AULA 28: A FUNÇÃO DE GREEN DA ELETRODINÂMICA, E OS "POTENCIAIS RETARDADOS"

Como vimos na aula passada, no calibre de Lorentz, os potenciais obedecem:

$$\boxed{\frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{\rho}{\epsilon}}$$ Vamos resolver esta equação

$$\frac{1}{c_0^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \mu\vec{J}$$

Para resolver a equação p/ φ, vamos recorrer à transformada de Fourier:

$$f(x) \longrightarrow \tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{ikx} f(x)$$

$$\updownarrow$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ikx}\, \tilde{f}(k)$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{-ik(x-x')} = \delta(x-x')$$

3D: $f(\vec{x}) \rightarrow \tilde{f}(\vec{k}) = \int d^3x\, e^{i\vec{k}\cdot\vec{x}} f(\vec{x})$

$$f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{x}}\, \tilde{f}(\vec{k})$$

Porém, queremos estudar situações nas quais <u>tudo</u> pode depender do <u>tempo</u> também!

$f \to f(t,\vec{x}) \longrightarrow$ Transf. de Fourier p/ t

$$\vec{x} \longleftrightarrow \vec{k}$$

$$t \longleftrightarrow \omega$$

$$\Rightarrow f(t,\vec{x}) \to \tilde{f}(\omega,\vec{k}) = \int dt\, e^{i\omega t} \int d^3x\, e^{i\vec{k}\cdot\vec{x}}\, f(t,\vec{x})$$

$$\updownarrow$$

$$f(t,\vec{x}) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{x}}\, \tilde{f}(\omega,\vec{k})$$

<u>com</u>: $$\int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}(\vec{x}-\vec{x}')} = \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\int \frac{d\omega}{2\pi}\, e^{-i\omega(t-t')} = \delta(t-t')$$

Vamos agora transformar a Eq. p/ φ:

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{\rho}{\epsilon_0} \leftarrow \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{x}}\, \tilde{\rho}(\omega,\vec{k})$$

$$\varphi(t,\vec{x}) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{k}\cdot\vec{x}}\, \tilde{\varphi}(\omega,\vec{k})$$

Mas $\frac{\partial^2}{\partial t^2}\left(e^{-i\omega t}\right) = -\omega^2 e^{-i\omega t}$, $\nabla^2\left(e^{-i\vec{k}\cdot\vec{x}}\right) = -\vec{k}^2 e^{-i\vec{k}\cdot\vec{x}}$, e

Essa equação fica então:

$$\int \frac{d\omega}{(2\pi)} e^{-i\omega t} \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \left[\left(-\frac{\omega^2}{c^2} + \vec{k}^2 \right) \tilde{\varphi}(\omega,\vec{k}) = \frac{\tilde{\rho}(\omega,\vec{x})}{\epsilon_0} \right]$$

Ou seja,

$$\boxed{\tilde{\varphi}(\omega,\vec{k}) = \frac{1}{\epsilon_0} \frac{\tilde{\rho}(\omega,\vec{x})}{-\frac{\omega^2}{c^2} + \vec{k}^2}}$$

Bem, isso é "quase" uma solução — exceto que o que desejamos é $\varphi(t,\vec{x})$, e não $\tilde{\varphi}(\omega,\vec{k})$! Mas agora, basta transformar "de volta" para o espaço real:

$$\varphi(t,\vec{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \int \frac{d^3\vec{k}}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \frac{1}{\epsilon_0} \frac{\tilde{\rho}(\omega,\vec{k})}{-\frac{\omega^2}{c^2} + \vec{k}^2}$$

onde

$$\tilde{\rho}(\omega,\vec{k}) = \int dt' \int d^3x'\, e^{i\omega t'} e^{i\vec{k}\cdot\vec{x}'} \rho(t',\vec{x}')$$

Portanto,

$$\varphi(t,\vec{x}) = \frac{1}{\epsilon_0} \int dt' \int d^3x'\, \rho(t',\vec{x}') \times \underbrace{\int \frac{d\omega}{2\pi} \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{-i\omega(t-t')}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^2 - \omega^2/c^2}}_{G(t,\vec{x};t',\vec{x}')}$$

Vamos agora calcular $G(t,\vec{x};t',\vec{x}')$.
Definindo

$$\Delta t = t - t' \;;\quad \Delta\vec{x} = \vec{x} - \vec{x}'$$

$$\Rightarrow G = \frac{1}{(2\pi)^4}\int dw\, e^{-iw\Delta t}\int d^3k\, \frac{e^{-i\vec{k}\cdot\Delta\vec{x}}}{\underbrace{\vec{k}^2}_{\text{só o módulo}}-w^2/c^2}$$

Podemos escolher o sistema de referência $\vec{k}'$ de tal forma que $\hat{k}_z$ aponta na direção $\Delta\vec{x}$.

$$\Rightarrow \vec{k}\cdot\Delta\vec{x} = k\cdot\Delta x\cdot\cos\theta_k$$

$$\int d\varphi_k \rightarrow 2\pi$$

$$\Rightarrow G = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} dw\, e^{-iw\Delta t}\times 2\pi\int_0^{\infty} dk\cdot k^2\cdot\int_0^{\pi} d\theta\cdot \operatorname{sen}\theta\, \frac{e^{-ik\Delta x\cdot\cos\theta}}{k^2-w^2/c^2}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} dw\, e^{-iw\Delta t}\times\int_0^{\infty}\frac{dk\cdot k^2}{k^2-w^2/c^2}\times\underbrace{\int_{-1}^{1} d\mu\, e^{-ik\Delta x\cdot\mu}}$$

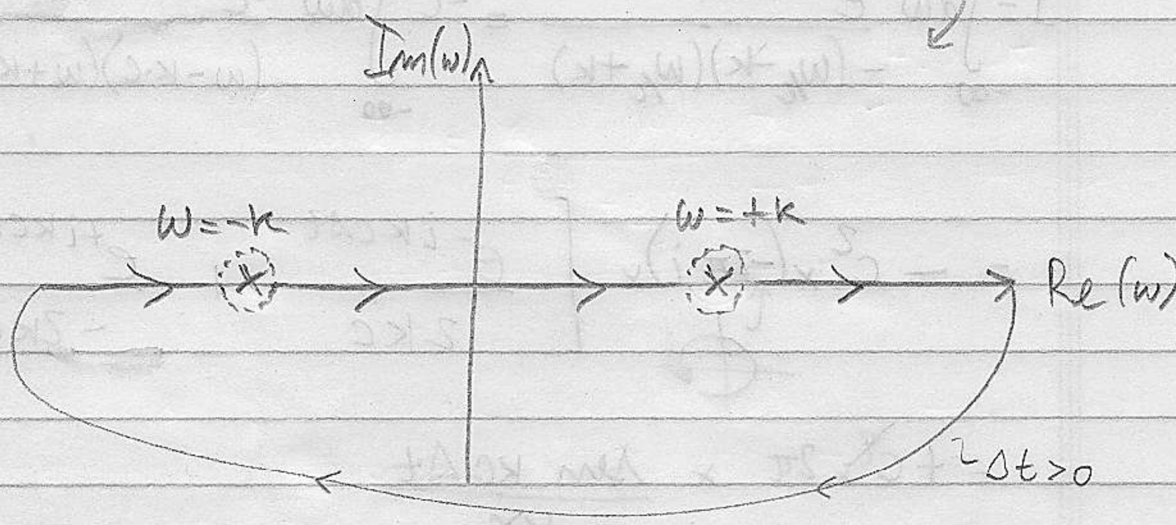
$$\frac{1}{-ik\Delta x}\, e^{-ik\Delta x\cdot\mu}\Big|_{-1}^{1} =$$

$$= \frac{e^{ik\Delta x}-e^{-ik\Delta x}}{ik\Delta x}$$

$$= 2\,\frac{\operatorname{sen} k\Delta x}{k\Delta x} \qquad \left[\,\dot{=}\,2 j_0(k\Delta x)\right]$$

Portanto,

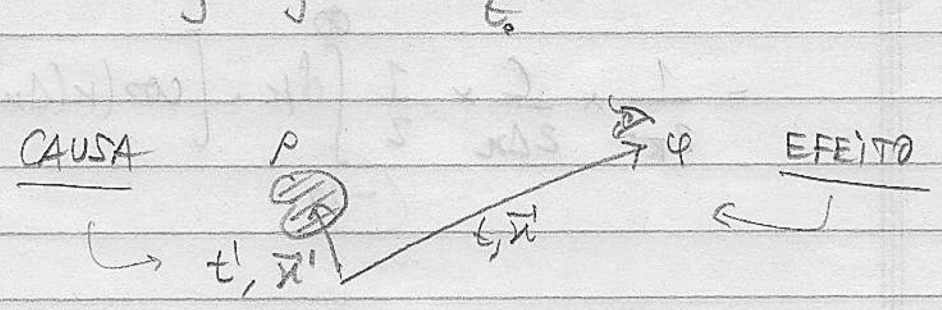
$$G = \frac{1}{(2\pi)^3} \times \int_0^{\infty} dk \times k^2 \times 2\,\frac{\operatorname{sen} k\Delta x}{k\Delta x} \underbrace{\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega\Delta t}}{k^2 - \omega^2/c^2}}_{I(k,\Delta t) = 2\pi i \sum \text{resíduos}}$$



O fator $e^{-i\omega\Delta t}$ é quem nos diz onde vamos "fechar" o circuito. Caso $t > t'$, $\Delta t > 0$, e daí $\mathrm{Im}(\omega) < 0 \Rightarrow$ fecha "por baixo". Caso $t < t'$, $\Delta t < 0$, e $\mathrm{Im}(\omega) > 0 \Rightarrow$ fecha "por cima".

Mas vejamos:

$$\varphi(t, \vec{x}) = \int dt' \int d^3x'\, \frac{\rho(t', \vec{x}')}{\epsilon_0} \times G(t, \vec{x}; t', \vec{x}')$$



A FÍSICA nos diz que a resposta (o efeito) deve vir depois da causa, $\Delta t > 0$.

Incluindo ambos os polos dentro do circuito, obtemos:

$$I = \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\Delta t}}{-(\omega/c - k)(\omega/c + k)} = -c^2 \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega\Delta t}}{(\omega - k\cdot c)(\omega + kc)}$$

$$= -c^2 \times (-2\pi i) \times \left[\frac{e^{-ikc\Delta t}}{2kc} + \frac{e^{+ikc\Delta t}}{-2kc} \right]$$

$$= +c^{\cancel{2}} \times 2\pi \times \frac{\text{sen}\, kc\Delta t}{k\cancel{c}}$$

Portanto,

$$G = \frac{4\pi c}{(2\pi)^3} \times \int_0^{\infty} dk \cdot k^2 \times \frac{\text{sen}\, k\Delta x}{k\Delta x} \, \frac{\text{sen}\, kc\Delta t}{\cancel{k}c}$$

$$= \frac{1}{2\pi^2} \frac{c}{\Delta x} \int_0^{\infty} dk \times \frac{\cos(k(\Delta x - c\Delta t)) - \cos(k(\Delta x + c\Delta t))}{2}$$

$$= \frac{1}{2\pi^2} \times \frac{c}{2\Delta x} \times \frac{1}{2} \int_{-\infty}^{\infty} dk \cdot \left[\cos(k(\Delta x - c\Delta t)) - \cos(k(\Delta x + c\Delta t)) \right]$$

Podemos expressar isso como:

$$G = \frac{c}{\Delta r} \times \frac{1}{8\pi^2} \times \mathrm{Re}\left[\int_{-\infty}^{\infty} dk \left(e^{ik(\Delta r - c\Delta t)} - e^{ik(\Delta r + c\Delta t)}\right)\right]$$

Agora, lembremos que $\int \frac{dk}{2\pi} e^{iky} = \delta(y) = \mathrm{Re}[\delta(y)]$

$$\Rightarrow G(t, \vec{r}; t', \vec{r}') = \frac{c}{\Delta r} \frac{1}{8\pi^2} \times \left[2\pi\,\delta(\Delta r - c\Delta t) - 2\pi\,\delta(\Delta r + c\Delta t)\right]$$

Mas $\Delta r = |\vec{r} - \vec{r}'| > 0$, e $\Delta t > 0$,
então $\delta(\Delta r + c\Delta t) = 0$ <u>sempre</u> $(\Delta r + c\Delta t \neq 0!)$

$$\Rightarrow \boxed{G(\Delta t, \Delta\vec{r}) = \frac{c}{\Delta r} \times \frac{1}{4\pi} \times \delta(\Delta r - c\Delta t)}$$

"FUNÇÃO DE GREEN RETARDADA" DO ELETROMAGNETISMO

$$\delta(\Delta r - c\Delta t) \Longrightarrow \Delta t = \frac{\Delta r}{c}$$

$$t - t' = \frac{\Delta r}{c}$$

$$t' = t - \frac{\Delta r}{c}$$

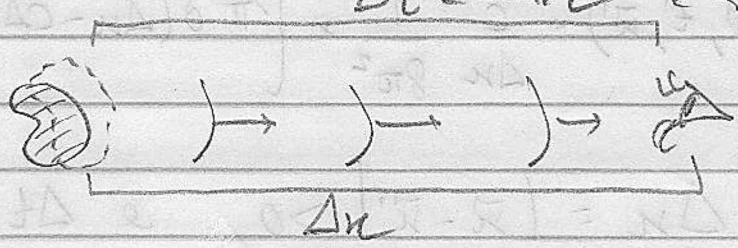
$$\delta(\Delta r - c\Delta t) = \frac{1}{c}\,\delta\left(\frac{\Delta r}{c} - t + t'\right)$$

Retornando p/ a expressão p/ φ:

$$\varphi(t,\vec{x}) = \int dt' \int d^3\vec{x}'\, \frac{\rho(t',\vec{x}')}{\epsilon} \times G$$

δ: $t' = t - \frac{\Delta x}{c}$

$$\boxed{\varphi(t,\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3\vec{x}'\, \frac{\rho(t' = t - \frac{\Delta x}{c}, \vec{x}')}{|\vec{x} - \vec{x}'|}}$$



"Retardo" do sinal p/ chegar em $\vec{x}$, no instante t.

Mas... e para $\vec{A}$?

$\Rightarrow$ "Mesmas equações, mesmas soluções!"

$$\Rightarrow \boxed{\vec{A}(t,\vec{x}) = \frac{\mu_0}{4\pi} \int d^3\vec{x}'\, \frac{\vec{J}(t' = t - \frac{\Delta x}{c}, \vec{x}')}{|\vec{x} - \vec{x}'|}}$$

<u>OBS</u>: Usamos o calibre de Lorentz, mas esse <u>resultado</u> (que sinais se propagam à velocidade c) é geral!