Chapter 1

Electromagnetic Torque

1.1 Magnetic Flux Density Approach

Rotor surface magnetic flux density ($\vec{B_r}$) is written in (1.1), considering rotor references as given in Fig. 1.1 and considering the following conditions:

- for $-l/2 \leq z \leq -l/2$ and $r = r_r$, B_r is written as in 1.1;
- for $z < -l/2$ or $z > l/2$, $B_r = 0$.

$$\vec{B_r}(\theta, r, z) = B_r(\theta)\vec{a_r} \tag{1.1}$$

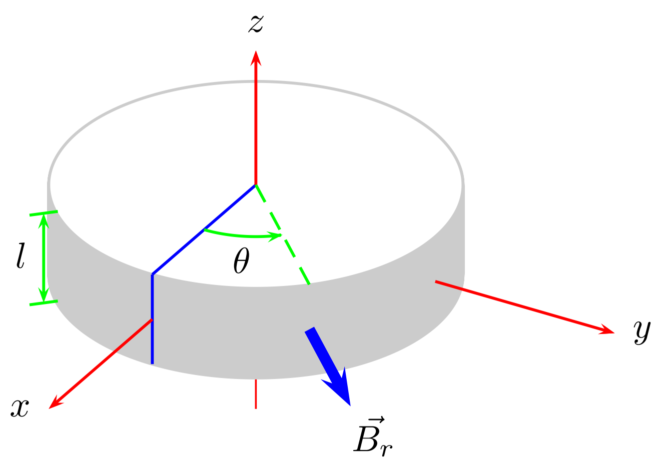

Figure 1.1: Rotor references.

Considering the rotor of Fig. 1.1 in the situation of Fig. 1.2. Now, the flux density at each conductor is given by (1.2). So, $\theta_1 = \alpha + \theta_s - \theta_r$ and $\theta_2 = -\alpha + \theta_s - \theta_r$.

$$\vec{B}(\theta) = B(\theta)\vec{a_r} \tag{1.2}$$

The force in a set of conductors, perpendicular to the plane of rotor, is given by (1.3), which consider the effective current is those conductors (i_{ST}), not the total. This effective current is given by (1.4).

$$\vec{F} = \vec{i_{ST}} \times \vec{B} \cdot l \tag{1.3}$$

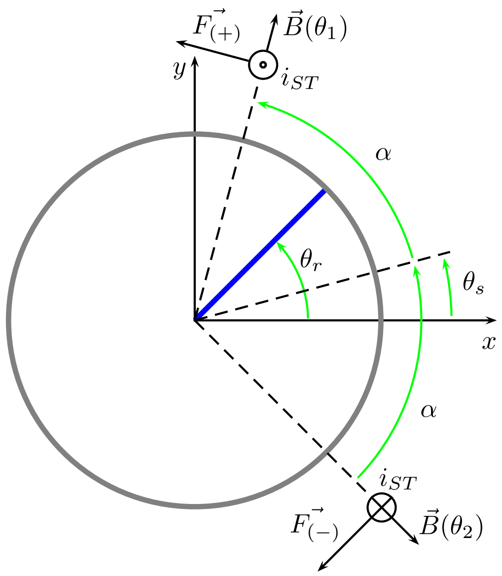

Figure 1.2: Stator references for a given rotor position (θ_r).

$$i_{ST} = N_s k_s i_s \tag{1.4}$$

$$B(\theta) = \sum_{n=-\infty}^{\infty} \bar{b}_n e^{jn\theta} \tag{1.5}$$

$$\begin{aligned} \vec{F_{(+)}} &= N_s k_s i_s l B(\alpha + \theta_s - \theta_r)\,(\vec{a_z} \times \vec{a_r}) \\ \vec{F_{(-)}} &= N_s k_s i_s l B(-\alpha + \theta_s - \theta_r)\,(-\vec{a_z} \times \vec{a_r}) \end{aligned} \tag{1.6}$$

Rotor electromagnetic torque (1.7) is a reaction to forces given by (1.6).

$$\vec{T}_{1s}(\theta_r) = -\left((r\vec{a_r}) \times \vec{F_{(+)}} + (r\vec{a_r}) \times \vec{F_{(-)}}\right) \tag{1.7}$$

$$\vec{T}_{1s}(\theta_r) = -N_s k_s i_s r l\,(B(\alpha + \theta_s - \theta_r) - B(-\alpha + \theta_s - \theta_r))\,\vec{a_z} \tag{1.8}$$

$$B(\alpha + \theta_s - \theta_r) - B(-\alpha + \theta_s - \theta_r) = \sum_{n=-\infty}^{\infty} \left(\bar{b}_n e^{jn(\alpha+\theta_s-\theta_r)} - \bar{b}_n e^{jn(-\alpha+\theta_s-\theta_r)}\right) \tag{1.9}$$

$$\vec{T}_{1s}(\theta_r) = -2jN_s k_s i_s r l \sum_{n=-\infty}^{\infty} \bar{b}_n \sin n\alpha\, e^{jn(\theta_s-\theta_r)} \tag{1.10}$$

The magnetic flux in a spire is Φ_s.

$$\Phi_s(\theta_r) = N_s k_s \int_{-l}^{l} \int_{-\alpha+\theta_s}^{\alpha+\theta_s} B(\theta - \theta_r)\, r\, d\theta\, dz = N_s k_s r l \int_{-\alpha+\theta_s}^{\alpha+\theta_s} \left(\sum_{n=-\infty}^{\infty} \bar{b}_n e^{jn(\theta-\theta_r)}\right) d\theta \tag{1.11}$$

$$\int_{-\alpha+\theta_s}^{\alpha+\theta_s} \sum_{n=-\infty}^{\infty} \bar{b}_n e^{jn(\theta-\theta_r)} d\theta = \sum_{n=-\infty}^{\infty} \bar{b}_n \int_{-\alpha+\theta_s}^{\alpha+\theta_s} e^{jn(\theta-\theta_r)} d\theta \tag{1.12}$$

$$\int_{-\alpha+\theta_s}^{\alpha+\theta_s} e^{jn(\theta-\theta_r)}d\theta = \left.\frac{e^{jn(\theta-\theta_r)}}{jn}\right|_{-\alpha+\theta_s}^{\alpha+\theta_s} = \frac{1}{jn}\left[e^{jn(\alpha+\theta_s-\theta_r)} - e^{jn(-\alpha+\theta_s-\theta_r)}\right] =$$
$$= \frac{1}{jn}e^{jn(\theta_s-\theta_r)}\left(e^{jn\alpha} - e^{-jn\alpha}\right) = \frac{1}{jn}e^{jn(\theta_s-\theta_r)}2j\sin n\alpha = \frac{2\sin n\alpha}{n}e^{jn(\theta_s-\theta_r)} \tag{1.13}$$

$$\Phi_s(\theta_r) = N_s k_s r l \sum_{n=-\infty}^{\infty} \frac{\bar{b}_n 2\sin n\alpha}{n} e^{jn(\theta_s-\theta_r)} \tag{1.14}$$

$$\bar{\Phi}_{sn} = \frac{N_s k_s r l \bar{b}_n 2\sin n\alpha}{n} e^{jn(\theta_s-\theta_r)} \tag{1.15}$$

$$\bar{b}_n = \frac{n\bar{\Phi}_{sn}}{2N_s k_s r l \sin n\alpha} \tag{1.16}$$

$$\vec{T}_{1s}(\theta_r) = -2N_s k_s i_s r l \left(\sum_{n=-\infty}^{\infty} j\frac{n\bar{\Phi}_{sn}\sin n\alpha}{2N_s k_s r l \sin n\alpha} e^{jn(\theta_s-\theta_r)}\right)\vec{a_z} \tag{1.17}$$

$$\vec{T}_{1s}(\theta_r) = -i_s \left(\sum_{n=-\infty}^{\infty} jn\bar{\Phi}_{sn} e^{jn(\theta_s-\theta_r)}\right)\vec{a_z} \tag{1.18}$$

As $B(\theta)$ is Real and Even[1] (series $\bar{b}_n$ is purely real and $\bar{b}_n = \bar{b}_{(-n)}$), simplifications shown in (1.19) are valid, because of (1.15).

$$\sum_{n=-\infty}^{\infty} jn\bar{\Phi}_{sn}e^{jn(\theta_s-\theta_r)} = \sum_{n=1}^{\infty} jn\bar{\Phi}_{sn}e^{jn(\theta_s-\theta_r)} + 0 + \sum_{n=1}^{\infty} j(-n)\bar{\Phi}_{s(-n)}e^{-jn(\theta_s-\theta_r)} =$$
$$= \sum_{n=1}^{\infty} jn\bar{\Phi}_{sn}\left(e^{jn(\theta_s-\theta_r)} - e^{jn(\theta_s-\theta_r)}\right) = -\sum_{n=1}^{\infty} n\bar{\Phi}_{sn}2\sin n(\theta_s-\theta_r) \tag{1.19}$$

Simplifications shown in (1.19) leads to final electromagnetic torque equation due to one stator coil in (1.20).

$$\vec{T}_{1s} = i_s \sum_{n=1}^{\infty} 2\bar{\Phi}_{sn} n \sin n(\theta_s-\theta_r)\vec{a_z} \tag{1.20}$$

Equation (1.20) implies that for a coil at $\theta_s = 0$ and for rotor in its origin, rotor magnetic and stator magnetic fluxes are aligned, rotor is in its stable position.

Another simplification is made in (1.21) by the following: if $f(t)$ is Real and $f(t) \Leftrightarrow F(n)$, then $F^*(n) = F(-n)$.

$$F(n) = jn\bar{\Phi}_{sn}e^{jn(\theta_s-\theta_r)}$$

$$F^*(n) = -jn\bar{\Phi}^*_{sn}e^{-jn(\theta_s-\theta_r)}$$

$$F^*(n) = F(-n) \tag{1.21}$$

$$\sum_{n=-\infty}^{\infty} F(n) = \sum_{n=1}^{\infty} F(n) + 0 + \sum_{n=1}^{\infty} F(-n) = \sum_{n=1}^{\infty} F(n) + \sum_{n=1}^{\infty} F^*(n)$$

[1] I do not like this hypothesis. If $B(\theta)$ is not Even?

$$\sum_{n=-\infty}^{\infty} jn\bar{\Phi}_{sn}e^{jn(\theta_s-\theta_r)} = \sum_{n=-\infty}^{\infty} jn\left(\bar{\Phi}_{sn}e^{jn(\theta_s-\theta_r)} - \bar{\Phi}^*_{sn}e^{-jn(\theta_s-\theta_r)}\right) \tag{1.22}$$

$$\bar{\Phi}_{sn} = \Re(\bar{\Phi}_{sn}) + j\Im(\bar{\Phi}_{sn}) = a + jb \quad \text{and} \quad \theta = \theta_s - \theta_r$$

$$\bar{\Phi}_{sn}e^{jn(\theta_s-\theta_r)} - \bar{\Phi}^*_{sn}e^{-jn(\theta_s-\theta_r)} = (a+jb)e^{jn\theta} - (a-jb)e^{-jn\theta} = \tag{1.23}$$

$$= a\left(e^{jn\theta} - e^{-jn\theta}\right) + jb\left(e^{jn\theta} + e^{-jn\theta}\right) = 2ja\sin n\theta + 2jb\cos n\theta$$

With simplifications shown in (1.23), $B(\theta)$ can be even, odd, or any. Using (1.23), it leads to (1.24), (1.25) and (1.26).

$$\vec{T}_{1s} = i_s\sum_{n=1}^{\infty} 2n\left(\Re(\bar{\Phi}_{sn})\sin n(\theta_s-\theta_r) + \Im(\bar{\Phi}_{sn})\cos n(\theta_s-\theta_r)\right)\vec{a_z} \tag{1.24}$$

$$\vec{T}_{1s} = i_s\sum_{n=1}^{\infty} 2n\left(\frac{(\bar{\Phi}_{sn}+\bar{\Phi}^*_{sn})}{2}\sin n(\theta_s-\theta_r) + \frac{(\bar{\Phi}_{sn}-\bar{\Phi}^*_{sn})}{2j}\cos n(\theta_s-\theta_r)\right)\vec{a_z} \tag{1.25}$$

$$\vec{T}_{1s} = i_s\sum_{n=1}^{\infty} 2n\Im\left(\bar{\Phi}_{sn}e^{jn(\theta_s-\theta_r)}\right)\vec{a_z} \tag{1.26}$$

Comparing (1.18) to (1.26)...

1.2 Back Electromotive Force Approach

$$\Phi_s'(\theta_r) = \frac{d\Phi_s(\theta_r)}{d\theta_r} \Rightarrow \bar{\Phi}'_{sn} = -jn\bar{\Phi}_{sn} \tag{1.27}$$

From (1.14) and (1.27):

$$\Phi_s'(\theta_r) = -2N_sk_srl\sum_{n=-\infty}^{\infty}\bar{b}_n\sin n\alpha e^{jn(\theta_s-\theta_r)} \tag{1.28}$$

$$T_{1s} = i_s\Phi_s'(\theta_r) \tag{1.29}$$

$$T_{1s} = -2ji_sN_sk_srl\sum_{n=-\infty}^{\infty}\bar{b}_n\sin n\alpha\, e^{jn(\theta_s-\theta_r)} \tag{1.30}$$

1.3 Three Phases

Three phases: a at position $\theta_s = 0$ rd, b at $\theta_s = 2\pi/3$ rd and c at $\theta_s = -2\pi/3$ rd.

$$T_{el} = i_a\sum_{n=-\infty}^{\infty} -jn\bar{\Phi}_{sn}e^{-jn\theta_r} + i_b\sum_{n=-\infty}^{\infty} -jn\bar{\Phi}_{sn}e^{jn\left(\frac{2\pi}{3}-\theta_r\right)} + i_c\sum_{n=-\infty}^{\infty} -jn\bar{\Phi}_{sn}e^{jn\left(-\frac{2\pi}{3}-\theta_r\right)} \tag{1.31}$$

$$T_{el} = i_a\sum_{n=-\infty}^{\infty} -jn\bar{\Phi}_{sn}e^{-jn\theta_r} + i_b\sum_{n=-\infty}^{\infty} -jn\bar{\Phi}_{sn}e^{-jn\theta_r}e^{jn\frac{2\pi}{3}} + i_c\sum_{n=-\infty}^{\infty} -jn\bar{\Phi}_{sn}e^{-jn\theta_r}e^{-jn\frac{2\pi}{3}} \tag{1.32}$$

$$T_{el} = -\sum_{n=-\infty}^{\infty} \left(i_a jn\bar{\Phi}_{sn} e^{-jn\theta_r} + i_b jn\bar{\Phi}_{sn} e^{-jn\theta_r} e^{jn\frac{2\pi}{3}} + i_c jn\bar{\Phi}_{sn} e^{-jn\theta_r} e^{-jn\frac{2\pi}{3}} \right) \tag{1.33}$$

$$T_{el} = -\sum_{n=-\infty}^{\infty} jn\bar{\Phi}_{sn} e^{-jn\theta_r} \left(i_a + i_b e^{jn\frac{2\pi}{3}} + i_c e^{-jn\frac{2\pi}{3}} \right) \tag{1.34}$$

Defining $\bar{i}_{\alpha\beta n}$ as in (1.35), it is possible to write electromagnetic torque as in (1.37).

$$\bar{i}_{\alpha\beta n} = i_a + i_b e^{jn\frac{2\pi}{3}} + i_c e^{-jn\frac{2\pi}{3}} \tag{1.35}$$

$$T_{el} = -\sum_{n=-\infty}^{\infty} jn\,\bar{i}_{\alpha\beta n}\bar{\Phi}_{sn} e^{-jn\theta_r} \tag{1.36}$$

Or:

$$T_{el} = \sum_{n=-\infty}^{\infty} \bar{i}_{\alpha\beta n}\bar{\Phi}'_{sn} e^{-jn\theta_r} \tag{1.37}$$

For $\bar{i}_{\alpha\beta n}$:

$$\bar{i}_{\alpha\beta n} = \begin{cases} 0 & \text{for} \quad n = 3k \\ i_a + i_b e^{j\frac{2\pi}{3}} + i_c e^{-j\frac{2\pi}{3}} & \text{for} \quad n = 3k+1 \\ i_a + i_b e^{-j\frac{2\pi}{3}} + i_c e^{j\frac{2\pi}{3}} & \text{for} \quad n = 3k-1 \end{cases} \tag{1.38}$$

Appendix A

Fourier Series Properties

A.1 Real series

For a real function $f(t)$, periodic, satisfying (A.1) as T as smaller and positive as possible, it is possible to write (A.2) and (A.3), where a_0, a_n, b_n, c_n and θ_n are shown in

$$f(t) = f(t+T) \tag{A.1}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t \ \ , \ \omega = \frac{2\pi}{T} \tag{A.2}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \theta_n) \tag{A.3}$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \tag{A.4}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt \tag{A.5}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t dt \tag{A.6}$$

$$c_n^2 = a_n^2 + b_n^2 \tag{A.7}$$

$$\theta_n = \arctan\frac{b_n}{a_n} \tag{A.8}$$

Appendix B

Complex Series

Orthogonality:

$$\int_0^T e^{jk\omega t} e^{-jl\omega t}\, dt = \begin{cases} T & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases} \tag{B.1}$$

If $f(t) \Leftrightarrow \{F_n\}$ then $\mathcal{F}\{F(t)\} = F_n$ and:

$$F_n = \frac{1}{T} \int_0^T f(t)\, e^{-jn\omega t}\, dt \tag{B.2}$$

$$\mathcal{F}\{\text{sq}(t)\} = \frac{2}{j\pi n} e^{jn\omega t} \tag{B.3}$$

$$\mathcal{F}\{\sin \omega t\} = \frac{1}{2j} e^{j\omega t} - \frac{1}{2j} e^{-j\omega t} \tag{B.4}$$

Properties:

- if $f(t)$ is purely real, then: $F_{-n} = F_n^*$;
- if $f(-t) = f(t)$ (**even** function) and purely real, then: $F_{-n} = F_n$;
- if $f(-t) = -f(t)$ (**odd** function) and purely real, then: $F_{-n} = -F_n$;

B.1 Realationship Between Real and Complex Coefficients

For a purely real function $f(t)$:

$$a_0 = F_0 \tag{B.5}$$

$$a_n = F_n + F_{-n} \quad \text{for} \quad n \neq 0 \tag{B.6}$$

$$b_n = j\,(F_n - F_{-n}) \quad \text{for} \quad n \neq 0 \tag{B.7}$$

$$F_n = \frac{1}{2}\,(a_n - jb_n) \quad \text{for} \quad n > 1 \tag{B.8}$$

$$F_{-n} = \frac{1}{2}\,(a_n + jb_n) \quad \text{for} \quad n > 1 \tag{B.9}$$

And:

$$F_n = F^*_{-n} \tag{B.10}$$

Appendix C

Parseval's Theorem

$$\frac{1}{T}\int_0^T f^2(t)\,dt = \sum_{n=-\infty}^{\infty} |F_n|^2 \tag{C.1}$$