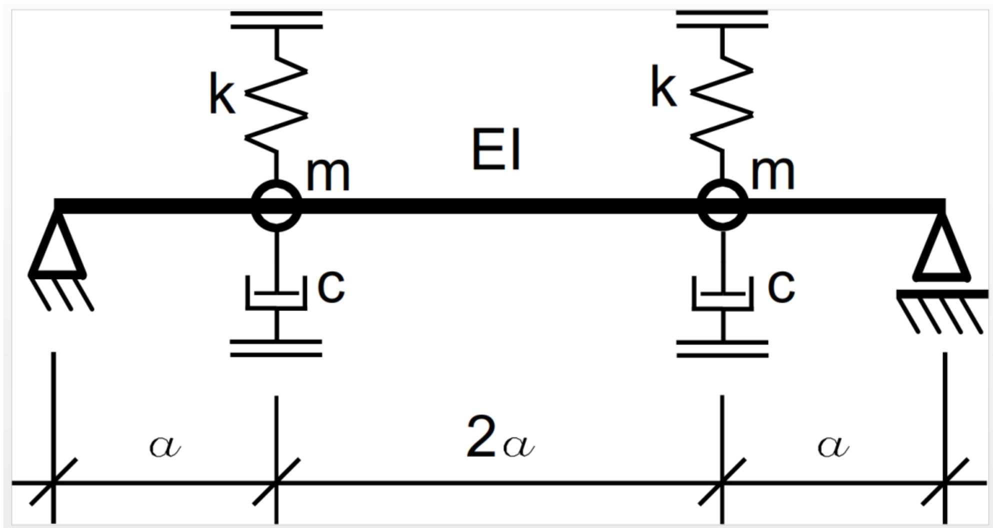
Imposição das condições iniciais

$$\mathbf{Y}_0 = \mathbf{\Phi}^{-1}\mathbf{U}_0 = \left[\underbrace{\left(\mathbf{M}^*\right)^{\mathbf{-1}}\mathbf{\Phi}^{\mathbf{T}}\mathbf{M}}_{\mathbf{\Phi}^{-1}}\right]\mathbf{U}_0 \quad \text{e} \quad \dot{\mathbf{Y}}_0 = \mathbf{\Phi}^{-1}\dot{\mathbf{U}}_0 = \left[\underbrace{\left(\mathbf{M}^*\right)^{\mathbf{-1}}\mathbf{\Phi}^{\mathbf{T}}\mathbf{M}}_{\mathbf{\Phi}^{-1}}\right]\dot{\mathbf{U}}_0$$

$$Y_{i0} = \frac{\Phi_i^{\,T}\mathbf{M}\mathbf{U_0}}{M_i^{\,*}} \qquad \dot{Y}_{i0} = \frac{\Phi_i^{\,T}\mathbf{M}\dot{\mathbf{U}}_\mathbf{0}}{M_i^{\,*}}$$

Método da superposição Modal

Exemplo



$$EI = 7000 Nm^2;\ m = 480kg;\ a = 1m;\ k = 31500 N/m$$
$$c = 1000 Ns/m$$

$$\begin{aligned} \omega_1 &= 8,75 rad/s \\ \omega_2 &= 12,37\, rad/s \end{aligned}$$

$$\phi_1 = \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\} \qquad \phi_2 = \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\}$$

Método da superposição Modal

Exemplo

$$\mathbf{R} = \left\{ \begin{array}{c} 8000 \\ 4000 \end{array} \right\} \mathrm{sen}\, 10,6t \qquad \mathbf{U}_0 = \left\{ \begin{array}{c} 0,30 \\ 0 \end{array} \right\} m \qquad \dot{\mathbf{U}}_0 = \left\{ \begin{array}{c} 0 \\ 3 \end{array} \right\} m/s$$

$$M_1^* = 960 \quad K_1^* = 73500 \quad C_1^* = 2000 \quad \xi_1 = 0,12 \quad \omega_{D_1} = 8,69$$

$$M_2^* = 960 \quad K_2^* = 146898 \quad C_2^* = 2000 \quad \xi_2 = 0,08 \quad \omega_{D_2} = 12,33$$

$$\mathbf{R}^* = \mathbf{\Phi}^T \mathbf{R} = \mathbf{R}_0^* \mathrm{sen}\, 10,6t$$

$$\mathbf{R}_0^* = \left\{ \begin{array}{c} 12000 \\ 4000 \end{array} \right\} \text{em N}$$

$$\mathbf{Y}_0 = \left\{ \begin{array}{c} 0,15 \\ 0,15 \end{array} \right\} m \qquad \dot{\mathbf{Y}}_0 = \left\{ \begin{array}{c} 1,5 \\ -1,5 \end{array} \right\} m/s$$

Método da superposição Modal

Exemplo

$$Y_i(t) = e^{-\xi_i\omega_i t}\rho_i\cos\left(\omega_{Di}t - \theta_i\right) + \bar{\rho}_i\text{sen}\left(\bar{\omega}t - \bar{\theta}_i\right)$$

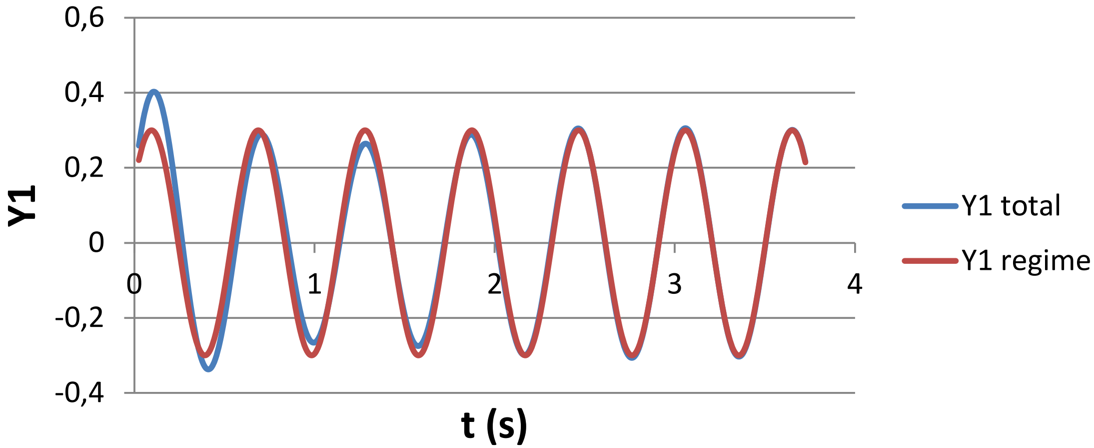
$$\bar{\rho}_i = \frac{R_{0i}^*}{K_i^*}D_i \qquad D_i = \frac{1}{\sqrt{\left(1-\beta_i^2\right)^2 + \left(2\xi_i\beta_i\right)^2}} \qquad \beta_i = \frac{\bar{\omega}}{\omega_i} \qquad \bar{\theta}_i = \arctan\frac{2\xi_i\beta_i}{1-\beta_i^2}$$

$$\rho_i = \sqrt{\left(Y_{0i} + \bar{\rho}_i\text{sen}\bar{\theta}_i\right)^2 + \left[\frac{\dot{Y}_{0i} - \bar{\omega}\bar{\rho}_i\cos\bar{\theta}_i + \xi_i\omega_i\left(Y_{0i} + \bar{\rho}_i\text{sen}\bar{\theta}_i\right)}{\omega_{Di}}\right]^2}$$

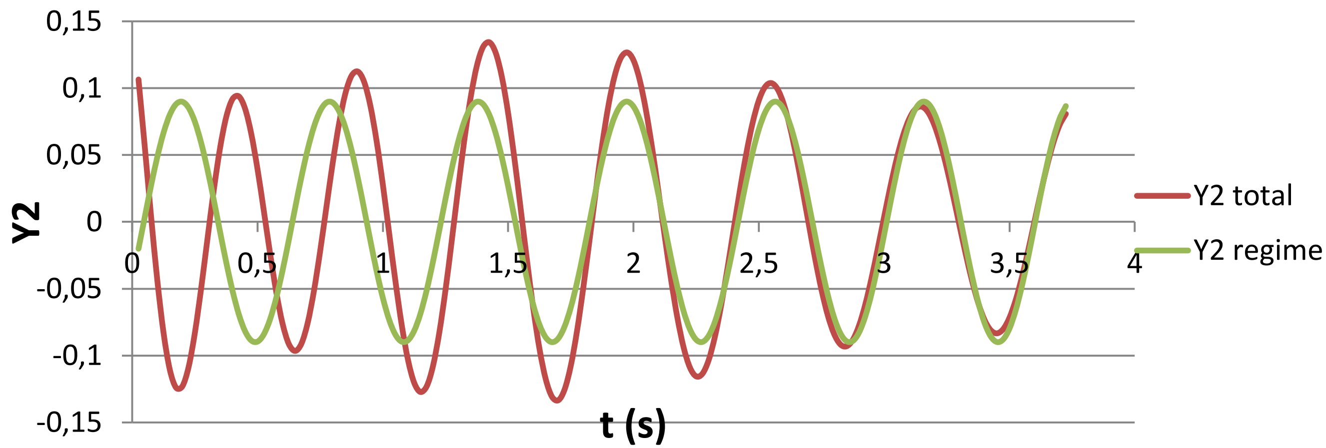
$$\theta_i = \arctan\frac{\dot{Y}_{0i} - \bar{\omega}\bar{\rho}_i\cos\bar{\theta}_i + \xi_i\omega_i\left(Y_{0i} + \bar{\rho}_i\text{sen}\bar{\theta}_i\right)}{\omega_{Di}\left(Y_{0i} + \bar{\rho}_i\text{sen}\bar{\theta}_i\right)}$$

Método da superposição Modal

Exemplo



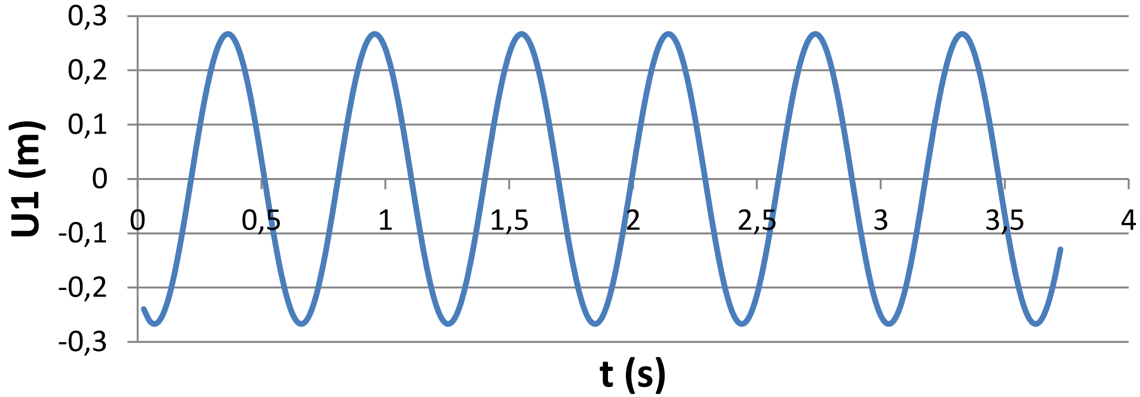
$$Y_1(t) = e^{-1,05t}\,0,14\cos(8,69t - 1,50) + 0,30\,\text{sen}(10,6t + 0,56)$$



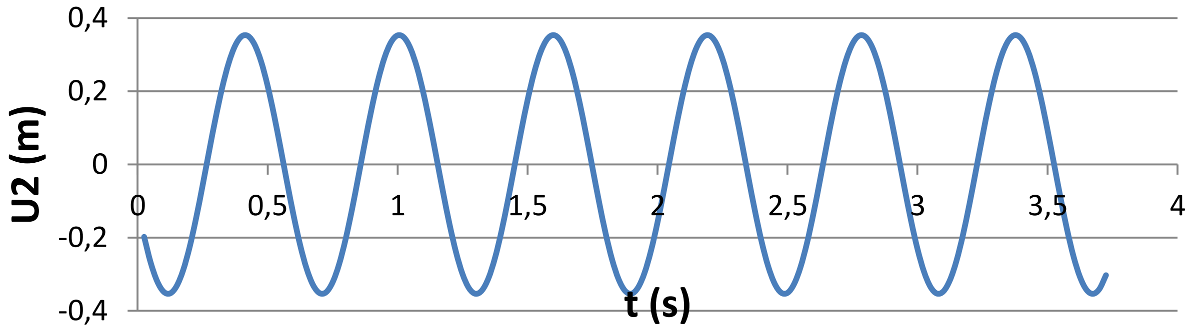
$$Y_2(t) = e^{-0,99t}\,0,26\cos(12,33t + 0,74) + 0,09\,\text{sen}(10,6t - 0,49)$$

Método da superposição Modal

Exemplo



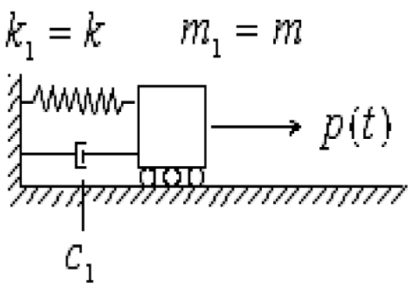
$$U_1(t) = 0{,}30\text{sen}(10{,}6t + 0{,}56) + 0{,}09\text{sen}(10{,}6t - 0{,}49) = 0{,}35\text{sen}(10{,}6t + 0{,}34)$$



$$U_2(t) = 0{,}30\text{sen}(10{,}6t + 0{,}56) - 0{,}09\text{sen}(10{,}6t - 0{,}49) = 0{,}27\text{sen}(10{,}6t + 0{,}85)$$

Método das Superposição Modal

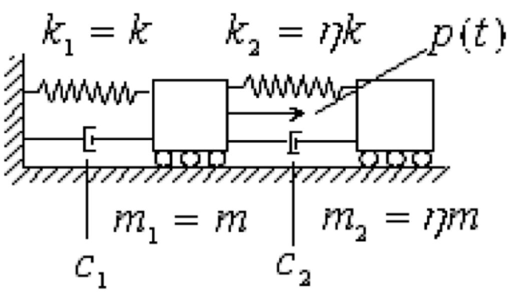
Controle Passivo



$$p\left(t\right) = p_0 \operatorname{sen} \omega_0 t \qquad \omega_0 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k}{m}}$$

$$\xi = \frac{c_1}{2m_1\omega_0} = 0,05$$

$$\rho = D\left(\frac{p_0}{k}\right) = \frac{1}{2\xi}\left(\frac{p_0}{k}\right) = 10\left(\frac{p_0}{k}\right)$$



$$k_2 = \eta k \text{ e } m_2 = \eta m$$

$$\eta = 0,1$$

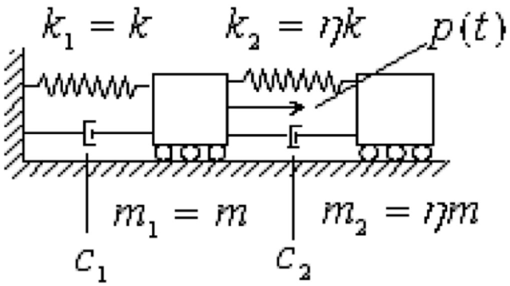
$$\omega_1 = 0,854\omega_0, \qquad \{\phi_1\} = \begin{Bmatrix} 0,2702 \\ 1 \end{Bmatrix}$$

$$\omega_2 = 1,171\omega_0, \qquad \{\phi_2\} = \begin{Bmatrix} -0,3702 \\ 1 \end{Bmatrix}$$

Amortecimento de Rayleigh $a_0 = 0$ e $a_1 = \frac{2\xi}{\omega_0}$

Método das Superposição Modal

Controle Passivo



$$
\begin{aligned}
M_1^* &= \boldsymbol{\phi}_1^T \mathbf{M} \boldsymbol{\phi}_1 = 0,17298m \\
K_1^* &= \boldsymbol{\phi}_1^T \mathbf{K} \boldsymbol{\phi}_1 = 0,12625k \\
C_1^* &= a_1 \mathbf{K}_1^* = 0,25250\frac{k\xi}{\omega_0} = 0,01263\frac{k}{\omega_0} \\
R_1^* &= \boldsymbol{\phi}_1^T \left\{ \begin{array}{c} p_0 \operatorname{sen} \omega_0 t \\ 0 \end{array} \right\} = 0,2702 p_0 \operatorname{sen} \omega_0 t;
\end{aligned}
$$

$$
\begin{aligned}
M_2^* &= \boldsymbol{\phi}_2^T \mathbf{M} \boldsymbol{\phi}_2 = 0,23702m, \\
K_2^* &= \boldsymbol{\phi}_2^T \mathbf{K} \boldsymbol{\phi}_2 = 0,32475k, \\
C_2^* &= a_1 \mathbf{K}_2^* = 0,64950\frac{k\xi}{\omega_0} = 0,03248\frac{k}{\omega_0}, \\
R_2^* &= \boldsymbol{\phi}_2^T \left\{ \begin{array}{c} p_0 \operatorname{sen} \omega_0 t \\ 0 \end{array} \right\} = -0,3702 p_0 \operatorname{sen} \omega_0 t.
\end{aligned}
$$

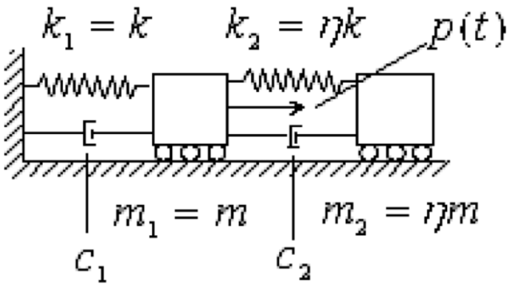
$$\ddot{Y}_1 + 2\xi_1\omega_1\dot{Y}_1 + \omega_1^2 Y_1 = \frac{R_1^*}{M_1^*} = 1,56174\frac{p_0}{m}\operatorname{sen}\omega_0 t,$$

$$\ddot{Y}_2 + 2\xi_2\omega_2\dot{Y}_2 + \omega_2^2 Y_2 = \frac{R_2^*}{M_2^*} = -1,56174\frac{p_0}{m}\operatorname{sen}\omega_0 t,$$

$$
\begin{aligned}
\xi_1 &= \frac{C_1^*}{2M_1^*\omega_1} = 0,854308\xi = 0,0427, \\
\xi_2 &= \frac{C_2^*}{2M_2^*\omega_2} = 1,170536\xi = 0,0585.
\end{aligned}
$$

Método das Superposição Modal

Controle Passivo



Resposta em regime estacionário

$$
\begin{aligned}
Y_1 &= D_1 \frac{R_1^*}{M_1^* \omega_1^2} \operatorname{sen}\left(\omega_0 t - \bar{\theta}_1\right) = 2,13983 \frac{p_0}{k} D_1 \operatorname{sen}\left(\omega_0 t - \bar{\theta}_1\right), \\
Y_2 &= D_2 \frac{R_2^*}{M_2^* \omega_2^2} \operatorname{sen}\left(\omega_0 t - \bar{\theta}_2\right) = -1,13983 \frac{p_0}{k} D_2 \operatorname{sen}\left(\omega_0 t - \bar{\theta}_2\right),
\end{aligned}
$$

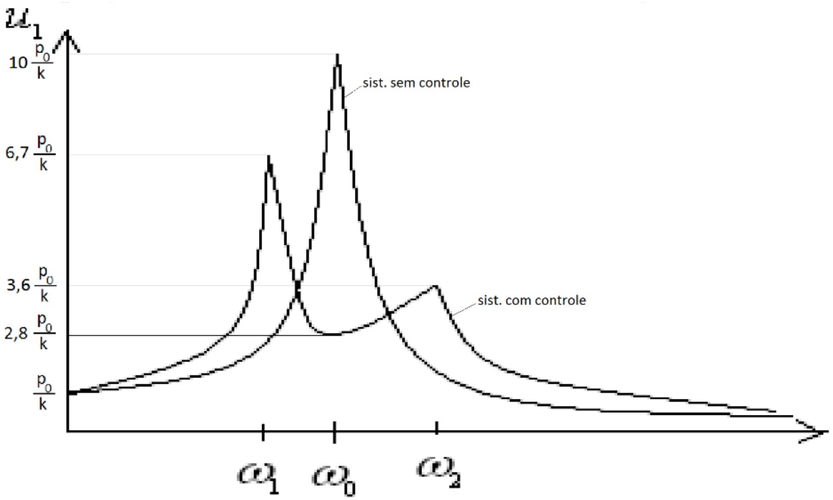
$$
\beta_1 = \frac{\omega_0}{\omega_1} = 1,171 \quad e \quad \beta_2 = \frac{\omega_0}{\omega_2} = 0,854
$$

$$
\begin{aligned}
D_1(\beta_1;\xi_1) &= 2,601 & \bar{\theta}_1(\beta_1;\xi_1) &= -0,263 \\
D_2(\beta_2;\xi_2) &= 3,466 & \bar{\theta}_2(\beta_2;\xi_2) &= 0,353.
\end{aligned}
$$

$$
\begin{cases}
u_1 = 0,2702 Y_1 - 0,3702 Y_2, \\
u_2 = Y_1 + Y_2,
\end{cases}
\quad \Longrightarrow \quad
\begin{cases}
\begin{aligned}
u_1 &= 1,504 \frac{p_0}{k} \operatorname{sen}\left(\omega_0 t + 0,263\right) + 1,463 \frac{p_0}{k} \operatorname{sen}\left(\omega_0 t - 0,353\right) \\
&= 2,827 \left(\frac{p_0}{k}\right) \operatorname{sen}\left(\omega_0 t + 0,039\right); \\
u_2 &= 5,566 \frac{p_0}{k} \operatorname{sen}\left(\omega_0 t + 0,263\right) - 3,951 \frac{p_0}{k} \operatorname{sen}\left(\omega_0 t - 0,353\right) \\
&= 3,270 \left(\frac{p_0}{k}\right) \operatorname{sen}\left(\omega_0 t - 1,036\right).
\end{aligned}
\end{cases}
$$

Método das Superposição Modal

Controle Passivo



Integração direta no domínio do tempo

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}(t) \quad \text{com} \quad \mathbf{U}(0) = \mathbf{U_0} \qquad \mathbf{\dot{U}}(0) = \mathbf{\dot{U}_0}$$

Método de Runge-Kutta de 4ª ordem

$$\mathbf{X} = \mathbf{U} \quad \text{e} \quad \mathbf{Y} = \mathbf{\dot{U}}$$

$$\begin{cases} \mathbf{\dot{X}} = \mathbf{F}(\mathbf{Y}) = \mathbf{Y} \\ \mathbf{\dot{Y}} = \mathbf{G}(\mathbf{X}, \mathbf{Y}, t) = \mathbf{M}^{-1}\left[\mathbf{R}(t) - \mathbf{KX} - \mathbf{CY}\right] \end{cases}$$

$$\mathbf{X}_{i+1} = \mathbf{X}_i + \frac{\Delta t}{6}\left(\mathbf{F}_1 + 2\mathbf{F}_2 + 2\mathbf{F}_3 + \mathbf{F}_4\right)$$

$$\mathbf{Y}_{i+1} = \mathbf{Y}_i + \frac{\Delta t}{6}\left(\mathbf{G}_1 + 2\mathbf{G}_2 + 2\mathbf{G}_3 + \mathbf{G}_4\right)$$

$$\left\{\begin{aligned}
\mathbf{F}_1 &= \mathbf{F}(\mathbf{Y}_i) \\
\mathbf{F}_2 &= \mathbf{F}\left(\mathbf{Y}_i + \frac{\Delta t}{2}\mathbf{G}_1\right) \\
\mathbf{F}_3 &= \mathbf{F}\left(\mathbf{Y}_i + \frac{\Delta t}{2}\mathbf{G}_2\right) \\
\mathbf{F}_4 &= \mathbf{F}\left(\mathbf{Y}_i + \Delta t\mathbf{G}_3\right) \\
\mathbf{G}_1 &= \mathbf{G}\left(\mathbf{Y}_i, \mathbf{\dot{Y}}_i, t\right) \\
\mathbf{G}_2 &= \mathbf{G}\left(\mathbf{X}_i + \frac{\Delta t}{2}\mathbf{F}_1, \mathbf{Y}_i + \frac{\Delta t}{2}\mathbf{G}_1, t\right) \\
\mathbf{G}_3 &= \mathbf{G}\left(\mathbf{X}_i + \frac{\Delta t}{2}\mathbf{F}_2, \mathbf{Y}_i + \frac{\Delta t}{2}\mathbf{G}_2, t\right) \\
\mathbf{G}_4 &= \mathbf{G}\left(\mathbf{X}_i + \Delta t\mathbf{F}_3, \mathbf{Y}_i + \Delta t\mathbf{G}_3, t\right)
\end{aligned}\right.$$

Integração direta no domínio do tempo

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}(t) \quad \text{com} \quad \mathbf{U}(0) = \mathbf{U_0} \qquad \mathbf{\dot{U}}(0) = \mathbf{\dot{U}_0}$$

$$\mathbf{\ddot{U}}(0) = \mathbf{M}^{-1}\left[\mathbf{R}(0) - \mathbf{KU_0} - \mathbf{C\dot{U}_0}\right]$$

Método de Euler-Gauss

$$\hat{\mathbf{K}}\mathbf{U}_{i+1} = \hat{\mathbf{R}}_{i+1},$$

$$\hat{\mathbf{K}} = \mathbf{K} + \frac{4}{\Delta t^2}\mathbf{M} + \frac{2}{\Delta t}\mathbf{C},$$

$$\hat{\mathbf{R}}_{i+1} = \mathbf{R}_{i+1} + \mathbf{M}\left(\frac{4}{\Delta t^2}\mathbf{U}_i + \frac{4}{\Delta t}\dot{\mathbf{U}}_i + \ddot{\mathbf{U}}_i\right) + \mathbf{C}\left(\frac{2}{\Delta t}\mathbf{U}_i + \dot{\mathbf{U}}_i\right).$$

$$\ddot{\mathbf{U}}_{i+1} = \frac{4}{\Delta t^2}\left(\mathbf{U}_{i+1} - \mathbf{U}_i\right) - \frac{4}{\Delta t}\dot{\mathbf{U}}_i - \ddot{\mathbf{U}}_i,$$

$$\dot{\mathbf{U}}_{i+1} = \dot{\mathbf{U}}_i + \frac{\Delta t}{2}\ddot{\mathbf{U}}_i + \frac{\Delta t}{2}\ddot{\mathbf{U}}_{i+1}.$$

Integração direta no domínio do tempo

$$\mathbf{M}\ddot{\mathbf{U}}+\mathbf{C}\dot{\mathbf{U}}+\mathbf{K}\mathbf{U}=\mathbf{R}(t) \text{ com } \mathbf{U}(0)=\mathbf{U}_0 \qquad \dot{\mathbf{U}}(0)=\dot{\mathbf{U}}_0$$

$$\ddot{\mathbf{U}}(0)=\mathbf{M}^{-1}\left[\mathbf{R}(0)-\mathbf{K}\mathbf{U}_0-\mathbf{C}\dot{\mathbf{U}}_0\right]$$

Método de Newmark

$$\hat{\mathbf{K}}\mathbf{U}_{i+1}=\hat{\mathbf{R}}_{i+1},$$

$$\hat{\mathbf{K}}=\mathbf{K}+\frac{1}{\alpha\Delta t^2}\mathbf{M}+\frac{\delta}{\alpha\Delta t}\mathbf{C},$$

$$\hat{\mathbf{R}}_{i+1}=\begin{array}{c}\mathbf{R}_{i+1}+\mathbf{M}\left(\frac{1}{\alpha\Delta t^2}\mathbf{U}_i+\frac{1}{\alpha\Delta t}\dot{\mathbf{U}}_i+\left(\frac{1}{2\alpha}-1\right)\ddot{\mathbf{U}}_i\right)\\+\mathbf{C}\left(\frac{\delta}{\alpha\Delta t}\mathbf{U}_i+\left(\frac{\delta}{\alpha}-1\right)\dot{\mathbf{U}}_i+\left(\frac{\delta}{\alpha}-2\right)\frac{\Delta t}{2}\ddot{\mathbf{U}}_i\right)\end{array}.$$

$$\ddot{\mathbf{U}}_{i+1}=\frac{1}{\alpha\Delta t^2}\left(\mathbf{U}_{i+1}-\mathbf{U}_i\right)-\frac{1}{\alpha\Delta t}\dot{\mathbf{U}}_i-\left(\frac{1}{2\alpha}-1\right)\ddot{\mathbf{U}}_i,$$

$$\dot{\mathbf{U}}_{i+1}=\dot{\mathbf{U}}_i+(1-\delta)\,\Delta t\ddot{\mathbf{U}}_i+\delta\Delta t\ddot{\mathbf{U}}_{i+1}.$$

Integração direta no domínio do tempo

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}(t) \text{ com } \mathbf{U}(0) = \mathbf{U_0} \qquad \mathbf{\dot{U}}(0) = \mathbf{\dot{U}_0}$$

$$\mathbf{\ddot{U}}(0) = \mathbf{M}^{-1}\left[\mathbf{R}(0) - \mathbf{KU_0} - \mathbf{C\dot{U}_0}\right]$$

Método de Newmark é incondicionalmente estável,
para análise linear, se:

$$\delta \geq \frac{1}{2} \qquad \alpha \geq \frac{1}{4}\left(\frac{1}{2} + \delta\right)^2$$

Método de Newmark coincide com o Método de Euler-Gauss se:

$$\delta = \frac{1}{2} \text{ e } \alpha = \frac{1}{4}$$

Método da aceleração linear:

$$\delta = \frac{1}{2} \text{ e } \alpha = \frac{1}{6}$$

NÃO É INCONDICIONALMENTE ESTÁVEL!!!

Integração direta no domínio do tempo

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}(t) \quad \text{com} \quad \mathbf{U}(0) = \mathbf{U_0} \qquad \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_\mathbf{0}$$

$$\ddot{\mathbf{U}}(0) = \mathbf{M}^{-1}\left[\mathbf{R}(0) - \mathbf{K}\mathbf{U_0} - \mathbf{C}\dot{\mathbf{U}}_\mathbf{0}\right]$$

Método de Wilson-θ

$$\hat{\mathbf{K}}\mathbf{U}_\theta = \hat{\mathbf{R}}_\theta,$$

$$\hat{\mathbf{K}} = \mathbf{K} + \frac{6}{(\theta\Delta t)^2}\mathbf{M} + \frac{3}{\theta\Delta t}\mathbf{C},$$

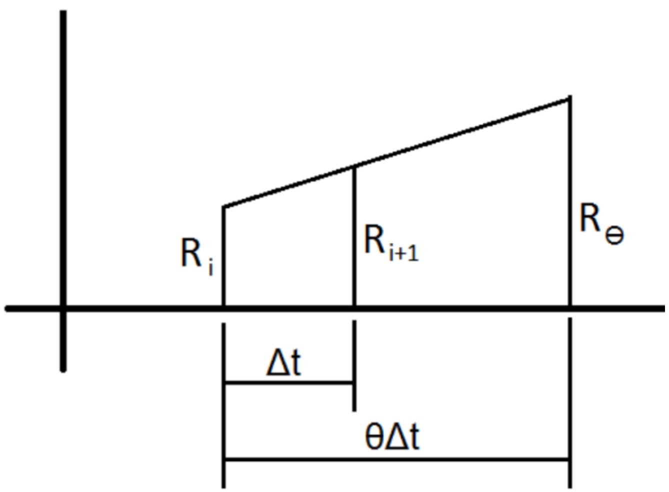
$$\begin{aligned}\hat{\mathbf{R}}_\theta \;&=\; \mathbf{R}_\theta + \mathbf{M}\left(\frac{6}{(\theta\Delta t)^2}\mathbf{U}_i + \frac{6}{\theta\Delta t}\dot{\mathbf{U}}_i + 2\ddot{\mathbf{U}}_i\right) \\ &\quad + \mathbf{C}\left(\frac{3}{\theta\Delta t}\mathbf{U}_i + 2\dot{\mathbf{U}}_i + \frac{\theta\Delta t}{2}\ddot{\mathbf{U}}_i\right).\end{aligned}$$

Integração direta no domínio do tempo

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}(t) \quad \text{com} \quad \mathbf{U}(0) = \mathbf{U_0} \qquad \dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_\mathbf{0}$$

$$\ddot{\mathbf{U}}(0) = \mathbf{M}^{-1}\left[\mathbf{R}(0) - \mathbf{K}\mathbf{U_0} - \mathbf{C}\dot{\mathbf{U}}_\mathbf{0}\right]$$

Método de Wilson-θ é incondicionalmente estável para $\theta > 1,37$



$$\mathbf{R}_\theta = \mathbf{R}_i + \theta\left(\mathbf{R}_{i+1} - \mathbf{R}_i\right)$$

$$\ddot{\mathbf{U}}_\theta = \frac{6}{(\theta\Delta t)^2}\left(\mathbf{U}_\theta - \mathbf{U}_i\right) - \frac{6}{\theta\Delta t}\dot{\mathbf{U}}_i - 2\ddot{\mathbf{U}}_i,$$

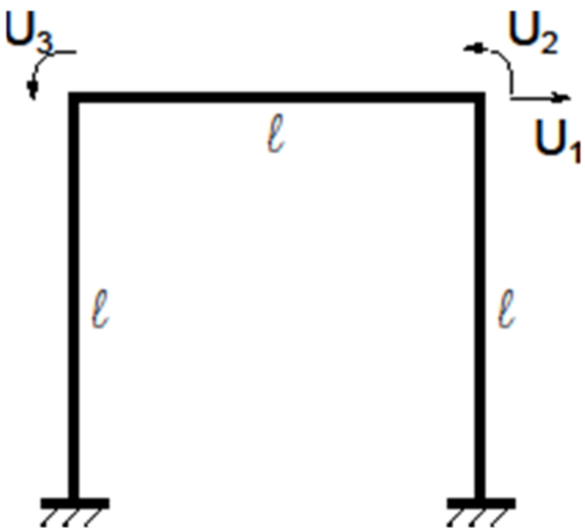
$$\ddot{\mathbf{U}}_{i+1} = \ddot{\mathbf{U}}_i + \frac{\ddot{\mathbf{U}}_\theta - \ddot{\mathbf{U}}_i}{\theta},$$

$$\dot{\mathbf{U}}_{i+1} = \dot{\mathbf{U}}_i + \Delta t\ddot{\mathbf{U}}_i + \frac{\Delta t}{2\theta}\left(\ddot{\mathbf{U}}_\theta - \ddot{\mathbf{U}}_i\right),$$

$$\mathbf{U}_{i+1} = \mathbf{U}_i + \Delta t\dot{\mathbf{U}}_i + \frac{(\Delta t)^2}{2}\ddot{\mathbf{U}}_i + \frac{(\Delta t)^2}{6\theta}\left(\ddot{\mathbf{U}}_\theta - \ddot{\mathbf{U}}_i\right).$$

Integração direta no domínio do tempo

Exemplo



$$R_1(t) = 1000\,\mathrm{sen}\,\bar{\omega}t \;\;(\mathrm{em}\;\; N)$$

$$\bar{\omega} = \omega_1 = 32{,}1 rad/s$$

$$EI = 80000 Nm^2$$

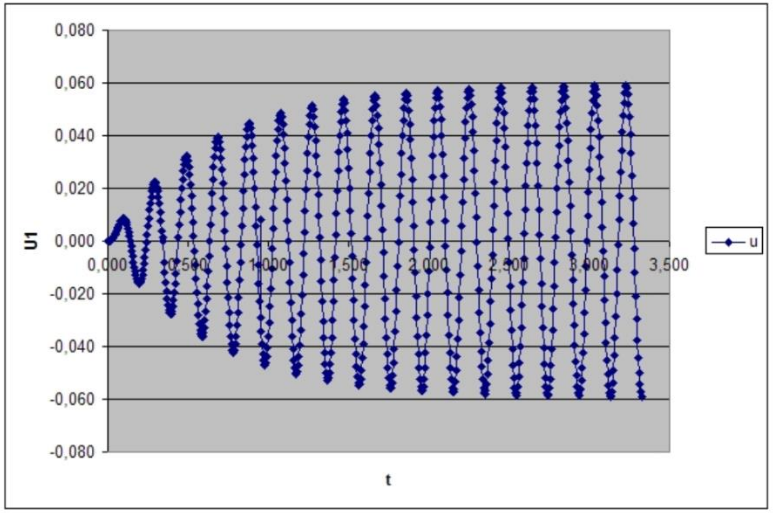
$$\ell = 2m$$

$$\rho A = 50 kgm^{-1}$$

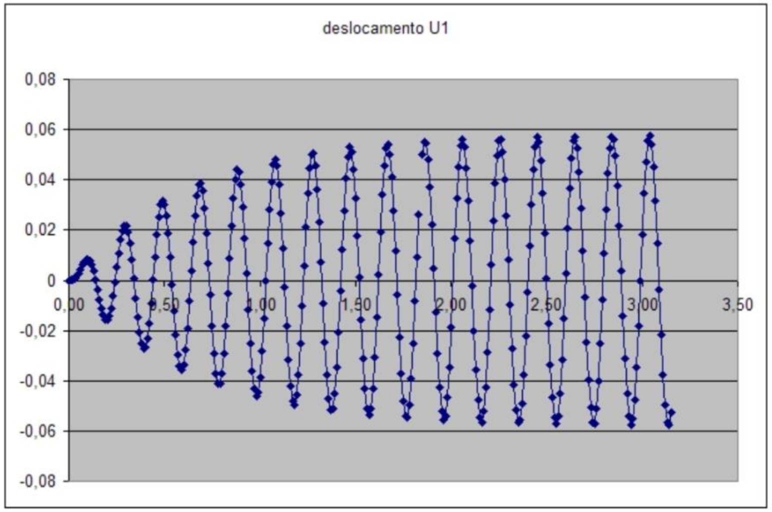
Amortecimento de Rayleigh com $a_0 = 0$ e $a_1 = 3,12 \times 10^{-3}s$

Integração direta no domínio do tempo

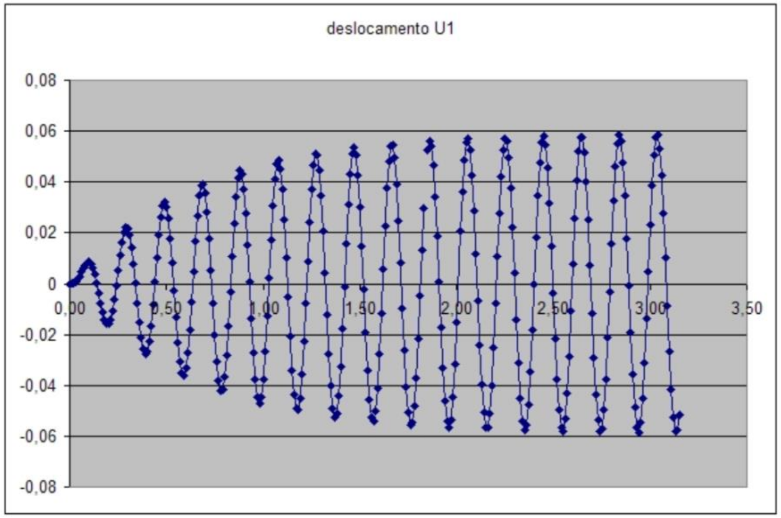
Exemplo



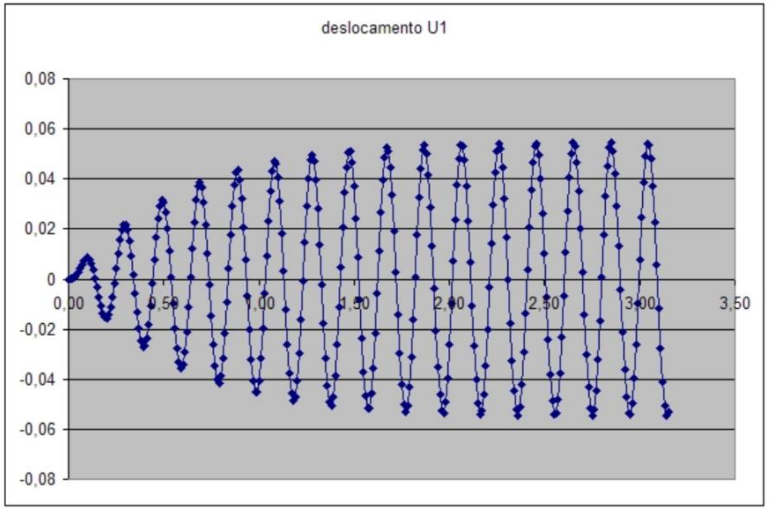
Runge-Kutta



Euler-Gauss



Acel. Linear



Wilson-θ