4300303
Eletromagnetismo I
Segundo Semestre 2014

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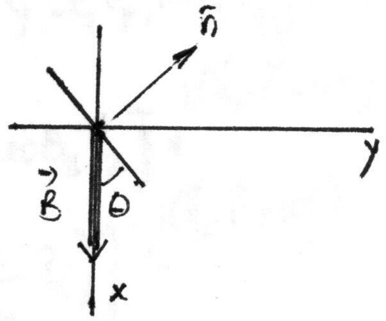
1. a) $\mathcal{E} = \oint_{\mathcal{C}} \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$; $I = \frac{\mathcal{E}}{R}$



$$\vec{B}\cdot d\vec{S} = B\hat{e}_x\cdot(dS\,\hat{n}) = B\,dS\cos\left(\frac{\pi}{2}+\theta\right)$$

$$\therefore \vec{B}\cdot d\vec{S} = -B\,dS\sin\theta = -B\sin(\omega t)\,dS$$

$$\therefore \mathcal{E} = \frac{d}{dt}B\sin(\omega t)\int dS \quad \therefore \quad \boxed{I = \frac{Ba^2\omega}{R}\cos(\omega t)}$$

b) $\vec{\tau} = \vec{m}\times\vec{B} = (Ia^2\hat{n})\times B\hat{e}_x \quad \therefore \vec{\tau} = BIa^2\,\hat{n}\times\hat{e}_x$; $\hat{n}\times\hat{e}_x = -\hat{e}_z\overset{\sin\left(\frac{\pi}{2}+\theta\right)}{\cancel{\sin(\theta)}}$

$$\therefore \vec{\tau} = -\frac{B^2a^4\omega}{R}\cos(\omega t)\sin\left(\frac{\pi}{2}+\omega t\right)\hat{e}_z \qquad \boxed{\therefore \vec{\tau} = -\frac{B^2a^4\omega}{R}\cos^2(\omega t)\,\hat{e}_z}$$

2. a) $r_s = \frac{n_1\cos\theta_B - n_2\cos\left(\frac{\pi}{2}-\theta_B\right)}{n_1\cos\theta_B + n_2\cos\left(\frac{\pi}{2}-\theta_B\right)} = \frac{n_1\cos\theta_B - n_2\sin\theta_B}{n_1\cos\theta_B + n_2\sin\theta_B} = \frac{n_1 - n_2\tan\theta_B}{n_1 + n_2\tan\theta_B}$

; $\tan\theta_B = \frac{n_2}{n_1} \quad \therefore \quad \boxed{r_s = \frac{n_1^2-n_2^2}{n_1^2+n_2^2}}$

$$t_s = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{2n_1\cos\theta_B}{n_1\cos\theta_B + n_2\sin\theta_B} = \frac{2n_1}{n_1 + n_2\tan\theta_B}$$

$$\boxed{t_s = \frac{2n_1^2}{n_1^2+n_2^2}}$$

$$t_p = \frac{2n_1\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1} = \frac{2n_1\cos\theta_1}{n_1\,\mathrm{sen}\,\theta_1 + n_2\cos\theta_1} = \frac{2n_1}{n_1\,\mathrm{tg}\,\theta_B + n_2}$$

$$\therefore\ t_p = \frac{2n_1}{n_1\frac{n_2}{n_1} + n_2} \quad\therefore\ t_p = \frac{n_1}{n_2}$$

$$\vec{E}_2 = t_s\vec{E}_{1s} + t_p\vec{E}_{1p} = \left[t_s E_s\hat{e}_x + t_p E_p\cos\theta_2\,\hat{e}_y - t_p E_p\,\mathrm{sen}\,\theta_2\,\hat{e}_z\right]e^{i(\vec{k}_2\cdot\vec{r}-\omega t)}$$

$$\therefore\ \vec{E}_2 = \left[\frac{2n_1^2}{n_1^2+n_2^2}E_s\hat{e}_x + \frac{n_1}{n_2}E_p\,\mathrm{sen}\,\theta_B - \frac{n_1}{n_2}E_p\cos\theta_B\,\hat{e}_z\right]e^{i(\vec{k}_2\cdot\vec{r}-\omega t)}$$

$$= \left[\frac{2n_1^2}{n_1^2+n_2^2}E_s\hat{e}_x + \frac{n_1}{n_2}E_p\,\mathrm{sen}\,\theta_B\left(\hat{e}_y - \frac{\cos\theta_B}{\mathrm{sen}\,\theta_B}\hat{e}_z\right)\right]e^{i(\vec{k}_2\cdot\vec{r}-\omega t)}$$

$$\therefore\ \boxed{\vec{E}_2 = \left[\frac{2n_1^2}{n_1^2+n_2^2}E_s\hat{e}_x + \frac{n_1}{n_2}E_p\,\mathrm{sen}\,\theta_B\left(\hat{e}_y - \frac{n_1}{n_2}\hat{e}_z\right)\right]e^{i(\vec{k}_2\cdot\vec{r}-\omega t)}}$$

3. a)

$$\nabla^2\phi + k^2\phi = -g \qquad [\times G]$$

$$\ominus$$

$$\nabla^2 G + k^2 G = -\delta(\vec{r}-\vec{r}\,') \qquad [\times\phi]$$

$$\Rightarrow\ G\nabla^2\phi - \phi\nabla^2 G = \phi\,\delta(\vec{r}-\vec{r}\,') - G\,g(\vec{r})$$

Integrando ambos os lados da equação resultante sobre um volume V, temos

$$\int_V\left[G\nabla^2\phi - \phi\nabla^2 G\right]d\tau = \int\phi\,\delta(\vec{r}-\vec{r}\,')\,d\tau - \int G\,g\,d\tau$$

Utilizando a Segunda Identidade de Green (dada no formulário), e usando a integral sobre uma função δ, temos

$$\phi(\vec{r}\,') - \int G(\vec{r},\vec{r}\,')\,g(\vec{r})\,d\tau = \int_S\left[G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right]ds$$

Tomando $S\to\infty$ e impondo que $\phi(R\to\infty)=0$; $G(R\to\infty)=0$, obtemos, finalmente

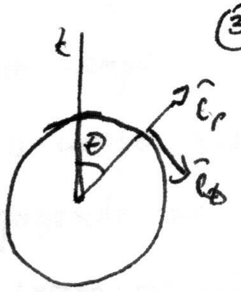
$$\boxed{\phi(\vec{r}) = \int_V G(\vec{r},\vec{r}\,')\,g(\vec{r}\,')\,d\tau'}$$

onde trocamos os símbolos das variáveis $\vec{r}$ e $\vec{r}\,'$.

③

b) $\vec{A}(\vec{r},t) = -\dfrac{\mu_0 p_0 \omega}{4\pi r} \sin\left[\omega\left(t - \dfrac{r}{c}\right)\right]\hat{e}_z$

$\vec{B} = \nabla \times \vec{A}$



Em coordenadas esféricas, $\hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$

$\therefore$

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin\left[\omega\left(t - \frac{r}{c}\right)\right]\left(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta\right)$$

$\therefore$

$$\vec{B} = -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{-\frac{\partial}{\partial r}\left[\sin\left(\omega t - \frac{\omega r}{c}\right)\sin\theta\right] - \frac{\partial}{\partial \theta}\left[\frac{1}{r}\sin\left(\omega t - \frac{\omega r}{c}\right)\cos\theta\right]\right\}\hat{e}_\varphi$$

$\therefore$

$$\vec{B} = -\frac{\mu_0 p_0 \omega}{4\pi r}\left[\frac{\omega}{c}\cos\left(\omega t - \frac{\omega r}{c}\right)\sin\theta + \frac{1}{r}\sin\left(\omega t - \frac{\omega r}{c}\right)\sin\theta\right]\hat{e}_\varphi$$

O primeiro termo cai com $1/r$ e, portanto, é um campo de radiação. Mas o segundo cai com $1/r^2$, não caracterizando um campo de radiação. Portanto

$$\boxed{\vec{B}_{rad} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left(\omega t - \frac{\omega r}{c}\right)\hat{e}_\varphi}$$

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4. $\vec{E} = E_0 \cos\left[\omega\left(\dfrac{y}{c} - t\right)\right]\hat{e}_x + E_0 \sin\left[\omega\left(\dfrac{x}{c} - t\right)\right]\hat{e}_y$

a) $-\dfrac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \left[\left(\dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z}\right)\hat{e}_x + \left(\dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x}\right)\hat{e}_y + \left(\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right)\hat{e}_z\right]$

$\therefore -\dfrac{\partial \vec{B}}{\partial t} = E_0 \dfrac{\omega}{c}\left[\cos\left[\omega\left(\dfrac{x}{c} - t\right)\right] + \sin\left[\omega\left(\dfrac{y}{c} - t\right)\right]\right]\hat{e}_z$

$\therefore \vec{B} = \dfrac{E_0}{c}\left[\sin\left[\omega\left(\dfrac{x}{c} - t\right)\right] - \cos\left[\omega\left(\dfrac{y}{c} - t\right)\right]\right]\hat{e}_z$

$\vec{S} = \dfrac{1}{\mu_0}\vec{E} \times \vec{B} = \dfrac{1}{\mu_0}\dfrac{E_0^2}{c}\left[\cos^2\left[\omega\left(\dfrac{y}{c} - t\right)\right]\hat{e}_y + \sin^2\left[\omega\left(\dfrac{x}{c} - t\right)\right]\hat{e}_x\right]$

$\Rightarrow$ $\cancel{\vec{S} = \dfrac{E_0^2}{\mu_0 c}\left[\cos^2\left(\dfrac{y}{c} - t\right)\hat{e}_y + \sin^2\right.}$ $\vec{S} = \dfrac{E_0^2}{\mu_0 c}\left[\cos^2\left[\omega\left(\dfrac{y}{c} - t\right)\right]\hat{e}_y + \sin^2\left[\omega\left(\dfrac{x}{c} - t\right)\right]\hat{e}_x\right]$

OBS Será aceita a solução de quem percebeu que o primeiro campo corresponde a uma onda se propagando na direção y com o campo elétrico na direção x e o segundo campo a uma onda se propagando na direção x com o campo elétrico na direção y. Como o campo magnético tem que ficar à direita do campo elétrico, com relação à direção de propagação, os dois campos magnéticos poderiam ser imediatamente escritos e também os vetores de Poynting correspondentes

b) Nesta solução usaremos a seguinte notação:

$$E_x = E_0 \cos\left[\omega\left(\frac{y}{c}-t\right)\right]; \quad E_y = E_0 \sin\left[\omega\left(\frac{x}{c}-t\right)\right]$$

$$B_z = \frac{E_0}{c}\left[\sin\left[\omega\left(\frac{x}{c}-t\right)\right] - \cos\left[\omega\left(\frac{y}{c}-t\right)\right]\right]$$

Então

$$T_{xx} = \epsilon_0\left[E_x E_x - \frac{1}{2}(E_x^2+E_y^2)\right] + \frac{1}{\mu}\left[0 - \frac{B_z^2}{2}\right] = \frac{\epsilon_0}{2}\left[E_x^2 - E_y^2\right] - \frac{B_z^2}{2\mu}$$

$$T_{xy} = \epsilon_0\left[E_x E_y\right] + \frac{1}{\mu_0} B_x B_y = \epsilon_0 E_x E_y$$

$$T_{xz} = \epsilon_0\left[E_x E_z\right] + \frac{1}{\mu} B_x B_z = 0$$

$$T_{yx} = \epsilon_0\left[E_y E_x\right] + \frac{1}{\mu} B_y B_x = \epsilon_0 E_y E_x = T_{xy}$$

$$T_{yy} = \epsilon_0\left[E_y E_y - \frac{1}{2}(E_x^2+E_y^2)\right] + \frac{1}{\mu}\left[0 - \frac{B_z^2}{2}\right] = \frac{\epsilon_0}{2}\left[-E_x^2 + E_y^2\right] - \frac{B_z^2}{2\mu}$$

$$T_{yz} = 0$$

$$T_{zx} = 0$$

$$T_{zy} = 0$$

$$T_{zz} = \frac{1}{\mu}\left[B_z B_z - \frac{1}{2}B_z^2\right] = \frac{B_z^2}{2\mu_0}$$