Aula 11

14.2-4

Mensagem	Probabilidade	
m_1	$1/2$	0
m_2	$1/4$	1
m_3	$1/8$	20
m_4	$1/16$	21
m_5	$1/32$	220
m_6	$1/64$	221
m_7	$1/64$	222

$1/2$	0	$1/2$	0
$1/4$	1	$1/4$	1
$1/8$	20	$1/4$	2
$1/16$	21		
$1/16$	22		

$$L = \sum P_i L_i = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{8}\cdot 2 + \frac{1}{16}\cdot 2$$

$$+ \frac{1}{32}\cdot 3 + \frac{1}{64}\cdot 3 + \frac{1}{64}\cdot 3$$

$$= \frac{32+16+16+8+6+3+3}{64} = \frac{84}{64} = \frac{21}{16} \text{ (dígitos ternários)}$$

Do problema 15.2-1, $H(m) = \frac{63}{32}$ bits $= \frac{63}{32} \frac{1}{\log_2 3}$ dígitos ternários

$= 1{,}242$ dígitos ternários

②

$$\eta = \frac{H(m)}{L} \times 100\% = \frac{1,242}{21/16} = 94,64\%$$

Redundância:

$$\gamma = (1-\eta) \cdot 100 = 5,36\%$$

14.2-5

mensagem	Prob.					
m_1	1/3 1	1/3 1	1/3 1	1/3 1	1/3 1	2/3 0
m_2	1/3 00	1/3 00	1/3 00	1/3 00	1/3 00	1/3 1
m_3	1/9 011	1/9 011	1/9 011	2/9 010	1/3 01	
m_4	1/9 0100	1/9 0100	1/9 0100	1/9 011		
m_5	1/27 01011	2/27 01010	1/9 0101			
m_6	1/27 010100	1/27 01011				
m_7	1/27 010101					

$$L = \sum P_i L_i = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{9} \cdot 3 + \frac{1}{9} \cdot 4 + \frac{1}{27} \cdot 5 + \frac{1}{27} \cdot 6 + \frac{1}{27} \cdot 6 = \frac{9+18+9+12+5+6+6}{27}$$

$$= \frac{65}{27} \text{ bits}$$

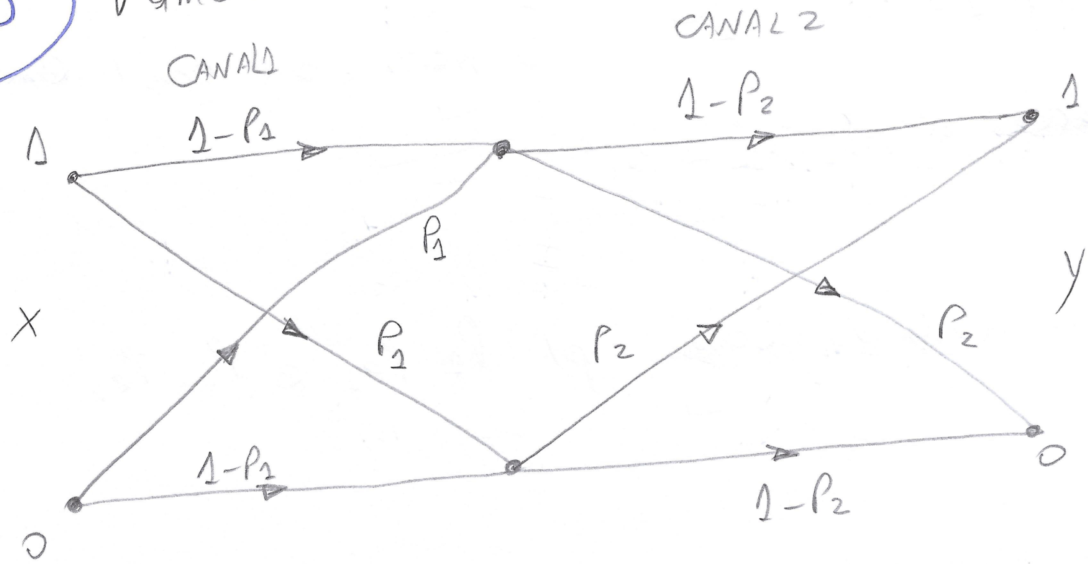
Do Exercício 14.2-2,

$H(m) = 2{,}289$ bits

Eficiência: $\eta = \frac{H(m)}{L} \times 100\% = \frac{2{,}289}{65/27} = 95{,}07\%$

Redundância $\gamma = (1-\eta) \cdot 100\% = 4{,}92\%$

14.4.3 Vamos considerar o caso geral da cascata



$P_{Y|X}(1|1) = (1-P_1)(1-P_2) + P_1 P_2 = 1 - P_1 - P_2 + 2P_1P_2$

$P_{Y|X}(0|1) = (1-P_1)P_2 + P_1(1-P_2) = P_2 - P_1P_2 + P_1 - P_1P_2$

$= P_1 + P_2 - 2P_1P_2$

Assim, a matriz de canal da cascata é

$$\begin{bmatrix} 1 - P_1 - P_2 + 2P_1P_2 & P_1 + P_2 - 2P_1P_2 \\ P_1 + P_2 - 2P_1P_2 & 1 - P_1 - P_2 + 2P_1P_2 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 1 - P_1 & P_1 \\ P_1 & 1 - P_1 \end{bmatrix}}_{\text{Matriz de canal p/ Canal 1}} \underbrace{\begin{bmatrix} 1 - P_2 & P_2 \\ P_2 & 1 - P_2 \end{bmatrix}}_{\text{Matriz de canal p/ Canal 2}} = M_1 \cdot M_2$$

(a) Para $P_1 = P_2 = P_e$,

$M_1 = M_2 = M$ e a matriz de canal em cascata é $M_1 \cdot M_2 = M^2$.

(b) Está mostrado acima p/ $P_1 = P_{e_1}$ e $P_2 = P_{e_2}$

(c) Considere a cascata de K canais idênticos separado na cascata dos 1os. K-1 canais com o último

$$\longrightarrow \boxed{M_{K-1}} \longrightarrow \boxed{M} \longrightarrow$$

Do resultado do item (b), $M_K = M_{K-1} \cdot M$, válido para qualquer K

Já mostramos que p/ $K=2$, $M_2 = M^2$ (item a). Por indução,

$$M_3 = M \cdot M^2 = M^3$$

$$M_4 = M \cdot M^3 = M^4$$

$$\vdots$$

$$\boxed{M_K = M^K}$$

Para $M=3$

$$M_3 = M^3 = \begin{bmatrix} 1-P_e & P_e \\ P_e & 1-P_e \end{bmatrix}^3 =$$

$$= \underbrace{\begin{bmatrix} 1-2P_e+2P_e^2 & 2P_e-2P_e^2 \\ 2P_e-2P_e^2 & 1-2P_e+2P_e^2 \end{bmatrix}}_{M^2} \cdot \underbrace{\begin{bmatrix} 1-P_e & P_e \\ P_e & 1-P_e \end{bmatrix}}_{M}$$

$$= \begin{bmatrix} & \overbrace{(1-2P_e+2P_e^2)\cdot(P_e)+(2P_e-2P_e^2)\cdot(1-P_e)}^{P_E} \\ & \end{bmatrix}$$

$$P_E = P_e - \cancel{2P_e^2} + \cancel{2P_e^3} + 2P_e - \cancel{2P_e^2} - 2P_e^2 + \cancel{2P_e^3}$$

$$= 4P_e^3 - 6P_e^2 + 3P_e \rightarrow \boxed{P_E = 3P_e - 6P_e^2 + 4P_e \approx \underline{3P_e}}$$

KP_e

No exemplo 10.7,

$$1-P_E = (1-P_e)^3 + \frac{\cancel{3!}^{\,3}}{\cancel{2!(1)!}} P_e^2 (1-P_e)$$

$$= 1 - \cancel{3P_e} + 3P_e^2 - P_e^3 + 3P_e^2 - 3P_e^3$$

$$= 1 - 3P_e + 6P_e^2 - 4P_e^3 \quad e$$

$$\boxed{P_E = 3P_e - 6P_e^2 + 4P_e^3}$$

OK!

d) $$C_S = 1 - \left[P_E \log \frac{1}{P_E} + (1-P_E) \log \frac{1}{1-P_E} \right]$$

Com $P_E \approx kP_e$

$$\boxed{C_S = 1 - \left[kP_e \log \frac{1}{kP_e} + (1-kP_e) \log \frac{1}{1-kP_e} \right]}$$

14-4.4 Fazendo $q = 1-p$ a matriz de canal é

$x \backslash y$	y_1 0	y_2 1	y_3 E
x_1 0	q	0	p
x_2 1	0	q	p

Sejam

$x_1 = 0$ $\quad y_1 = 0$

$x_2 = 1$ $\quad y_2 = 1$

$y_3 = E$

$$P(y_1) = P(y_1|x_1)\cdot P(x_1) + P(y_1|x_2)\cdot P(x_2) =$$

$$= 0\cdot\frac{1}{2} + q\cdot\frac{1}{2} = q/2$$

$$P(y_2) = P(y_2|x_1)\cdot P(x_1) + P(y_2|x_2)\cdot P(x_2) =$$

$$= 0\cdot\frac{1}{2} + q\cdot 1/2 = q/2$$

$$P(y_3) = P(y_3|x_1)\cdot P(x_1) + P(y_3|x_2)\cdot P(x_2) = \frac{p}{2} + \frac{p}{2} = p$$

$$P(x_1|y_1) = \frac{P(y_1|x_1)\, P(x_1)}{P(y_1)} = \frac{q\cdot 1/2}{q/2} = 1$$

$$P(x_2|y_1) = \frac{P(y_1|x_2)\cdot P(x_2)}{P(y_1)} = 0$$

$$P(x_1|y_2) = \frac{P(y_2|x_1)\cdot P(x_1)}{P(y_2)} = 0$$

$$P(x_2|y_2) = \frac{P(y_2|x_2)\cdot P(x_2)}{P(y_2)} = \frac{q\cdot 1/2}{q/2} = 1$$

$$P(x_1|y_3) = \frac{P(y_3|x_1)\cdot P(x_1)}{P(y_3)} = \frac{p\cdot 1/2}{p} = 1/2 \quad \Big| \quad P(x_2|y_3) = 1/2$$

Assim,

(8)

$$\boxed{\begin{aligned} H(x) &= -\left[P(x_1)\log P(x_1) + P(x_2)\log P(x_2)\right] = \\ &= -\left[\tfrac{1}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\log\tfrac{1}{2}\right] = 1 \end{aligned}}$$

$$P(x_1, y_1) = P(x_1) \cdot P(y_1|x_1) = \frac{1}{2} \cdot q = q/2$$

$$P(x_1, y_2) = P(x_1) \cdot P(y_2|x_1) = 0$$

$$P(x_1, y_3) = P(x_1) \cdot P(y_3|x_1) = p/2$$

$$P(x_2, y_1) = P(x_2) \cdot P(y_1|x_2) = 0$$

$$P(x_2, y_2) = P(x_2) \cdot P(y_2|x_2) = q/2$$

$$P(x_2, y_3) = P(x_2) \cdot P(y_3|x_2) = p/2$$

$$H(x|y) = \sum_i \sum_j P(x_i, y_j) \log \frac{1}{P(x_i|y_j)} =$$

$$= \cancel{\frac{q}{2} \cdot \log 1} + 0 + \frac{p}{2} \log 2 + 0 + \cancel{\frac{q}{2} \cdot \log 1} + \frac{p}{2} \log 2$$

$$= p$$

$$I(x|y) = H(x) - H(x|y) = \underline{\underline{1-p}}$$

14.5-1 Temos que

$$H(x) = \int_{-M}^{M} p(x) \log \frac{1}{p(x)} dx = -\int_{-M}^{M} p(x) \log p(x) dx$$

Claro que $\int_{-M}^{M} p(x)\, dx = 1$

Fazendo como no texto,

$$F(x) = -p(x) \log p(x) \quad e \quad \varphi_1(x) = p(x)$$

$$\frac{\partial F}{\partial p} = -\log p(x) - \frac{p(x)}{p(x)} = -(1 + \log p(x))$$

$$\frac{\partial \varphi_1(x)}{\partial p} = 1$$

Substituindo em (14.37),

$$-(1 + \log p_0(x)) + \alpha_1 \cdot 1 = 0 \Rightarrow \log p_0(x) = \alpha_1 - 1$$

$$\Rightarrow \boxed{p_0(x) = e^{\alpha_1 - 1}}$$

Além disso

$$\int_{-M}^{M} p(x)\,dx = 1 \Rightarrow \int_{-M}^{M} e^{\alpha_1 - 1}\,dx = 2M\,e^{\alpha_1 - 1} = 1 \Rightarrow$$

$$\Rightarrow \boxed{e^{\alpha_1 - 1} = 1/2M}$$

Daí,

$$\boxed{p_0(x) = \frac{1}{2M}}$$

$$H(x) = +\int_{-M}^{M} p(x) \log \frac{1}{p(x)}\,dx = \int_{-M} \frac{1}{2M} \cdot \log 2M\,dx =$$

$$= \log 2M$$

15.5-2

$$\begin{cases} H(x) = -\int_0^{\infty} \underbrace{p(x) \log p(x)}_{F}\,dx \\ \int_0^{\infty} \underbrace{p(x)}_{\phi_1} dx = 1 \qquad \int_0^{\infty} \underbrace{x\,p(x)}_{\phi_2} = A \end{cases}$$

Fazendo $F(x) = -\,p(x) \log p(x)$ $\qquad \phi_1(x) = p(x)$

$$\Downarrow$$

$$\boxed{\frac{\partial F(x)}{\partial p} = -\left[\log p(x) + 1\right]} \qquad \frac{\partial \phi_1}{\partial p} = 1$$

$$\phi_2(x) = x \cdot p(x)$$

$$\frac{\partial \phi_2(x)}{\partial p} = x$$

Substituindo em (14.37),

$$-(1 + \log p_0(x)) + \alpha_1 \cdot 1 + \alpha_2 \cdot x = 0$$

$$\Rightarrow \quad p_0(x) = e^{\alpha_1 + \alpha_2 x - 1} = e^{\alpha_2 x} \cdot e^{\alpha_1 - 1}$$

Substituindo nas restrições,

$$\int_0^\infty p(x)\,dx = \int_0^\infty e^{\alpha_1 - 1} \cdot e^{\alpha_2 x}\,dx = e^{\alpha_1 - 1}\left[\frac{e^{\alpha_2 x}}{\alpha_2}\right]_0^\infty =$$

$$= e^{\alpha_1 - 1} \cdot \left(-\frac{1}{\alpha_2}\right) = 1 \Rightarrow \boxed{\alpha_2 = -e^{\alpha_1 - 1}}$$

$$\boxed{p_0(x) = -\alpha_2\, e^{\alpha_2 x}}$$

$$\int \underset{f}{x}\, \underset{g'}{e^{ax}} = \frac{x\,e^{ax}}{a} - \int \frac{e^{ax}}{a} = x\frac{e^{ax}}{a} - \frac{e^{ax}}{a^2} = \frac{e^{ax}}{a}(x - 1/a)$$

$$\int_0^\infty x\, p(x)\,dx = A \Rightarrow -\int_0^\infty x\,\alpha_2\, e^{\alpha_2 x}\,dx =$$

$$= -\cancel{\alpha_2}\left[\frac{e^{\alpha_2 x}}{\cancel{\alpha_2}}(x - 1/\alpha_2)\right]_0^\infty = -\frac{1}{\alpha_2} = A \Rightarrow \alpha_2 = -1/A$$

e

$$\boxed{p_0(x) = \frac{1}{A} e^{-x/A}} \quad 0 \leq x \leq \infty$$

ou

$$\boxed{p_0(x) = \frac{1}{A} e^{-x/A} \, u(x)}$$

$$H(x) = -\int_0^\infty \frac{1}{A} e^{-x/A} \log \left[\frac{1}{A} e^{-x/A} \right] dx =$$

$$= -\frac{1}{A} \int_0^\infty \left[e^{-x/A} \left(\left[\log e^{-x/A} \right] - \log A \right) \right] dx$$

$$= -\int_0^\infty p(x) \cdot \left[\left(-\frac{x}{A} \right) \log e - \log A \right] dx$$

$$= \log A \underbrace{\int_0^\infty p(x)\, dx}_{1} + \frac{\log e}{A} \underbrace{\int_0^\infty x \, p(x)\, dx}_{A} =$$

$$= \log A + \log e = \log Ae$$