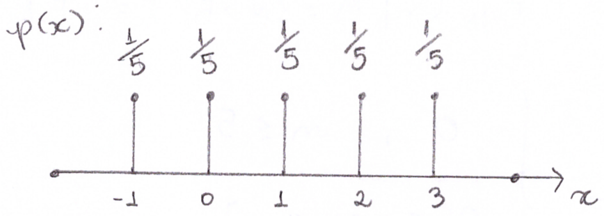
Resolução de exercícios sugeridos: Lista 3

10.



a) $Y = g(X) = |X| - 1$

$$\mu_Y = E[Y] = E[g(X)] = \sum_x g(x) \cdot p_X(x) \Rightarrow$$

$$\Rightarrow \mu_Y = \sum_{x=-1}^{3} (|x|-1) \cdot \frac{1}{5} = \frac{1}{5} \cdot (0 - 1 + 0 + 1 + 2) = \frac{2}{5}$$

b)

X	Y	$p_X(x)$
-1	0	1/5
0	-1	1/5
1	0	1/5
2	1	1/5
3	2	1/5

$\longrightarrow$

Y	$p_Y(y)$
-1	1/5
0	2/5
1	1/5
2	1/5

$$\Rightarrow p_Y(y) = \begin{cases} \frac{1}{5}, & y = -1, 1, 2 \\ \frac{2}{5}, & y = 0 \\ 0, & \text{c.c.} \end{cases}$$

ou:

$$p_Y(y) = \frac{2}{5}\delta(y) + \frac{1}{5}[\delta(y+1) + \delta(y-1) + \delta(y-2)]$$

com $\delta(m) = \begin{cases} 1, & m = 0 \\ 0, & \text{c.c.} \end{cases}$

c) $\sigma_Y^2 = Var[Y] = E[Y^2] - (E[Y])^2$

$$\hookrightarrow E[Y^2] = \sum_y y^2 \cdot p_Y(y) = \frac{1}{5} \cdot ((-1)^2 + 1^2 + 2^2) + \frac{2}{5} \cdot 0^2 = \frac{6}{5}$$

Logo, $\sigma_Y^2 = \frac{6}{5} - \left(\frac{2}{5}\right)^2 = \frac{6}{5} - \frac{4}{25} = \frac{30-4}{25} = \frac{26}{25}$.

d) Desigualdade de Chebyshev: $P(|X - \mu_X| \geq t) \leq \frac{Var(X)}{t^2}$

Logo, para a VA Y, temos:

$$P(|Y - \mu_Y| \geq t) \leq \frac{\sigma_Y^2}{t^2} \Rightarrow P(|Y - \mu_Y| \geq 2) \leq \frac{26}{25 \cdot 4} = \frac{26}{100} = \frac{13}{50}$$

17.

a) $F_{N|M}(n \mid M=\{N>5\}) = P[N \leq n \mid N>5] = \dfrac{P[(N \leq n) \cap (N>5)]}{P[N>5]} =$

$$= \begin{cases} 0 \ , & n \leq 5 \\ \dfrac{P[5 < N \leq n]}{P[N>5]} \ , & n > 5 \end{cases}$$

• $P[5 < N \leq n] = F_N(n) - F_N(5)$

$$f_{N|M}(n \mid M=\{N>5\}) = \frac{d}{dn} F_{N|M} = \begin{cases} 0 \ , & n \leq 5 \\ \dfrac{f_N(n)}{1 - F_N(5)} \ , & n > 5 \end{cases}$$

b) $E[N|M] = \displaystyle\int_{-\infty}^{+\infty} n \cdot f_{N|M}(n|M)\, dn = \frac{1}{1 - F_N(5)} \cdot \int_5^{+\infty} n \cdot f_N(n)\, dn$

• $\displaystyle\int_5^{+\infty} n \cdot f_N(n)\, dn = \frac{1}{\sqrt{2.\pi.2}} \int_5^{+\infty} n \cdot e^{-\frac{(n-4)^2}{2.2}} dn = \frac{1}{\sqrt{4\pi}} \int_5^{+\infty} n \cdot e^{-\frac{(n-4)^2}{4}} dn$

Sabemos integrar $\int x e^{-x^2} dx$, com a substituição $u = x^2$:

$du = 2x\, dx \Rightarrow \dfrac{du}{2} = x\, dx$, logo: $\displaystyle\int x e^{-x^2} dx = \frac{1}{2} \int e^{-u} du = \frac{-e^{-u}}{2} + k$

Vamos tentar o mesmo procedimento:

$u = \dfrac{(n-4)^2}{4} \Rightarrow du = \dfrac{2(n-4)}{4} dn = \left(\dfrac{n}{2} - 2\right) dn$

Temos que fazer o termo $\left(\dfrac{n}{2} - 2\right)$ aparecer na integral:

$$\frac{1}{\sqrt{4\pi}} \int_5^{+\infty} n \cdot e^{-\frac{(n-4)^2}{4}} dn = \frac{1}{\sqrt{4\pi}} \left(2 \int_5^{\infty} \frac{n}{2} \cdot e^{-\frac{(n-4)^2}{4}} dn - 2 \int_5^{\infty} 2 \cdot e^{-\frac{(n-4)^2}{4}} dn + 2 \int_5^{\infty} 2 \cdot e^{-\frac{(n-4)^2}{4}} dn \right)$$

$$= \frac{1}{\sqrt{4\pi}} \left(2 \cdot \int_5^{\infty} \left(\frac{n}{2} - 2\right) \cdot e^{-\frac{(n-4)^2}{4}} dn + 4 \int_5^{\infty} e^{-\frac{(n-4)^2}{4}} dn \right) = C$$

Fazendo a substituição na primeira integral, temos:

$$C = \frac{1}{\sqrt{4\pi}} \cdot \left(2 \cdot \int_{\frac{1}{4}}^{\infty} e^{-u}\, du + \underbrace{4 \int_{-\infty}^{+\infty} e^{-\frac{(n-4)^2}{4}}\, dn - 4 \int_{-\infty}^{5} e^{-\frac{(n-4)^2}{4}}\, dn}_{4 \int_{5}^{+\infty} e^{-\frac{(n-4)^2}{4}}\, dn} \right) =$$

$$\hookrightarrow \frac{(5-1)^2}{4}$$

$$= \frac{1}{\sqrt{\pi}} (-e^{-u}) \Big|_{-\frac{1}{4}}^{\infty} + 4 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi}} \cdot e^{-\frac{(n-4)^2}{4}}\, dn - 4 \int_{-\infty}^{5} \frac{1}{\sqrt{4\pi}} \cdot e^{-\frac{(n-4)^2}{4}}\, dn =$$

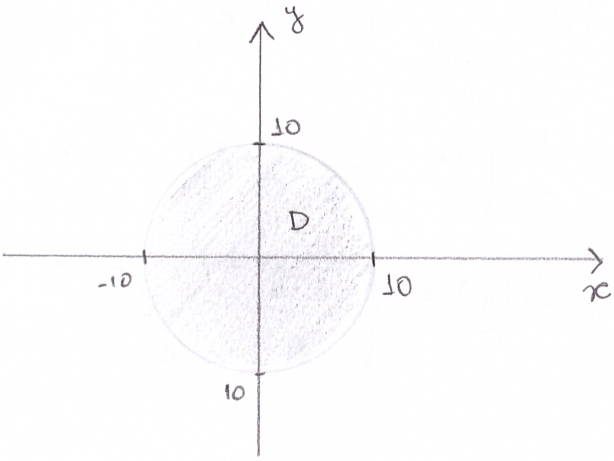
$$= \frac{e^{-1/4}}{\sqrt{\pi}} + 4\,(1 - F_N(5)).$$

Logo, $E[N|M] = \dfrac{1}{1 - F_N(5)} \cdot \left(\dfrac{e^{-1/4}}{\sqrt{\pi}} + 4\,(1 - F_N(5)) \right) \approx 5{,}81$

$1 - F_N(5) \approx 1 - 0{,}7579 = 0{,}2421$

c) $Var[N|M]$

25. $f_{XY}(x,y) = \begin{cases} c, & (x^2+y^2)^{1/2} \leq 10\,\mu m \\ 0, & c.c. \end{cases}$

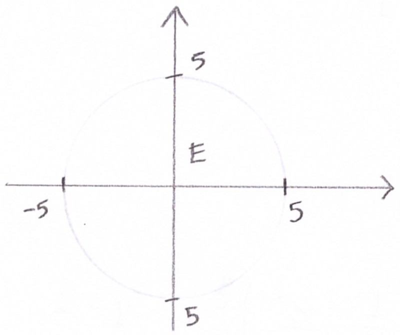


a) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1 \Rightarrow$

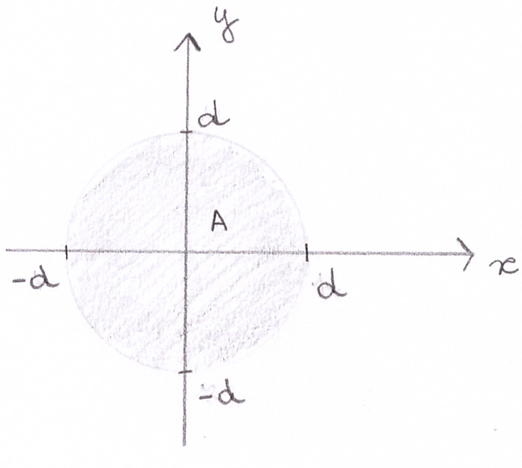
$\Rightarrow c \cdot \displaystyle\iint_D dx\,dy = 1 \Rightarrow$

$\Rightarrow c \cdot \pi \cdot 10^2 = 1 \Rightarrow c = \dfrac{1}{100\pi}$

b) $P[E < 5\,\mu m] = c \cdot \displaystyle\iint_E dx\,dy = \dfrac{\pi 5^2}{\pi 10^2} = \dfrac{1}{4}$



c) $D = \sqrt{X^2+Y^2} \Rightarrow D^2 = X^2+Y^2$



$F_D(d) = P[D \leq d] = P[X^2+Y^2 \leq d] = c \cdot \displaystyle\iint_A dx\,dy =$

$= \begin{cases} \dfrac{d^2}{100}, & d \leq 10\,\mu m \\ 1, & c.c. \end{cases}$

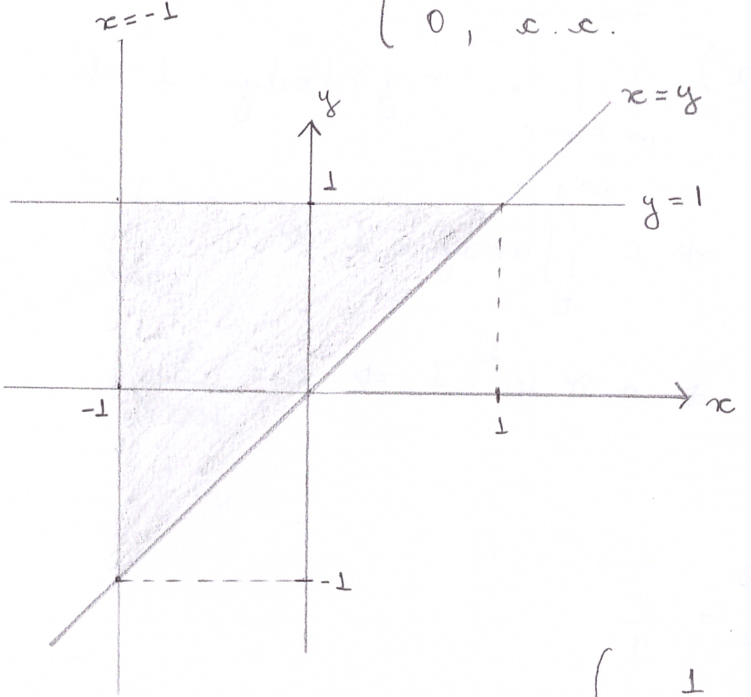
$f_D(d) = \dfrac{d}{dd} F_D(d) = \begin{cases} \dfrac{d}{50}; & d \leq 10\,\mu m \\ 0, & c.c. \end{cases}$

d) $f_X(x) = \displaystyle\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \dfrac{1}{100\pi}\int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}} dy = \dfrac{\sqrt{100-x^2}}{50\pi}, \quad -10 \leq x \leq 10\ (\mu m)$

e) $E[X] = \displaystyle\int_{-\infty}^{+\infty} x f_X(x)\,dx = \dfrac{1}{50\pi}\int_{-\infty}^{+\infty} \underbrace{x}_{\text{ímpar}} \cdot \overbrace{\sqrt{100-x^2}}^{\text{par}}\,dx = 0$

30.

$$f_{XY}(x,y) = \begin{cases} 1/2, & -1 \le x \le y \le 1 \\ 0, & c.c. \end{cases}$$



a) $f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx =$

$= \frac{1}{2}\int_{-1}^{y} dx = \frac{1}{2}(y+1), \quad -1 \le y \le 1$

$\therefore \; f_Y(y) = \begin{cases} \frac{1}{2}(y+1), & -1 \le y \le 1 \\ 0, & c.c. \end{cases}$

b) $f_{X|Y}(x,y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{y+1}, & -1 \le x \le y \le 1 \\ 0, & c.c. \end{cases}$

c) $E[X|Y=y] = \int_{-\infty}^{+\infty} x\, f_{X|Y}(x,y)\,dx = \frac{1}{y+1} \cdot \int_{-1}^{y} x\,dx = \frac{1}{y+1}\,\frac{x^2}{2}\Big|_{1}^{y} = \frac{y-1}{2}$