Aula 10

UV

14.1-1

$P_1 = 0.4$, $P_2 = 0.3$, $P_3 = 0.2$ e $P_4 = 0.1$

(a) $H(m) = -(P_1 \log_2 P_1 + P_2 \log_2 P_2 + P_3 \log_2 P_3 + P_4 \log_2 P_4) = 1,8464$ bits

(b) São 10^6 símbolos/s. Assim, a taxa de informação é de $1,8464 \cdot 10^6$ bits/s

14.1-2 (TV preto e branco)

Informação/elemento $= \log_2 10 = 3,322$ bits

Informação/quadro $= 3,32 \times 300\,000 = 9,966 \times 10^5$ bits

14.2-~~2~~1 → Entropia

$$H(m) = -\sum_{i=1}^{7} P_i \log_2 P_i = +\left(\frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \right.$$

$$+ \frac{1}{8}\log 8 + \frac{1}{16}\log 16 + \frac{1}{32}\log 32 + \frac{1}{64}\log 64 + \frac{1}{64}\log$$

$$\ldots m) = \frac{1}{2} + \frac{1}{2} + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \frac{1}{32}\cdot 5 + \frac{1}{64}\cdot 6 + \frac{1}{64}\cdot 6 =$$

$$1 + \frac{3}{8} + \frac{1}{4} + \frac{5}{32} + \frac{3}{32} + \frac{3}{32} = \frac{32+12+8+5+3+3}{32} = \frac{63}{32} \text{ bits}$$

Código

Mensagem	Probabilidade	Código	S1		S2		S3		S4		S5	
m1	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0	1/2	0
m2	1/4	10	1/4	10	1/4	10	1/4	10	1/4	10	1/2	1
m3	1/8	110	1/8	110	1/8	110	1/8	110	1/4	11		
m4	1/16	1110	1/16	1110	1/16	1110	1/8	111				
m5	1/32	11110	1/32	11110	1/16	1111						
m6	1/64	111110	1/32	11111								
m7	1/64	111111										

Comprimento médio.

$$L = \sum_i P_i L = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \frac{1}{8}\cdot 3 + \frac{1}{16}\cdot 4 + \frac{1}{32}\cdot 5 +$$

$$+ \frac{1}{64}\cdot 6 + \frac{1}{64}\cdot 6 = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{5}{32} + \frac{3}{32} + \frac{3}{32}$$

$$= \frac{63}{32} \text{ dígitos binários! } (= H(m))$$

Eficiência $\eta = \frac{H(m)}{L} \times 100 = \underline{\underline{100\%}}$

Redundância $\gamma = (100 - \eta) = \underline{\underline{0\%}}$

14.2-2

→ entropia

$$H(m) = -\sum_{i=1}^{7} P_i \log P_i =$$

$$= \frac{1}{3} \cdot \log 3 + \frac{1}{3} \log 3 + \frac{1}{9} \log 9 + \frac{1}{9} \log 9 + \frac{1}{27} \log 27 + \frac{1}{27} \log 27$$

$$+ \frac{1}{27} \log 27 = \log 3 \left[\frac{1}{3} + \frac{1}{3} + \frac{2}{9} + \frac{2}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} \right] =$$

$$= \log 3 \cdot \left[\frac{3+3+7}{9} \right] = \log_2 3 \cdot \frac{13}{9} = \underline{2{,}289 \text{ bits}}$$

$$= \frac{13}{9} \text{ (dígitos ternários)}$$

Código

Mensagem	Probabilidade	
m_1	1/3	0
m_2	1/3	1
m_3	1/9	20
m_4	1/9	21
m_5	1/27	220
m_6	1/27	221
m_7	1/27	222

1/3	0	1/3	0
1/3	1	1/3	1
1/9	20	1/3	2
1/9	21		
1/9	22		

→ Comprimento médio

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$$L = \sum_{i=1}^{7} P_i L_i = \frac{1}{3}.1 + \frac{1}{3}.1 + \frac{1}{9}.2 + \frac{1}{9}.2 + \frac{1}{27}.3 + \frac{1}{27}.3 +$$

$$+\frac{1}{27}.3 = \frac{18+12+9}{27} =$$

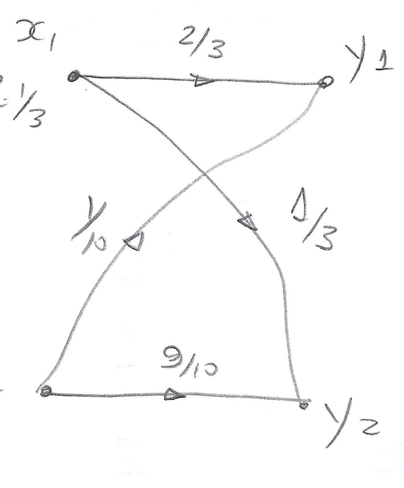
$$= \frac{39}{27} \text{ (dígitos ternários)}$$

$$= \frac{13}{9}$$

Eficiência $\eta = \frac{H(m)}{L} = 1 = 100\%$

Redundância $\gamma = (1-\eta) = 0$

14.4.1



$P_X(x_2) = 2/3$

$$P(y_1) = P(x_1)\,P(y_1|x_1) + P(x_2).P(y_1|x_1)$$

$$= 1/3 \cdot 2/3 + 2/3 \cdot 1/10 = \frac{2}{9} + \frac{1}{15} =$$

$$= \frac{20+6}{90} = \frac{26}{90}$$

$$= 13/45$$

$$P(y_2) = 1 - P(y_1) = 32/45$$

$$H(x) = P(x_1).\log\frac{1}{P(x_1)} + P(x_2).\log\frac{1}{P(x_2)} = \frac{1}{3}.\log_2 3 + \frac{2}{3}\log\frac{3}{2}$$

$\rightarrow$ $\boxed{H(x) = 0{,}9182 \text{ bits}}$

$\underline{H(x|y)}$ $\quad P(x_1|y_1) = \dfrac{P(y_1|x_1) \cdot P(x_1)}{P(y_1)} = \dfrac{2/3 \cdot 1/3}{13/45} =$

$$= \frac{2/9}{13/45} = \frac{2}{\cancel{9}} \cdot \frac{\overset{5}{\cancel{45}}}{13} = \frac{10}{13}$$

$$P(x_2|y_1) = \frac{P(y_1|x_2) \cdot P(x_2)}{P(y_1)} = \frac{\frac{1}{10} \cdot \frac{2}{3}}{\frac{13}{45}} = \frac{1}{\cancel{15}} \cdot \frac{\cancel{45}}{13} = 3/13$$

$$P(x_1|y_2) = \frac{P(y_2|x_1) \cdot P(x_1)}{P(y_2)} = \frac{1/3 \cdot 1/3}{32/45} = \frac{1}{\cancel{9}} \cdot \frac{45}{32} = 5/32$$

$$P(x_2|y_2) = \frac{P(y_2|x_2) \cdot P(x_2)}{P(y_2)} = \frac{\cancel{9}/\cancel{10} \cdot 2/\cancel{3}}{32/45} = \frac{3}{\cancel{5}} \cdot \frac{\overset{9}{\cancel{45}}}{32} = \frac{27}{32}$$

$$H(x|y_1) = P(x_1|y_1) \cdot \log \frac{1}{P(x_1|y_1)} + P(x_2|y_1) \cdot \log \frac{1}{P(x_2|y_1)}$$

$$= \frac{10}{13} \cdot \log_2 \frac{13}{10} + \frac{3}{13} \log_2 \frac{13}{3} = 0{,}7793$$

$$H(x|y_2) = P(x_1|y_2) \cdot \log \frac{1}{P(x_1|y_2)} + P(x_2|y_2) \log \frac{1}{P(x_2|y_2)}$$

$$= \frac{5}{32} \log_2 \frac{32}{5} + \frac{27}{32} \log_2 \frac{32}{27} = 0{,}6253$$

$$H(x|y) = P(y_1)\, H(x|y_1) + P(y_2)\, H(x|y_2)$$

$$= \frac{13}{45} \cdot 0{,}7793 + \frac{32}{45} \cdot 0{,}6253$$

$$= 0{,}6698 \qquad \boxed{H(x|y) = 0{,}6698 \text{ bits/símbolo}}$$

$$H(y) = \sum_i P(y_i) \log_2 \frac{1}{P(y_i)} = \frac{13}{45} \log_2 \frac{45}{13} + \frac{32}{45} \log_2 \frac{45}{32}$$

$$= 0{,}8673$$

$$\boxed{H(y) = 0{,}8673}$$

$$I(x|y) = H(x) - H(x|y) = 0{,}9182 - 0{,}6698 = 0{,}2484$$

$$H(y|x) = H(y) - I(x|y) = 0{,}8673 - 0{,}2484 =$$

$$= 0{,}6189 \text{ bits/símbolo}$$

1.4.4.2

A

a) A matriz do canal $P(y_j|x_i)$ é

$$\begin{array}{cc} & x_i \\ y_j & \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 1-p & p \end{bmatrix} \end{array}$$

e $P(y_1)=P$ e $P(y_2)=P(y_3)=\alpha$ $\boxed{(p\cdot \alpha+(1-p)\cdot\alpha)}$

Usamos então que $P(x_i|y_j) = \dfrac{P(y_j|x_i)\,P(x_i)}{P(y_j)}$

e obtemos a matriz

$$P(x_i|y_j) \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 1-p & p \end{bmatrix} \text{ e}$$

$$H(X) = \sum P(x_i)\log\frac{1}{P(x_i)} = -P\log P - 2\alpha\log\alpha$$

$$= -\left[P\log P + (1-P)\log\left(\frac{1-P}{2}\right)\right] = \Omega(P) + (1-P)$$

$$(\log 2 = 1)$$

(B)

$$H(x|y) = \sum_i \sum_j P(y_j) P(x_i|y_j) \log \frac{1}{P(x_i|y_j)}$$

$$= P \log 1 + q\left[p \log \frac{1}{p} + (1-p) \log \frac{1}{(1-p)}\right] +$$

$$+ q\left[(1-p) \log \frac{1}{1-p} + p \log \frac{1}{p}\right]$$

$$= 0 + 2q\,\Omega(p) = (1-p)\,\Omega(p)$$

$$\boxed{H(y) = H(x) = \Omega(p) + (1-p)}$$

$$I(x|y) = H(x) - H(x|y) = \Omega(P) + (1-P) - (1-P)\,\Omega(p)$$

$$= \Omega(P) + (1-P)\left[1 - \Omega(p)\right]$$

(b) Fazendo $\beta = 2^{\Omega(p)}$ ou $\Omega(p) = \log \beta$,

$$I(x|y) = \Omega(P) + (1-P)(1 - \log \beta)$$

C

$$\frac{d}{dP} I(x|y) = 0 \Rightarrow \frac{d}{dP}\left[\Omega(P) + (1-P)(1-\log \beta)\right] = 0$$

$$\to \frac{d}{dP}\left[P \log P + (1-P) - (1-P)\log(1-P)(1-\log\beta)\right] = 0$$

$$\Rightarrow \log P - \log(1-P) + [1 - \log\beta] = 0$$

Assim, $\log \frac{P}{1-P} = -1 + \log \beta$

Mas $-1 + \log\beta = -\log 2 + \log \beta = \log \beta/2$ e

$$\frac{P}{1-P} = \beta/2 \to P = \frac{\beta}{\beta+2} \text{ e } 1-P = \frac{2}{\beta+2}$$

$$C = MAX\ I(x|y) = \frac{\beta}{\beta+2}\log\frac{\beta+2}{\beta} + \frac{2}{\beta+2}\log\frac{\beta+2}{2} +$$

$$+ \frac{2}{\beta+2}\underbrace{[1-\log\beta]}_{\log 2/\beta} = \log\frac{\beta+2}{\beta}$$

$$\boxed{C = \log\frac{\beta+2}{\beta}}$$