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## Parsimonious Modeling of Yield Curves\*

### I. Introduction

The need for a parsimonious model of the yield curve was recognized by Milton Friedman (1977, p. 22) when he stated, "Students of statistical demand functions might find it more productive to examine how the whole term structure of yields can be described more compactly by a few parameters." The purpose of this paper is to introduce a simple, parsimonious model that is flexible enough to represent the range of shapes generally associated with yield curves: monotonic, humped, and S shaped. The ability of the model to fit U.S. Treasury bill yields and to predict the price of a long-term Treasury bond suggests to us that the model succeeds in the objective set by Friedman. Potential applications of parsimonious yield curve models include de-

This paper introduces a parametrically parsimonious model for yield curves that has the ability to represent the shapes generally associated with yield curves: monotonic, humped, and S shaped. We find that the model explains 96% of the variation in bill yields across maturities during the period 1981–83. The movement of the parameters through time reflects and confirms a change in Federal Reserve monetary policy in late 1982. The ability of the fitted curves to predict the price of the long-term Treasury bond with a correlation of .96 suggests that the model captures important attributes of the yield/maturity relation.

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mand functions (Friedman had in mind money demand), testing of theories of the term structure of interest rates, and graphic display for informative purposes.

The fitting of yield curves to yield/maturity data goes back at least to the pioneering efforts of David Durand (1942), whose method of fitting was to draw a monotonic envelope under the scatter of points in a way that seemed to him subjectively reasonable. Yield may be transformed to present value, and J. Huston McCulloch (1971, 1975) has proposed approximating the present value function by a piecewise polynomial spline fitted to price data. Gary Shea (1982, 1985) has shown that the resulting yield function tends to bend sharply toward the end of the maturity range observed in the sample. This would seem to be a most unlikely property of a true yield curve relation and also suggests that these models would not be useful for prediction outside the sample maturity range. Other researchers have fitted a variety of parametric models to yield curves, including Cohen, Kramer, and Waugh (1966), Fisher (1966), Echols and Elliott (1976), Dobson (1978), Heller and Khan (1979), and Chambers, Carleton, and Waldman (1984). Some of these are based on polynomial regression, and all include at least a linear term that would force extrapolated very long term rates to be unboundedly large (either positive or negative) despite their abilities to fit closely within the range of the data. Vasicek and Fong (1982) have recommended exponential splines as an alternative to polynomial splines. In a comparison of the two spline methodologies, Shea (1984) finds that exponential splines are subject to the same shortcomings that polynomial splines are, essentially because polynomial splines are used after a change of variables.

Students of the term to maturity structure of interest rates have invariably described yield curves that are essentially monotonic, humped, or, occasionally, S shaped. This consistency is strikingly evident in the long historical record of subjectively drawn curves presented by Wood (1983). A similar consistency is shown by the yield curves plotted in Malkiel (1966, pp. 13, 14) and in the *Treasury Bulletin*. This is true even of yield curves based on polynomial methods. For example, those plotted by Chambers et al. (1984) reveal the explosive tendencies of polynomials only toward the end of the fitted maturity range.

A class of functions that readily generates the typical yield curve shapes is that associated with solutions to differential or difference equations. The expectations theory of the term structure of interest rates provides heuristic motivation for investigating this class since, if spot rates are generated by a differential equation, then forward rates, being forecasts, will be the solution to the equations. For example, if the instantaneous forward rate at maturity  $m$ , denoted  $r(m)$ , is given by

the solution to a second-order differential equation with real and unequal roots, we would have

$$r(m) = \beta_0 + \beta_1 \cdot \exp(-m/\tau_1) + \beta_2 \cdot \exp(-m/\tau_2),$$

where  $\tau_1$  and  $\tau_2$  are time constants associated with the equation, and  $\beta_0$ ,  $\beta_1$ , and  $\beta_2$  are determined by initial conditions. This equation generates a family of forward rate curves that take on monotonic, humped, or S shapes depending on the values of  $\beta_1$  and  $\beta_2$  and that also have asymptote  $\beta_0$ . The yield to maturity on a bill, denoted  $R(m)$ , is the average of the forward rates

$$R(m) = 1/m \int_0^m r(x)dx,$$

and the yield curve implied by the model displays the same range of shapes.

Experimentation with fitting this model to bill yields suggested that it is overparameterized. As the values of  $\tau_1$  and  $\tau_2$  were varied, it was possible to find values of the  $\beta$ 's that gave nearly the same fit. Standard software for estimating nonlinear models failed to converge, giving another indication of overparameterization. A more parsimonious model that can generate the same range of shapes is given by the solution equation for the case of equal roots:

$$r(m) = \beta_0 + \beta_1 \cdot \exp(-m/\tau) + \beta_2[(m/\tau) \cdot \exp(-m/\tau)]. \quad (1)$$

This model may also be derived as an approximation to the solution in the unequal roots case by expanding in a power series in the difference between the roots. Model (1) may also be viewed as a constant plus a Laguerre function, which suggests a method for generalization to higher-order models. Laguerre functions consist of a polynomial times an exponential decay term and are a mathematical class of approximating functions; details may be found, for example, in Courant and Hilbert (1953, 1:93–97).

To obtain yield as a function of maturity for the equal roots, model (1) integrates  $r(\cdot)$  in (1) from zero to  $m$  and divides by  $m$ . The resulting function is

$$R(m) = \beta_0 + (\beta_1 + \beta_2) \cdot [1 - \exp(-m/\tau)]/(m/\tau) - \beta_2 \cdot \exp(-m/\tau), \quad (2)$$

which is also linear in coefficients, given  $\tau$ . The limiting value of  $R(m)$  as  $m$  gets large is  $\beta_0$  and as  $m$  gets small is  $(\beta_0 + \beta_1)$ , which are necessarily the same as for the forward rate function since  $R(m)$  is just an averaging of  $r(\cdot)$ . The range of shapes available for  $R(m)$  depends on a single parameter since for  $\tau = 1$ ,  $\beta_0 = 1$ , and  $(\beta_0 + \beta_1) = 0$  we have

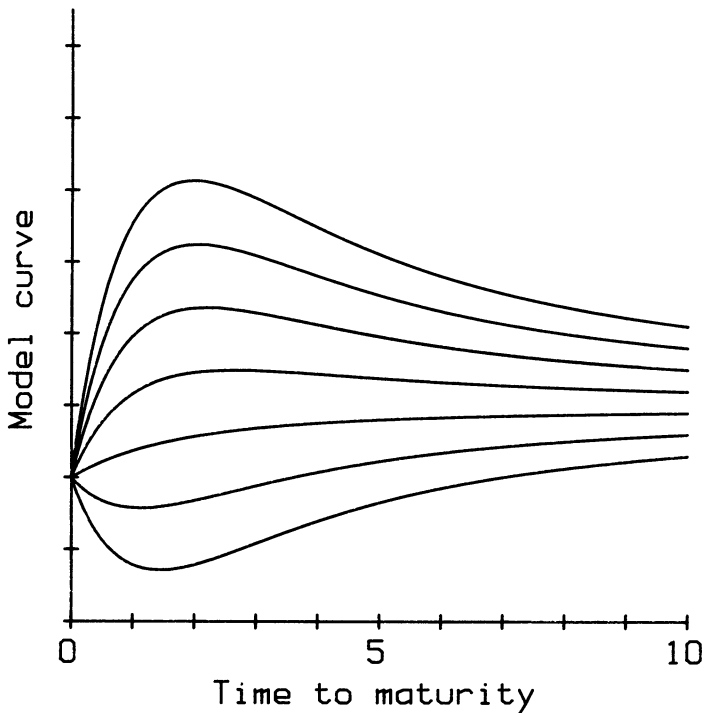


FIG. 1.—Yield curve shapes

$$R(m) = 1 - (1 - a) \cdot [1 - \exp(-m)]/m - a \cdot \exp(-m).$$

Allowing parameter  $a$  to take on values from minus six to 12 in equal increments generates the shapes displayed in figure 1, which include humps, S shapes, and monotonic curves. On the basis of the range of shapes available to us in the second order model, our operating hypothesis is that we will be able to capture the underlying relation between yield and term to maturity without resorting to more complex models involving more parameters.

Another way to see the shape flexibility of the second-order model is to interpret the coefficients of the model (1) as measuring the strengths of the short-, medium-, and long-term components of the forward rate curve (and hence of the yield curve). The contribution of the long-term component is  $\beta_0$ , that of the short-term component is  $\beta_1$ , and  $\beta_2$  indicates the contribution of the medium-term component. From figure 2 we see why these assignments are appropriate. The long-term component is a constant that does not decay to zero in the limit. The medium-term curve is the only function within this model that starts out at zero (and is therefore not short term) and decays to zero (and is therefore not long term). The short-term curve has the fastest decay of all func-

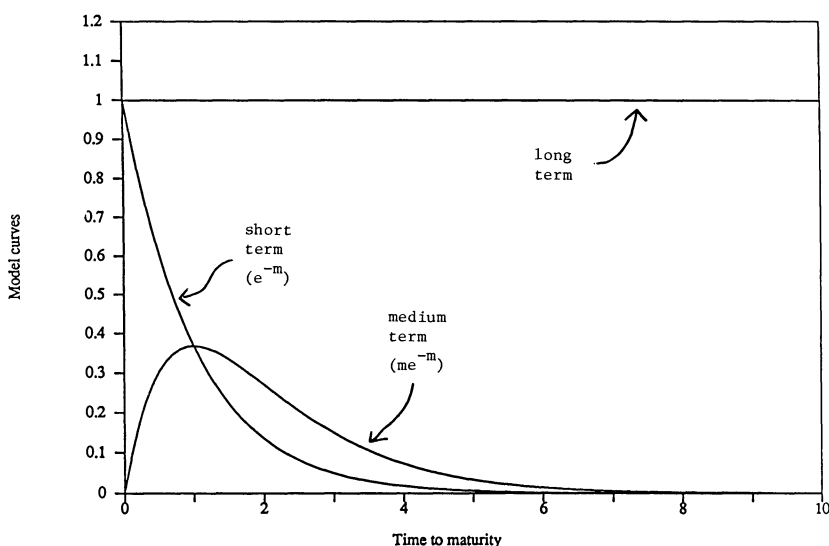


FIG. 2.—Components of the forward rate curve

tions within the model that decay monotonically to zero. It is easy to see how, with appropriate choices of weights for these components, we can generate a variety of yield curves based on forward rate curves with monotonic and humped shapes.

## II. Empirical Yield Curves for U.S. Treasury Bills

The objective of our empirical work is to assess the adequacy of the second-order model for describing the relation between yield and term to maturity for U.S. Treasury bills. By choosing bills for our pilot study, we hoped to avoid some of the complications associated with coupon bonds, such as differential rates of taxation for coupon income and capital gains. The data come from Federal Reserve Bank of New York quote sheets sampled on every fourth Thursday (excepting holidays) from January 22, 1981, through October 27, 1983, making 37 samples in all. The quote sheets give the bid and asked discount and bond equivalent yield for the bills in each maturity date outstanding as of the close of trading on the date of the quote sheet. The number of days to maturity is calculated from the delivery date, which is the following Monday for a Thursday transaction, until the maturity date. Typically, there are 32 maturities traded, which on these Thursdays works out to terms of from 3 days to 178 days in increments of 7 days, of 199 days, and, then, in 28-day increments to 339 days. On three dates there was also a 1-year bill traded. The bid and asked discounts are calculated on the quote sheets as if there were a 360-day year and

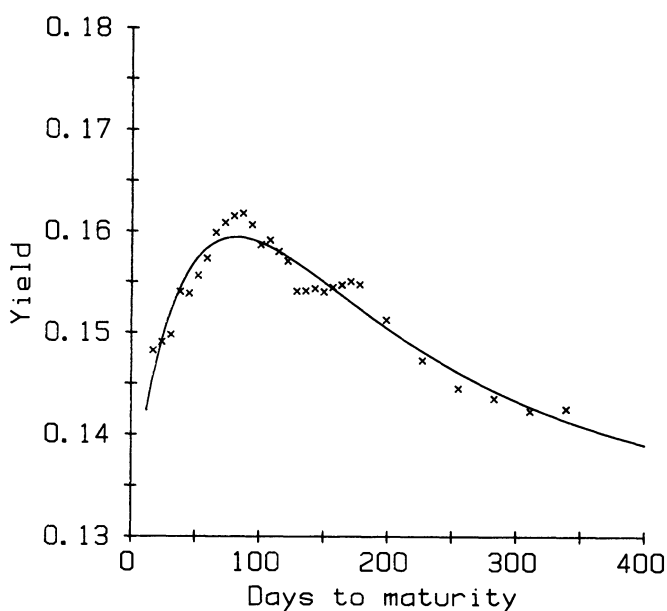


FIG. 3.—Data and fitted curve, January 22, 1981,  $\tau = 50$

are on a simple interest basis. Bill prices themselves are not displayed but are readily calculated from the discount yields. We have converted the asked discount to the corresponding price (that paid by an investor) and then calculated the continuously compounded rate of return from delivery date to maturity date annualized to a 365.25-day year. These yields are the data we fit to the yield curve model. Observations on the first two maturities, 3 and 10 days, are omitted because the yields are consistently higher, presumably because of relatively large transaction costs over a short term to maturity. This leaves 30 yield/maturity pairs observed on each of 34 market dates and 31 pairs on three dates. The data collected for the first date in our sample (January 22, 1981) are plotted in figure 3.

For purposes of fitting yield curves we have parameterized the model (2) in the form

$$R(m) = a + b \cdot [1 - \exp(-m/\tau)]/(m/\tau) + c \cdot \exp(-m/\tau). \quad (3)$$

For any provisional value of  $\tau$  we may readily calculate sample values of the two regressors. The best-fitting values of the coefficients  $a$ ,  $b$ , and  $c$  are then computed using linear least squares. Repeating this procedure over a grid of values for  $\tau$  produces the overall best-fitting values of  $\tau$ ,  $a$ ,  $b$ , and  $c$ . Recall that  $\tau$  is a time constant that determines the rate at which the regressor variables decay to zero. Plots of the data set reveal that the yield/maturity relation becomes quite flat in the

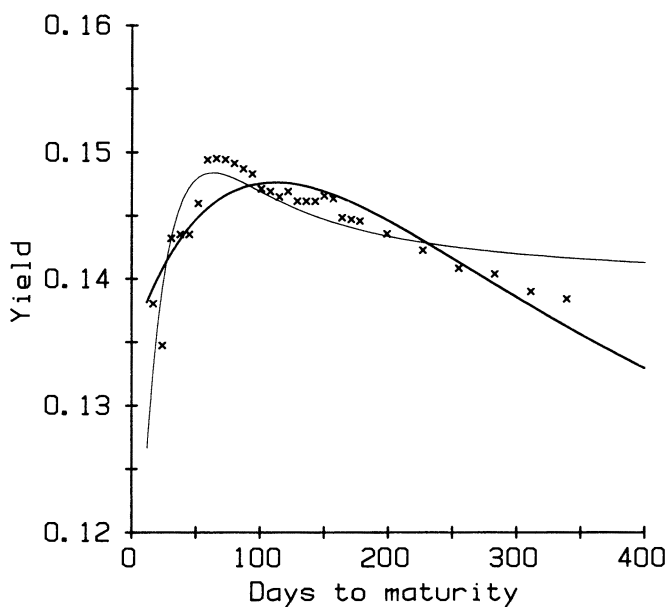


FIG. 4.—Data and fitted curves, February 19, 1981,  $\tau = 20$  and 100

range of 200–300 days (as in fig. 3), suggesting that best-fitting values of  $\tau$  would be in the range of 50–100. We consequently search over a grid from 10 to 200 in increments of 10 and also of 250, 300, and 365.

Small values of  $\tau$  correspond rapid decay in the regressors and therefore will be able to fit curvature at low maturities well while being unable to fit excessive curvature over longer maturity ranges. Correspondingly, large values of  $\tau$  produce slow decay in the regressors that can fit curvature over longer maturity ranges, but they will be unable to follow extreme curvature at short maturities. This trade-off is illustrated in figure 4, which shows the yields observed on February 19, 1981 (our second data set), and fitted curves for  $\tau = 20$  and 100. The best overall fit for this data set is given by  $\tau = 40$  (not plotted).

It is also quite clear from figure 4 that no set of values of the parameters would fit the data perfectly, nor is it our objective to find a model that would do so. A more highly parameterized model that could follow all the wiggles in the data is less likely to predict well, in our view, than a more parsimonious model that assumes more smoothness in the underlying relation than one observes in the data. There are a number of reasons why we would not expect the data to conform to the *true* underlying relation between yield and maturity even if we knew what it was. For example, there is not continuous trading in all bills, so published quotes may reflect transactions that occurred at different points in time during the trading day even though the quotes are supposed to



reflect conditions at the close of the market. Bills of specific maturities may sell at a discount or premium because of transaction cost differences. We hope that, by studying departures of the data from the fitted model, we will be able to identify *systematic* as well as idiosyncratic features of the data that the model is failing to capture.

The basic results for the second-order model fitted to each of the 37 data sets are presented in table 1. The stub column gives the data set number, column 1 the best-fitting value of  $\tau$ , column 2 the standard deviation of residuals in basis points (hundredths of a percent), and column 3 the value of  $R^2$ . Median values of these statistics over the 37 samples are given in the last row of the table. The first point worth noting is that the model accounts for a very large fraction of the variation in bill yields; median  $R^2$  is .959. The median standard deviation of residuals is 7.25 basis points, or .0725 percentage points, or a .000725 in yield. Standard deviations range from about 2 basis points to about 20. Best-fitting values of  $\tau$  have a median of 50. They occurred at the lower boundary of the search range ( $\tau = 10$ ) in two cases and at the upper boundary ( $\tau = 365$ ) in three cases. The fitted yield curve for the first data set is displayed in figure 3, showing an example of a humped shape.

It is clear from the pattern of deviations from the curve that residuals are not random but rather seem to exhibit some dependence along the maturity axis. We therefore refrain from making statements about the statistical significance of coefficient estimates on the basis of conventional standard errors. We will also be interested to see if such patterns are systematic across samples.

Although the best-fitting values of  $\tau$  vary considerably, as column 1 of table 1 shows, rather little precision of fit is lost if we impose the median value of 50 for  $\tau$  for all data sets. The resulting standard deviations appear in column 4 of table 1 and have a median value of 7.82 basis points, or only .57 basis points higher than when each data set was allowed to choose its own  $\tau$ . For a few data sets this constraint makes a noticeable difference, as in the case of data set 8 for August 6, 1981, in which a small value of  $\tau$  is able to account for a sharp drop in yields at maturities below 50 days. However, in the cases in which  $\tau$  was 365, the constraint costs little in terms of precision. The overall results suggest that little may be gained in practice by fitting  $\tau$  to each data set individually.

The lowest value of  $R^2$  recorded was 49.7 for data set 7 (July 9, 1981), while the highest was 99.6 for set 24 (October 28, 1982). The characteristics of the two data sets that led to this result are evident in figures 5 and 6, respectively. Data set 7 in figure 5 appears to be two data sets at different levels, which a smooth curve will have little ability to account for. This apparent discontinuity is unique in our sample and may reflect lack of late trading in the long sector of the market that day or, per-

TABLE 1 Measures of Model Fit

| Set No. | $\tau$ | Second-Order Model |       |                 | First Term Only   |
|---------|--------|--------------------|-------|-----------------|-------------------|
|         |        | SD at Best $\tau$  | $R^2$ | SD, $\tau = 50$ | SD at Best $\tau$ |
| 1       | 50     | 16.09              | 92.4  | 16.09           | 46.71             |
| 2       | 40     | 13.00              | 88.9  | 13.67           | 36.42             |
| 3       | 30     | 11.22              | 72.3  | 12.45           | 13.46             |
| 4       | 60     | 6.01               | 86.7  | 6.12            | 9.00              |
| 5       | 40     | 12.92              | 87.8  | 14.52           | 30.97             |
| 6       | 40     | 13.47              | 93.3  | 13.52           | 13.32             |
| 7       | 80     | 15.61              | 49.7  | 15.90           | 17.11             |
| 8       | 10*    | 10.43              | 81.7  | 22.42           | 23.00             |
| 9       | 20     | 19.85              | 88.8  | 20.34           | 19.56             |
| 10      | 50     | 18.33              | 95.2  | 18.33           | 18.10             |
| 11      | 30     | 4.88               | 98.8  | 6.11            | 6.95              |
| 12      | 300    | 12.28              | 93.8  | 12.43           | 12.16             |
| 13      | 50     | 7.76               | 99.4  | 7.76            | 7.67              |
| 14      | 30     | 11.08              | 98.0  | 11.32           | 11.22             |
| 15      | 60     | 10.51              | 95.7  | 10.75           | 15.20             |
| 16      | 10*    | 6.28               | 97.3  | 7.30            | 7.55              |
| 17      | 110    | 5.11               | 98.3  | 5.71            | 5.74              |
| 18      | 20     | 7.51               | 86.4  | 10.12           | 11.10             |
| 19      | 170    | 4.12               | 98.8  | 4.46            | 4.05              |
| 20      | 20     | 5.79               | 98.8  | 9.26            | 9.98              |
| 21      | 20     | 20.04              | 96.7  | 25.17           | 25.55             |
| 22      | 365*   | 15.08              | 98.3  | 15.84           | 15.41             |
| 23      | 40     | 10.01              | 99.1  | 11.65           | 14.78             |
| 24      | 30     | 2.91               | 99.6  | 5.13            | 6.17              |
| 25      | 20     | 7.25               | 97.4  | 7.45            | 7.34              |
| 26      | 100    | 5.18               | 93.9  | 5.33            | 5.09              |
| 27      | 300    | 3.71               | 97.3  | 4.03            | 3.65              |
| 28      | 50     | 5.38               | 95.5  | 5.38            | 5.28              |
| 29      | 110    | 6.72               | 85.6  | 6.90            | 6.59              |
| 30      | 70     | 1.95               | 98.0  | 2.10            | 2.21              |
| 31      | 365*   | 3.74               | 91.6  | 4.02            | 3.68              |
| 32      | 20     | 4.89               | 96.1  | 5.80            | 4.83              |
| 33      | 40     | 3.16               | 99.1  | 3.22            | 3.19              |
| 34      | 120    | 7.24               | 96.1  | 7.82            | 7.11              |
| 35      | 90     | 15.34              | 86.3  | 15.51           | 15.07             |
| 36      | 365*   | 5.53               | 95.9  | 6.17            | 5.43              |
| 37      | 180    | 3.01               | 99.0  | 4.25            | 2.97              |
| Median  | 50     | 7.25               | 95.9  | 7.82            | 9.00              |

NOTE.—Standard deviations are in basis points.

\* Best fit realized at boundary of range of search.

haps, clerical error (we refer to this as the “coffee break” data set). In contrast, data set 24 in figure 6 presents a very smooth, S-shaped pattern that is very precisely tracked by the model, leaving residuals with a standard deviation of only about 3 basis points.

The ability of the second-order model to generate hump shapes was one of its attractive attributes conceptually, but the question remains

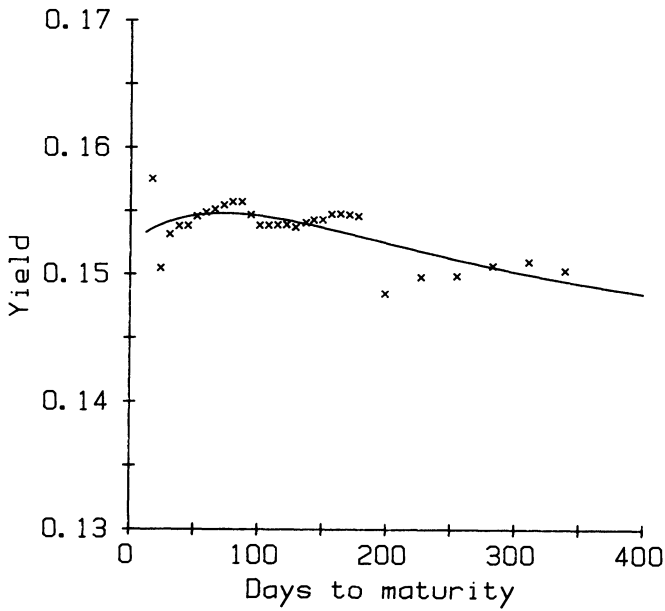


FIG. 5.—Data and fitted curves, July 9, 1981,  $\tau = 80$

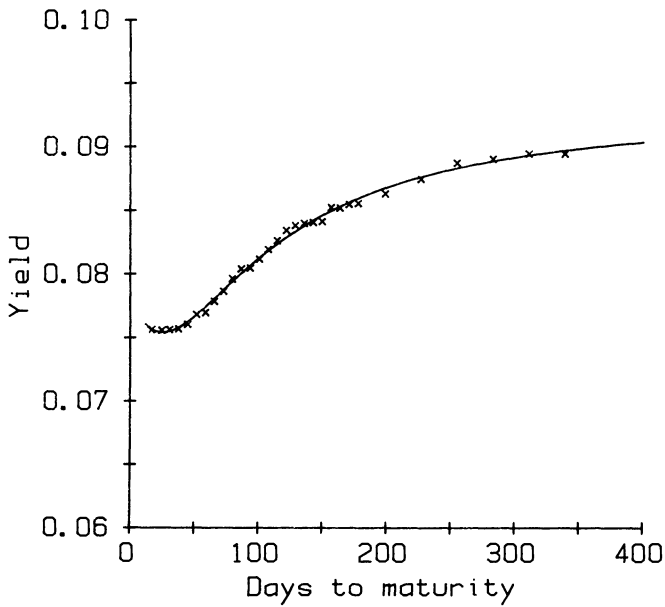


FIG. 6.—Data and fitted curve, October 28, 1982,  $\tau = 30$

whether this flexibility is important empirically. A simpler alternative model would be a simple exponential function for forward rates obtained by setting  $\beta_2$  equal to zero in equation (1). Only monotonic yield curves can be generated by this restricted model. The final column of table 1 shows the standard deviations of residuals resulting from imposing this constraint (but now allowing  $\tau$  to take its best-fitting value). The median over the 37 data sets is 9.00 basis points compared with the 7.25 reported for the unconstrained model. In some cases the standard deviation rises sharply. For example, it is no surprise that a monotonic curve does not fit the first data set well: the standard deviation rises from 16.09 to 46.71 basis points. In some cases the standard deviation is reduced slightly because the constrained model fits about as well and uses one less parameter.

Note that the loss of precision associated with the monotonic model is generally much greater for the first 23 data sets than for the later 14. The breaking point comes at October 1982. The ability of the second-order model to fit humped shapes was evidently much more important before this date than after. One way to see the evolution of the second-order model over time is to plot the three parameters of the forward rate function, as in figure 7. Recall from Section I that these can be associated with the short-term, medium-term, and long-term components of the model. The October 1982 breaking point is indicated by the

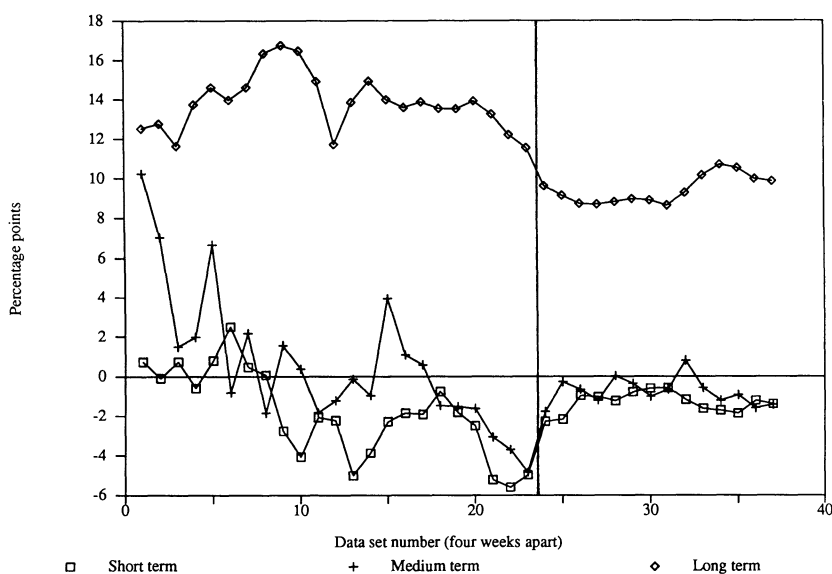


FIG. 7.—Time-series plots of forward rate components

vertical line. The variation in all the components is seen to have diminished after that date. The magnitudes of the short- and medium-term components are also much smaller after October 1982, and, without the medium-term component, the model becomes monotonic. Evidently, the shape of the yield curve abruptly became simpler and more stable. Note also from table 1 that the dispersion of the data around the fitted yield curves dropped sharply. We surmise that all these phenomena are the result of a change in Federal Reserve monetary policy in October 1982, at which time the Federal Reserve is said to have switched from stabilizing the monetary aggregates to stabilizing interest rates. Under the earlier regime, the market could reasonably have expected fairly sharp moves in interest rates in coming weeks as monetary aggregates were observed to deviate from target. Risk premiums associated with interest rate uncertainty may also have been larger and more variable. Under the new regime, the market would reasonably expect little change in interest rates. Simpler dynamics are associated with a simpler, lower-order yield curve. With prices moving less rapidly, dealer's quotes may also have become more accurate measures of the structure of yields at a point in time; hence, there was less dispersion in residuals from the fitted model. The evidence presented here clearly confirms that a major change in monetary policy did occur in October 1982.

### III. Analysis of Residuals: Maturity and Issue Effects

Plots of fitted yield curves against the data have suggested some dependence of residuals along the maturity axis. We would like to try to determine whether this is due to a systematic but nonsmooth influence of maturity on yield, which would show up in the pattern of residuals from our model. If such an effect persists through time, then we should be able to detect it in the average of the 37 residuals corresponding to a specific maturity. Figure 8 is a plot of the averaged residuals which fall in the range  $\pm 7$  basis points compared with a rough standard error of 1.2 basis points. The pattern seen here is not only average; it is typical. The first maturity is 17 days, and the large positive average residual reflects higher transaction costs per unit time. The fitted curves tend to be pulled up by this data point, leaving the next below the curve. The most unstable feature of the plot is a rising slope to just under 90 days, then a sharp drop, then a rise to just under 180 days, another sharp drop, and finally a rise to the longest maturity. These peaks correspond to the maturities of the bills auctioned by the Treasury. Roll (1970) has documented a similar pattern for average bid-asked spreads, which he attributed to lower dealer inventory costs for the highly active newly issued maturities. What is relevant to purchasers of bills is choice of asked yields across maturities. By fitting a smooth yield curve, we have made this pattern highly visible and have quantified it.

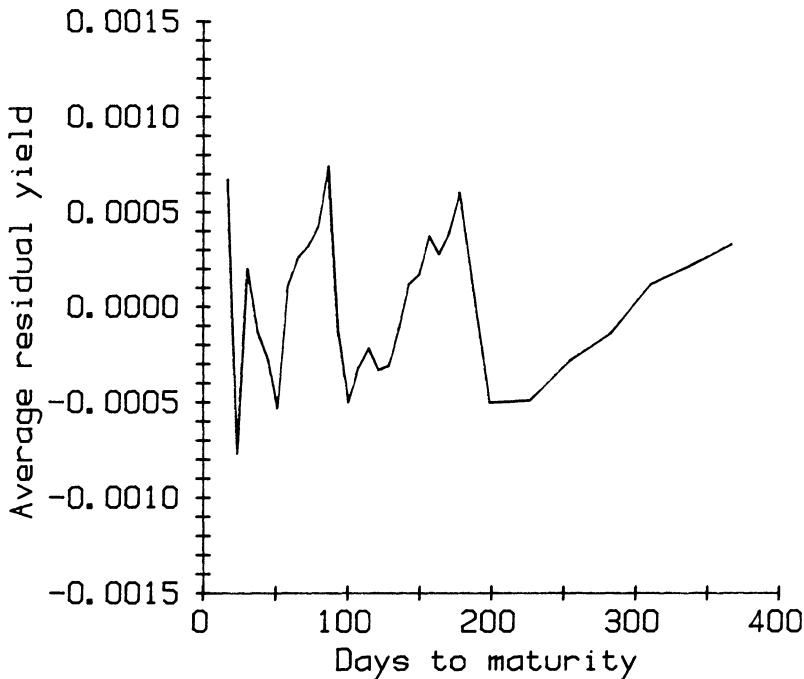


FIG. 8.—Averaged residuals by days to maturity

Issue effects are distinguished from maturity effects in that they pertain to the bills that mature on a particular date rather than to bills with a particular term to maturity. We found some evidence of issue effects since large residuals for a particular issue show some tendency to persist from one quote sheet to the next. Evidence for issue effects is less compelling than that for maturity effects but seems to warrant further investigation.

#### IV. Prediction out of Sample: Pricing a Long-Term Bond

One of our criteria for a satisfactory yield curve model is that it be able to predict yields beyond the maturity range of the sample used to fit it. An unreasonably exacting test would be to ask it to predict the yield or price of a long-term government bond, but this is what we have tried to do. The particular bond chosen is the 12 $\frac{3}{4}$ % coupon U.S. Treasury bond maturing in 2010 (callable in 2005) since this was the longest bond appearing on all our quote sheets.

We may estimate the price of a bond as the present value of the series of cash flows (coupon payments and principal repayment) discounted according to the yield curve value at the term of each payment. A bond can be thought of as a bundle of bills consisting of the

coupons with maturities spaced at 6-month intervals and the face value payment at the maturity date of the bond. Each component bill pays an amount equal to the semiannual coupon, except the last, which also pays the face value of the bond. Values from a yield curve can be used to discount each component bill in the stream. The resulting total value can be compared with the quoted price of the bond, adjusting first for accrued interest from the last coupon date, which the buyer must pay to the seller.

The predicted bond price will depend primarily on the portion of the yield curve that lies beyond the range of the sample bill data because at most only the first two semiannual coupon payments can be due within the 1-year maturity limit of U.S. Treasury bills. For our yield curve model with values of  $\tau$  around 50, the fitted curve flattens out considerably for maturities beyond a year. The first exponential term in the model goes from unity at zero maturity to .1369 at 365-days maturity, and the second term goes from unity to .0007 in the same interval. The pricing of the bond is therefore determined largely by the asymptotic level of the curve given by the intercept in the model,  $\beta_0$ . Equivalently, the value of the intercept must be close to the yield to maturity on the bond if the model is to price the bond accurately. When we allowed  $\tau$  to take its best-fitting value, two predictions went drastically awry: the twelfth (\$138.063 against an actual price of \$100.34) and the twenty-second (\$404.58 against an actual price of \$103.59). These were both models that had large values of  $\tau$  (see table 1). In both cases the bill yield data were fitted as the rising portion of a long hump with eventual decay to a much lower level, which was .079 for the twelfth model and -.025 for the twenty-second. The resulting discount rates are therefore too low and the predicted bond price correspondingly too high. Constraining  $\tau$  to a value of 50 in both cases costs little in standard deviation of fit (see table 1) but improves the predictions of the prices of the two bonds dramatically, to \$105.77 and \$102.52, respectively. Evidently, the value of  $\tau$  is best chosen by fitting across data sets rather than by selecting the value for each individual data set.

The relation between actual and predicted bond price is depicted as a time-series plot in figure 9 and as a scatter plot in figure 10. It is obvious that the correlation between actual and predicted price is high—numerically it is .963—but it is also clear that the predictions overshoot the actuals. The magnitude of overshooting is much larger than could be accounted for by favorable tax treatment for the bond when it is selling at a discount from face value. This suggests that our fitted curves may flatten out too rapidly. When yields generally were high and the yield curve downward sloping, the models overestimated long-term discount rates and therefore underestimated the price of the bond. The reverse was true when yields were relatively low and the yield

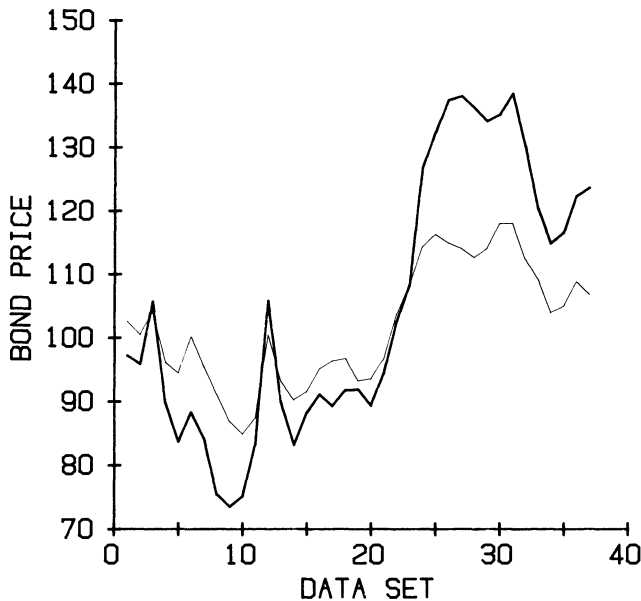


FIG. 9.—Time-series plot of actual and predicted (darker line) bond price

curve was upward sloping. Correcting the price predictions for these systematic biases by simple linear regressions, we obtain a standard deviation for the adjusted bond price prediction of only \$2.63.

What correspondence is there between the ability of a model to fit the bill yield data well and its accuracy in extrapolating beyond the sample to predict the yield on a bond? The short answer is none necessarily. A function may have the flexibility to fit data over a specific interval but may have very poor properties when extrapolated outside that interval. A cubic polynomial has the same number of parameters as does our model and indeed fits the bill yield data slightly better. The median standard deviation of residuals is only 7.1 basis points over the 37 data sets. However, we know that a cubic polynomial in maturity will head off to either plus infinity or minus infinity as maturity increases, the sign depending on the sign of the cubic term. It is clear, then, that, if we use a cubic polynomial yield curve to price out a bond, it will assign either very great present value or very little present value to distantly future payments. For our data set, the result is predicted bond prices that bunch in the intervals \$17–\$40 and \$384–\$408. The correlation between actual and predicted bond price is  $-0.020$ , so the polynomial model has no predictive value, although it fits the sample data very well.



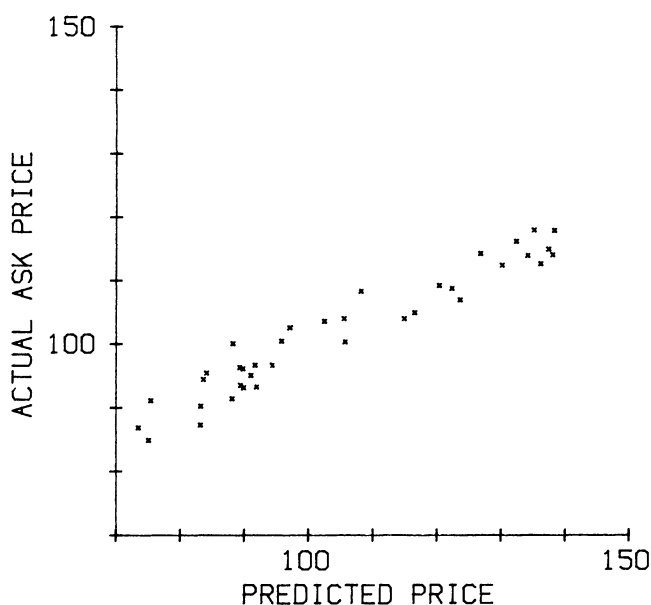


FIG. 10.—Scatter plot of actual and predicted bond price

## V. Conclusions

Our objective in this paper has been to propose a class of models, motivated by but not dependent on the expectations theory of the term structure, that offers a parsimonious representation of the shapes traditionally associated with yield curves. Pilot testing on U.S. Treasury bill data suggests that a very simple model with only a single-shape parameter is able to characterize the shape of the bill term structure. The model imposes sufficient smoothness to reveal a maturity-specific pattern that can be related to lower transaction costs for bills at the maturities issued by the Treasury. If the model reflects the basic shape of the term structure and not just a local approximation, then we should be able to predict yields or prices at maturities beyond the range of the sample. Confirming this, we find a high correlation between the present value of a long-term bond implied by the fitted curves and the actual reported price of the bond.

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