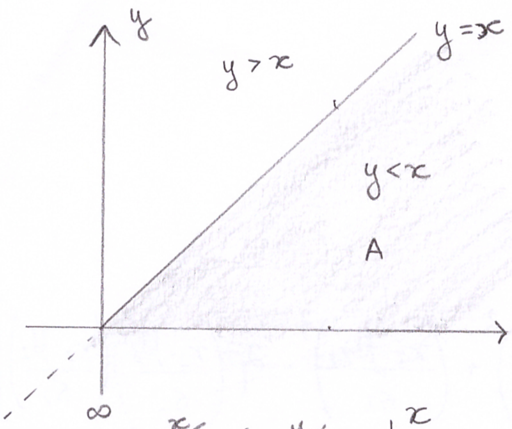
Resolução de exercícios sugeridos: Lista 4

5. $f_X(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & x \geq 0 \\ 0, & c.c. \end{cases}$ $\qquad f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2}, & y \geq 0 \\ 0, & c.c. \end{cases}$

a) $P[X > Y] = ?$



$P[X>Y] = P[(x,y) \in A] = \iint_A f_{XY}(x,y)\,dx\,dy$

Independência: $f_{XY}(x,y) = f_X(x).f_Y(y)$

Logo: $P[X>Y] = \int_0^\infty \int_0^x \frac{1}{3} e^{-x/3} . \frac{1}{2} e^{-y/2} dx\,dy =$

$= \int_0^\infty \frac{1}{3} e^{-x/3} (-e^{-y/2}) \Big|_0^x dx = \frac{1}{3} \int_0^\infty e^{-\frac{x}{3}} . (1 - e^{-\frac{x}{2}}) dx = \frac{1}{3} \int_0^\infty \left(e^{-\frac{x}{3}} - e^{-\frac{5x}{6}}\right) dx =$

$= (-e^{-\frac{x}{3}}) \Big|_0^\infty - \frac{1}{3} . \frac{\overset{2}{\cancel{6}}}{5} . (-e^{-\frac{5x}{6}}) \Big|_0^\infty = 1 - \frac{2}{5} = \frac{3}{5}$

b) $R_{XY} = E[XY] = E[X].E[Y] = 3.2 = 6$

c) $C_{XY} = E[XY] - E[X].E[Y] = 0$

6.

a) $Cov(X) = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}$ $\qquad \sigma_{XY} = E[XY] - \cancelto{0}{E[X]}.E[Y] = E[XY]$

b) $\begin{cases} Y_1 = X_1 - 2X_2 \\ Y_2 = 3X_1 + 4X_2 \end{cases}$

$E[Y_1] = E[X_1 - 2X_2] = E[X_1] - 2E[X_2] = 0$

$E[Y_2] = E[3X_1 + 4X_2] = 3E[X_1] + 4E[X_2] = 0$

$E[Y_1^2] = E[(X_1 - 2X_2)^2] = E[X_1^2 - 4X_1X_2 + 4X_2^2] = E[X_1^2] - 4E[X_1X_2] + 4.E[X_2^2] =$

$= 4 - 4.3 + 4.9 = 4 - 12 + 36 = 28$

$E[Y_2^2] = E[(3X_1 + 4X_2)^2] = (\ldots) = 252$

$E[Y_1Y_2] = E[(X_1 - 2X_2)(3X_1 + 4X_2)] = E[3X_1^2 - 8X_2^2 - 2X_1X_2] = 3.4 - 8.9 - 2.3 = -66$

8.

a) $(E[X])^2 = \mu_X^T \cdot \mu_X = \begin{bmatrix} 16 & 32 & 24 \\ 32 & 64 & 48 \\ 24 & 48 & 36 \end{bmatrix}$

$R_X = C_X + \mu_X^T \mu_X = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix}$

b) $f_{X_1X_2}(x_1, x_2) = \dfrac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \cdot \exp\left(-\dfrac{1}{2(1-\rho^2)}\left[\left(\dfrac{x_1 - \mu_{X_1}}{\sigma_1}\right)^2 + \left(\dfrac{x_2 - \mu_{X_2}}{\sigma_2}\right)^2 - 2\rho\left(\dfrac{x_1 - \mu_1}{\sigma_1}\right)\left(\dfrac{x_2 - \mu_2}{\sigma_2}\right)\right]\right)$

$\rho = \dfrac{\mathrm{Cov}(X_1, X_2)}{\sigma_1 \cdot \sigma_2} = \dfrac{-2}{4} = -\dfrac{1}{2}, \quad \rho^2 = \dfrac{1}{4}$

$\dfrac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} = \dfrac{1}{2\pi . 2 . 2 . \sqrt{1-\frac{1}{4}}} = \dfrac{1}{4\pi . \sqrt{3}}$

$\dfrac{1}{2(1-\rho^2)} = \dfrac{1}{2(1-\frac{1}{4})} = \dfrac{2}{3}$

$\therefore f_{X_1X_2}(x_1, x_2) = \dfrac{1}{4\pi\sqrt{3}} \cdot \exp\left(-\dfrac{2}{3}\left[\dfrac{(x_1-4)^2}{4} + \dfrac{(x_2-8)^2}{4} + \dfrac{(x_1-4)(x_2-8)}{4}\right]\right) =$

$= \dfrac{1}{4\pi\sqrt{3}} \cdot \exp\left(\dfrac{-(x_1-4)^2 - (x_2-8)^2 - (x_1-4)(x_2-8)}{6}\right)$

c) $P[X_1 > 8] = 1 - P[X_1 \leq 8] = 1 - F_{X_1}(8)$

10. $X \sim \text{Uniforme}(0,12)$ (ms)

a) $E[X] = 6$ ms

b) $Var[X] = 12\ \mu s$

c) $A = \sum_{i=1}^{12} X_i \Rightarrow$ tempo necessário para o processador obter toda informação do HD.

$$E[A] = E\left[\sum_{i=1}^{12} X_i\right] = \sum_{i=1}^{12} E[X_i] = 12 \cdot 6 = 72 \text{ ms}$$

d) $$Var[A] = Var\left[\sum_{i=1}^{12} X_i\right] = E\left[\left(\sum_{i=1}^{12} X_i\right)^2\right] - \left(E\left[\sum_{i=1}^{12} X_i\right]\right)^2$$

$$\left(\sum_{i=1}^{12} X_i\right)^2 = \sum_{i=1}^{12} X_i^2 + \sum_{i \neq j} X_i X_j \Rightarrow E\left[\left(\sum_{i=1}^{12} X_i\right)^2\right] = E\left[\sum_{i=1}^{12} X_i^2 + \sum_{i\neq j} X_i X_j\right] =$$

$$= \sum_{i=1}^{12} E[X_i^2] + \sum_{i \neq j} E[X_i X_j] = 12(12+6^2) + 12 \cdot 11 \cdot 6^2 = 12^2 + 12 \cdot 6^2 + 11 \cdot 12 \cdot 6^2 =$$
$$= 12^2 + 12^2 \cdot 6^2$$

logo, $Var[A] = 12^2 + 12^2 \cdot 6^2 - 12^2 \cdot 6^2 = 144\ \mu s^2$

e) Teorema do limite central: $S_n = \sum_{i=1}^{n} X_i$, $X_i \rightarrow$ VA independentes e identicamente distribuídas

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow[n \to \infty]{} Z \sim N(0,1)$$

$$A \approx Z \sim N(n \cdot \mu, n\sigma^2) = N(12 \cdot 6, 12^2) = N(72, 144)$$

$$P[A > 75\text{ms}] = 1 - P[A \leq 75\text{ms}] \approx 1 - F_Z(75\text{ms})$$

$$G = \frac{Z - 72}{12} \Rightarrow 12G + 72 = Z$$

$$P[A > 75\text{ms}] \approx P[Z > 75\text{ms}] = 1 - P[Z \leq 75\text{ms}] = 1 - P[12G + 72 \leq 75] \Rightarrow$$

$$\Rightarrow 1 - P\left[G \leq \frac{75-72}{12}\right] = 1 - P[G \leq 0{,}25] = 1 - (0{,}5 + 0{,}09871) = 0{,}4013$$

f) $$P_r[A < 48\text{ms}] \approx P[Z < 48\text{ms}] = P\left[G < \frac{48-72}{12}\right] = P[G < -2] = P_r[G > 2]$$

$$= 1 - P_r[G \leq 2] = 1 - (0{,}5 + 0{,}47725) = 0{,}02275$$