

# Modelagem Hierárquica

**Exemplo: Chapa em balanço**

**Escola Politécnica da Universidade de São Paulo**

**Disciplina: PEF3302 – Mecânica das Estruturas I**

# Descrição do problema físico

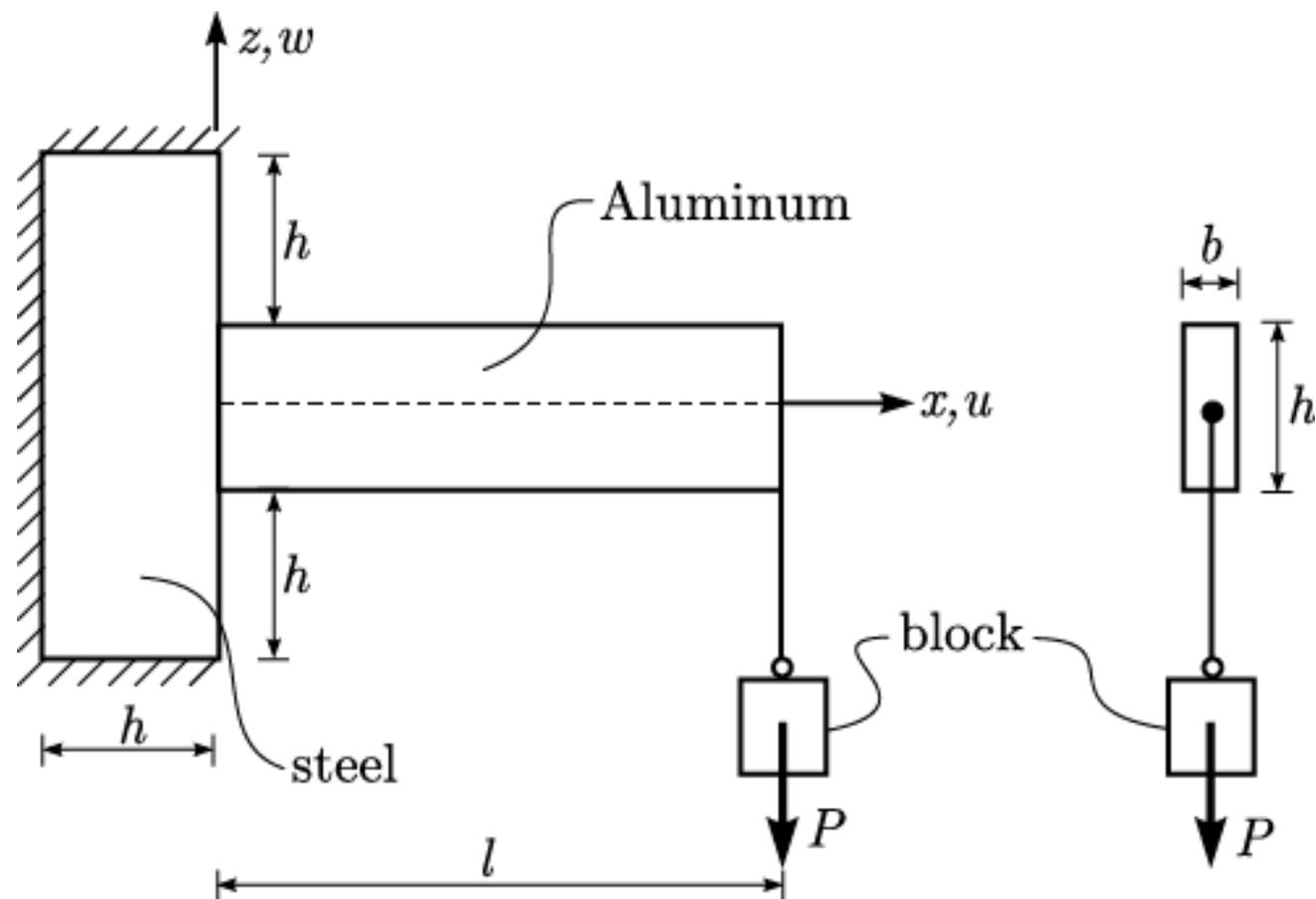


Figura 1

# Descrição do problema físico

**Modelo 1** – Barra de Bernoulli-Euler

**Modelo 2** – Barra de Timoshenko

**Modelos 3 a 6** – Modelos de EPT – Estado Plano de Tensão

**Relação  $l/h$**  – 100, 10 e 5

**Força  $P$**  – Normalizada para a tensão máxima prescrita pelo modelo de Bernoulli-Euler. Seja  $0,9 \cdot \tau_y$ , onde  $\tau_y$  é a tensão de escoamento do alumínio.

# Modelo de barra

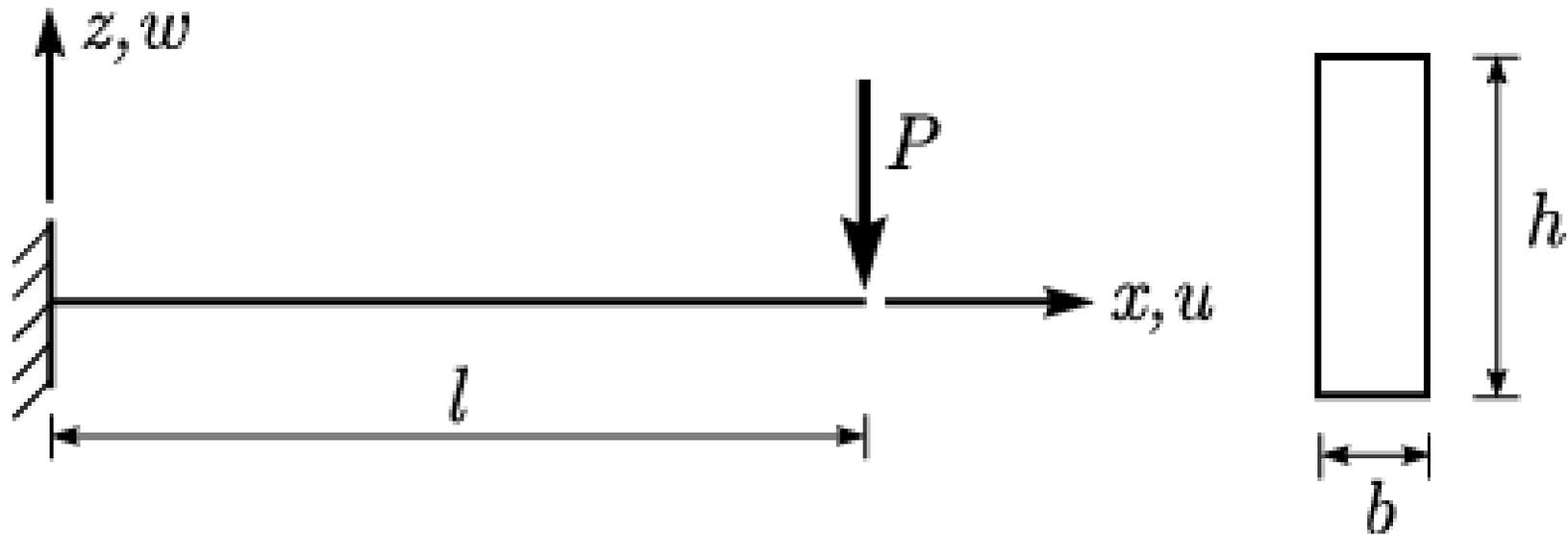


Figura 2

# Modelo de Estado Plano de Tensão

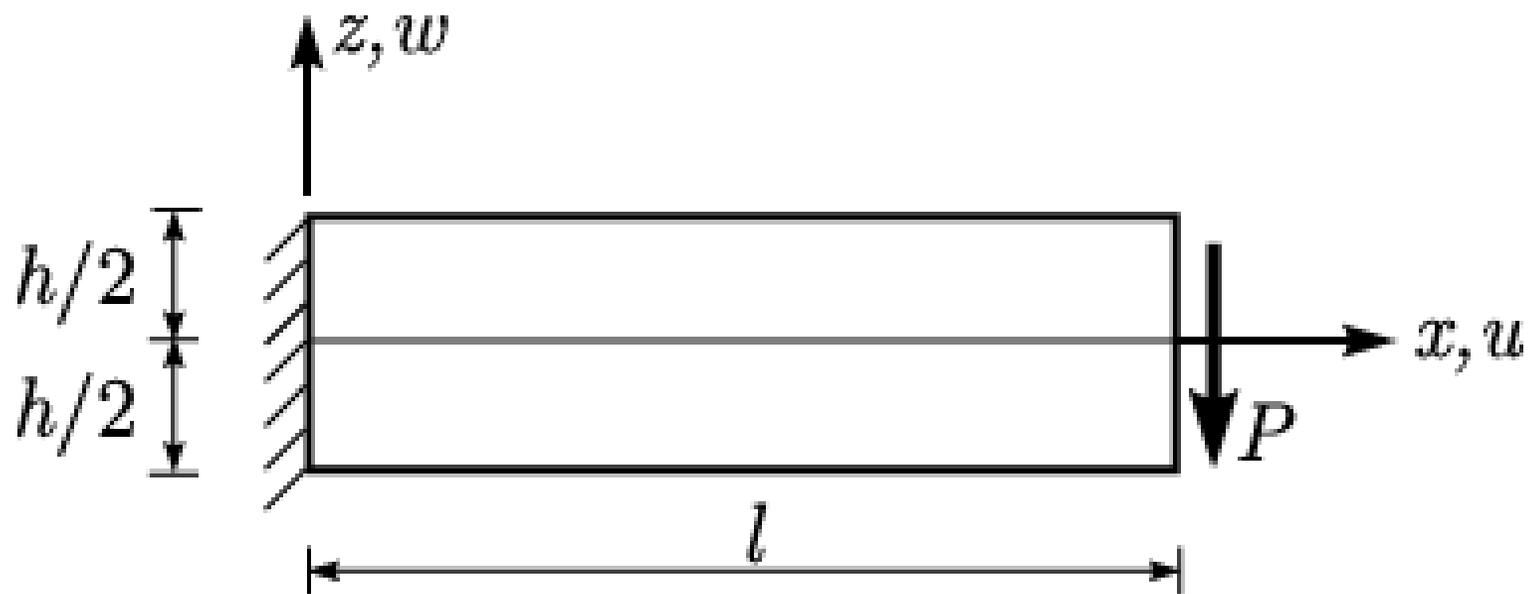


Figura 3

# Modelo de Estado Plano de Tensão

Condições de contorno em deslocamentos:

$$u(0, z) = 0$$

$$w(0, z) = 0$$

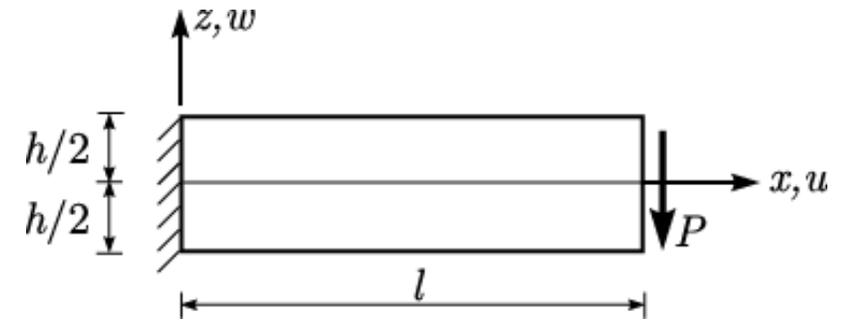
Condições de contorno em forças de superfície:

$$\mathbf{T}\mathbf{n} = \mathbf{f}^S$$

- Para as bordas livres:

$$\tau_{zz}(x, h/2) = \tau_{zz}(x, -h/2) = 0$$

$$\tau_{xz}(x, h/2) = \tau_{xz}(x, -h/2) = 0$$

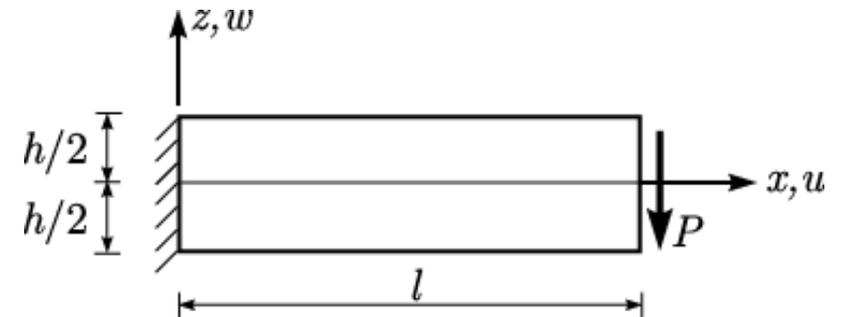


# Modelo de Estado Plano de Tensão

- Para a borda definida por  $x = \ell$ :

$$\tau_{xx}(\ell, z) = f_x^S(\ell, z) = 0$$

$$\tau_{xz}(\ell, z) = f_z^S(\ell, z)$$



$$\int_{L_P} -f_z^S(l, z) b \, dL = P$$

# Modelo de EPT simplificado

Determinar uma solução analítica para o modelo da Figura 2 é extremamente complexo. Considera-se alternativamente o seguinte modelo:

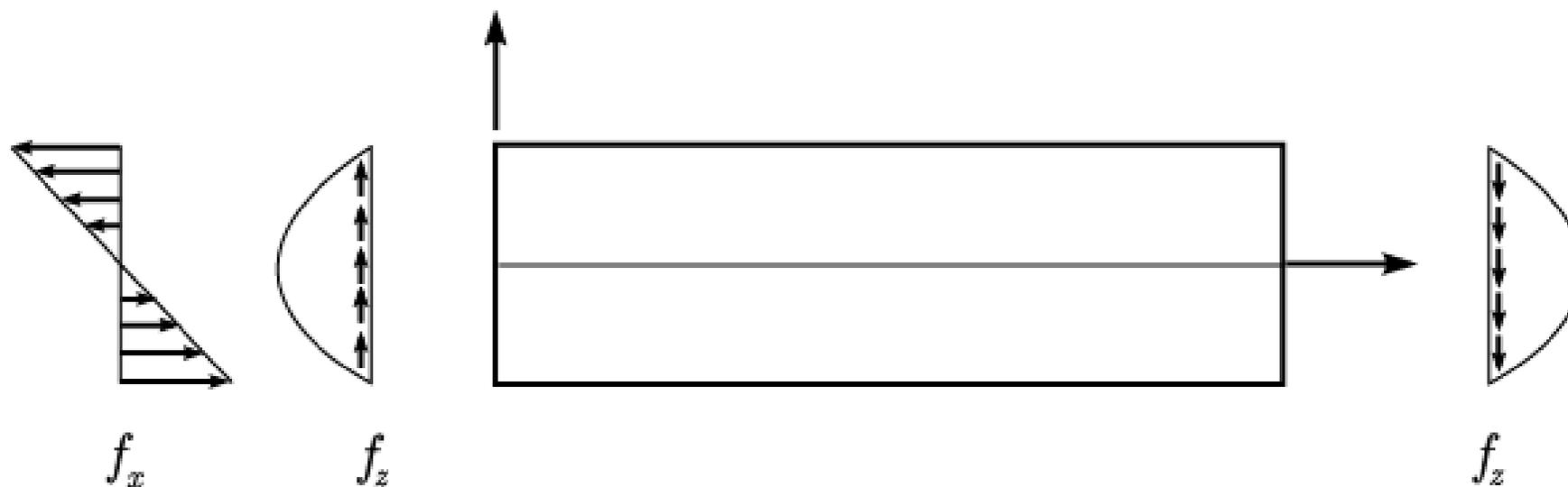


Figura 4

# Modelo de EPT simplificado

Assume-se a seguinte distribuição de tensões:

$$\tau_{xx}(x, z) = \frac{P(\ell - x)}{I} z = \frac{P(\ell - x)}{\frac{bh^3}{12}} z = \frac{12P(\ell - x)}{bh^3} z$$

$$\tau_{xz}(x, z) = -\frac{3P}{2bh} \left( 1 - \frac{4z^2}{h^2} \right) = -\frac{P}{2I} \left( \frac{h^2}{4} - z^2 \right)$$

$$\tau_{zz}(x, z) = 0$$

# Modelo de EPT simplificado

Seja o equilíbrio diferencial:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} = 0$$

# Modelo de EPT simplificado

E as equações constitutivas:

$$\varepsilon_{xx} = \frac{\tau_{xx}}{E} = \frac{P(\ell - x)}{EI} z$$

$$\varepsilon_{zz} = -\nu \frac{\tau_{xx}}{E} = -\frac{\nu P(\ell - x)}{EI} z$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = -\frac{P}{2GI} \left( \frac{h^2}{4} - z^2 \right)$$

# Modelo de EPT simplificado

Por integração, chega-se em:

$$u(x, z) = -\frac{P}{2EI} (x^2 - 2\ell x)z - \frac{\nu}{6EI} Pz^3 + \frac{Pz^3}{6GI} + dz + i$$

$$w(x, z) = -\frac{\nu P}{2EI} (\ell - x)z^2 + \frac{P}{6EI} (x^3 - 3\ell x^2) + ex + j$$

$$\gamma_{xz}(0,0) = e + d = -\frac{Ph^2}{8GI}$$

# Modelo de EPT simplificado

Condições de contorno em termos de deslocamento:

$$u(0,0) = 0$$

$$w(0,0) = 0$$

$$\frac{\partial u}{\partial z}(0,0) = 0$$

# Modelo de EPT simplificado

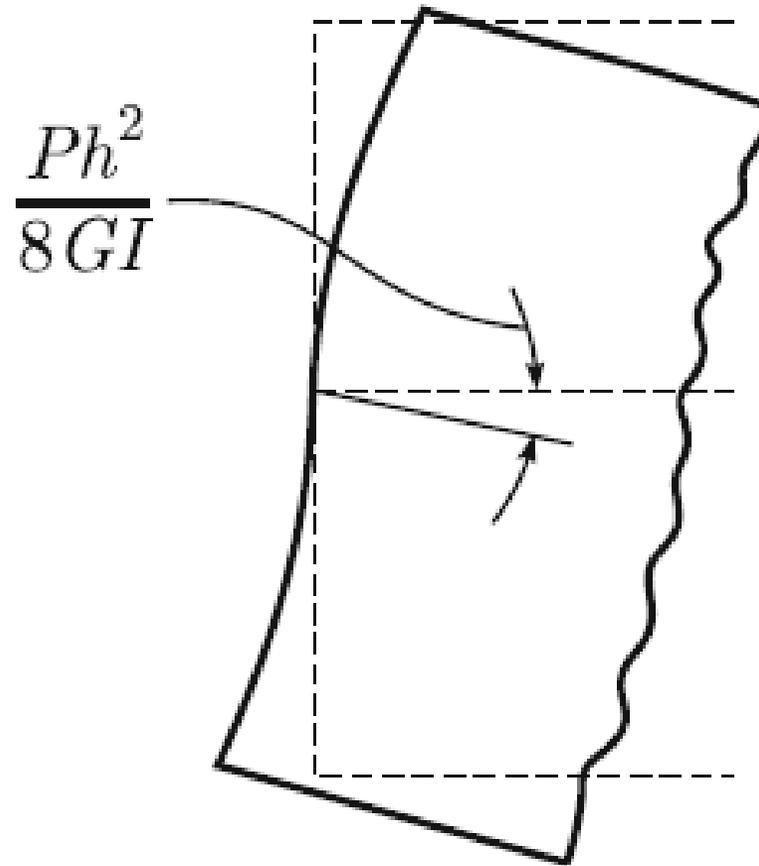


Figura 5 – Fator de magnificação de 30

# Modelo de EPT simplificado

Tabela 1

Model	$u$
Bernoulli-Euler	$-\frac{zP}{2EI}(x^2 - 2lx)$
Timoshenko	$-\frac{zP}{2EI}(x^2 - 2lx)$
Plane stress	$-\frac{zP}{2EI}(x^2 - 2lx) + \frac{Pz^3}{6EI}(2 + \nu)$

Model	$w$
Bernoulli-Euler	$\frac{P}{6EI}(x^3 - 3lx^2)$
Timoshenko	$\frac{P}{6EI}(x^3 - 3lx^2) - \frac{12+11\nu}{10(1+\nu)} \frac{P}{Gbh} x$
Plane stress	$\frac{P}{6EI}(x^3 - 3lx^2) - \frac{3}{2} \frac{P}{Gbh} x - \frac{\nu P}{2EI}(\ell - x)z^2$

# Comparativo de modelos analíticos

Define-se:

$$r_w = \left| \frac{\bar{w} - w_{BE}}{w_{BE}^{m\acute{a}x}} \right| \cdot 100$$

Onde:

$\bar{w}$  - Deslocamento do eixo segundo Timoshenko ou EPT simplificado

$w_{BE}$  - Deslocamento segundo Bernoulli-Euler

$w_{BE}^{m\acute{a}x}$  - Deslocamento máximo segundo Bernoulli-Euler (parâmetro de normalização)

# Comparativo de modelos analíticos

Simplificando-se:

$$r_w = \frac{k}{2} \left( \frac{h}{2} \right)^2 \frac{x}{\ell} \cdot 100$$

**Tabela 2**

	$r_w (x = \ell)$		
<b>Model</b>	$\ell/h = 5$	$\ell/h = 10$	$\ell/h = 100$
Timoshenko	3.12600	0.78150	0.007815
Plane stress	3.99000	0.99750	0.009975

# Modelo de Elementos Finitos

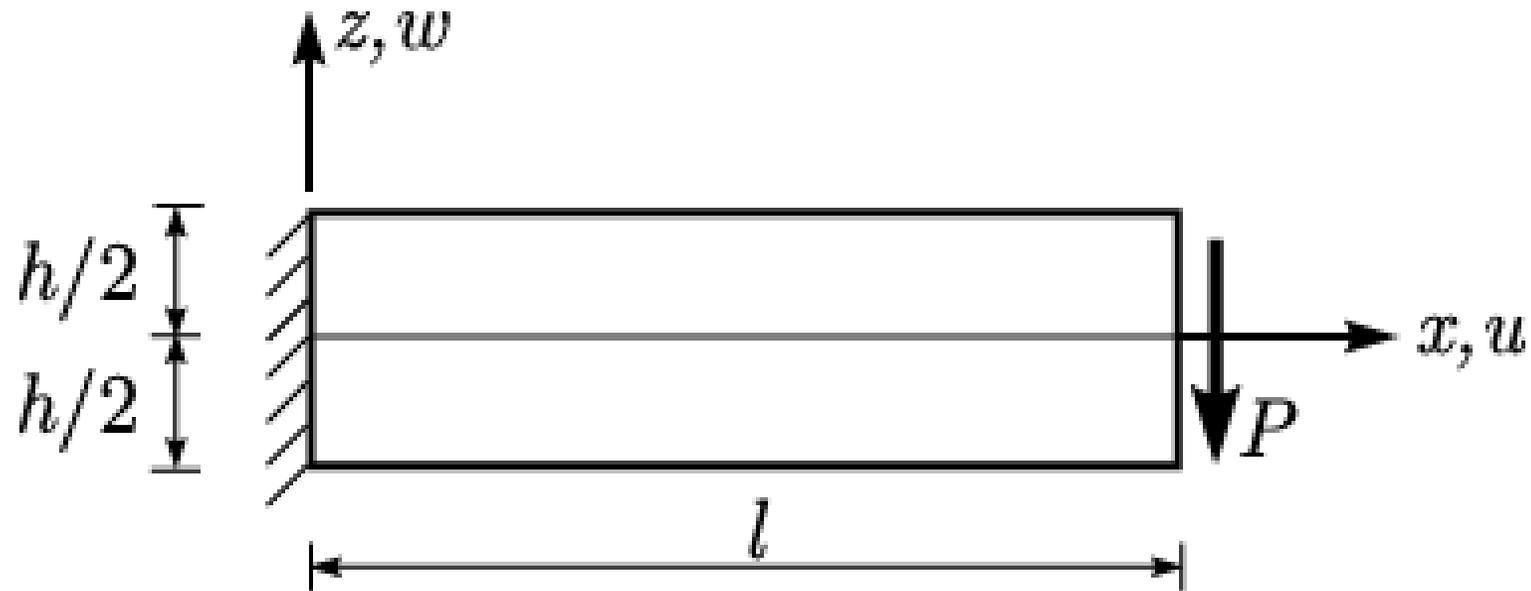


Figura 6

# Modelo de Elementos Finitos

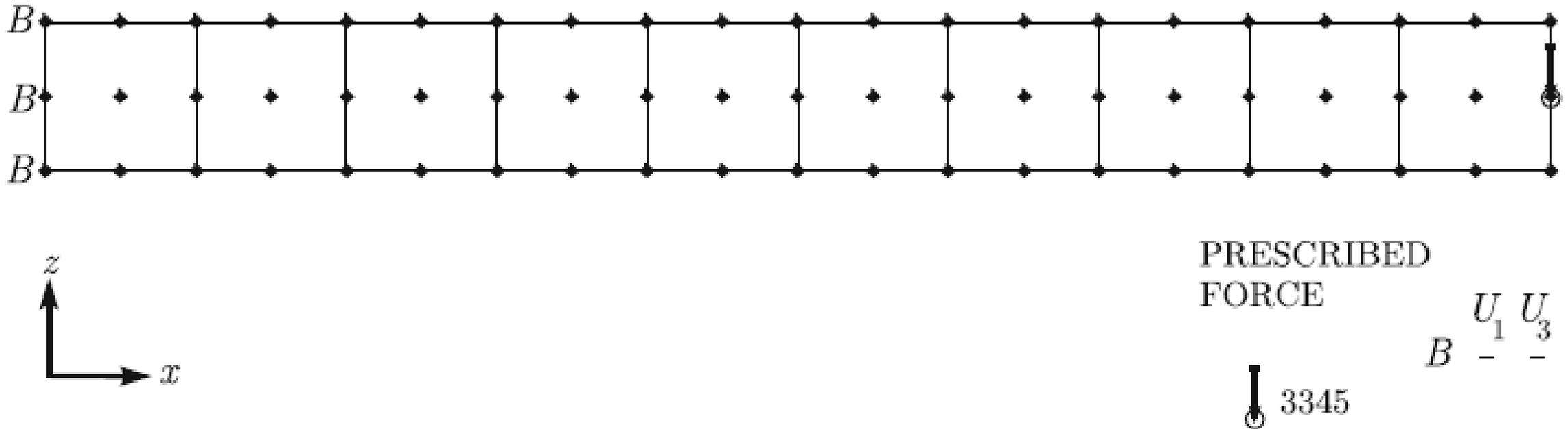


Figura 7

# Modelo de Elementos Finitos

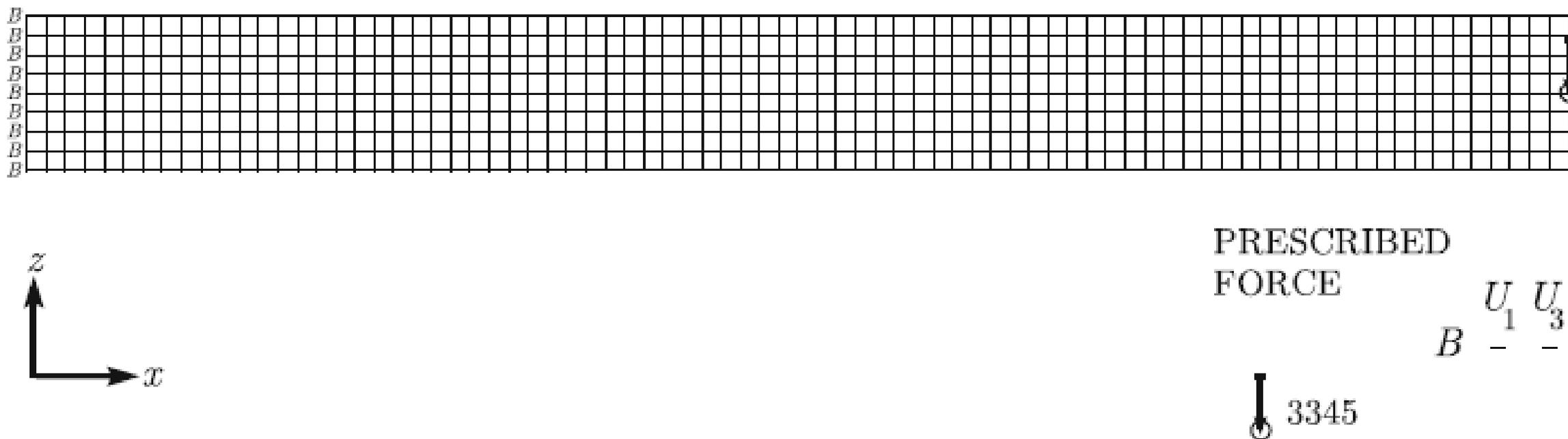


Figura 8

# Comparativo de modelos

**Tabela 3**

Model	w at tip	normalized w.r.t. the plane stress model
Bernoulli-Euler	-0.01911	0.99012
Timoshenko	-0.01926	0.99786
Plane stress	-0.01930	1.00000
FE plane stress (1x10)	-0.01915	0.99195
FE plane stress (2x20)	-0.01921	0.99497
FE plane stress (4x40)	-0.01922	0.99583
FE plane stress (8x80)	-0.01923	0.99620
FE plane stress (64x640)	-0.01924	0.99690

# Comparativo de modelos

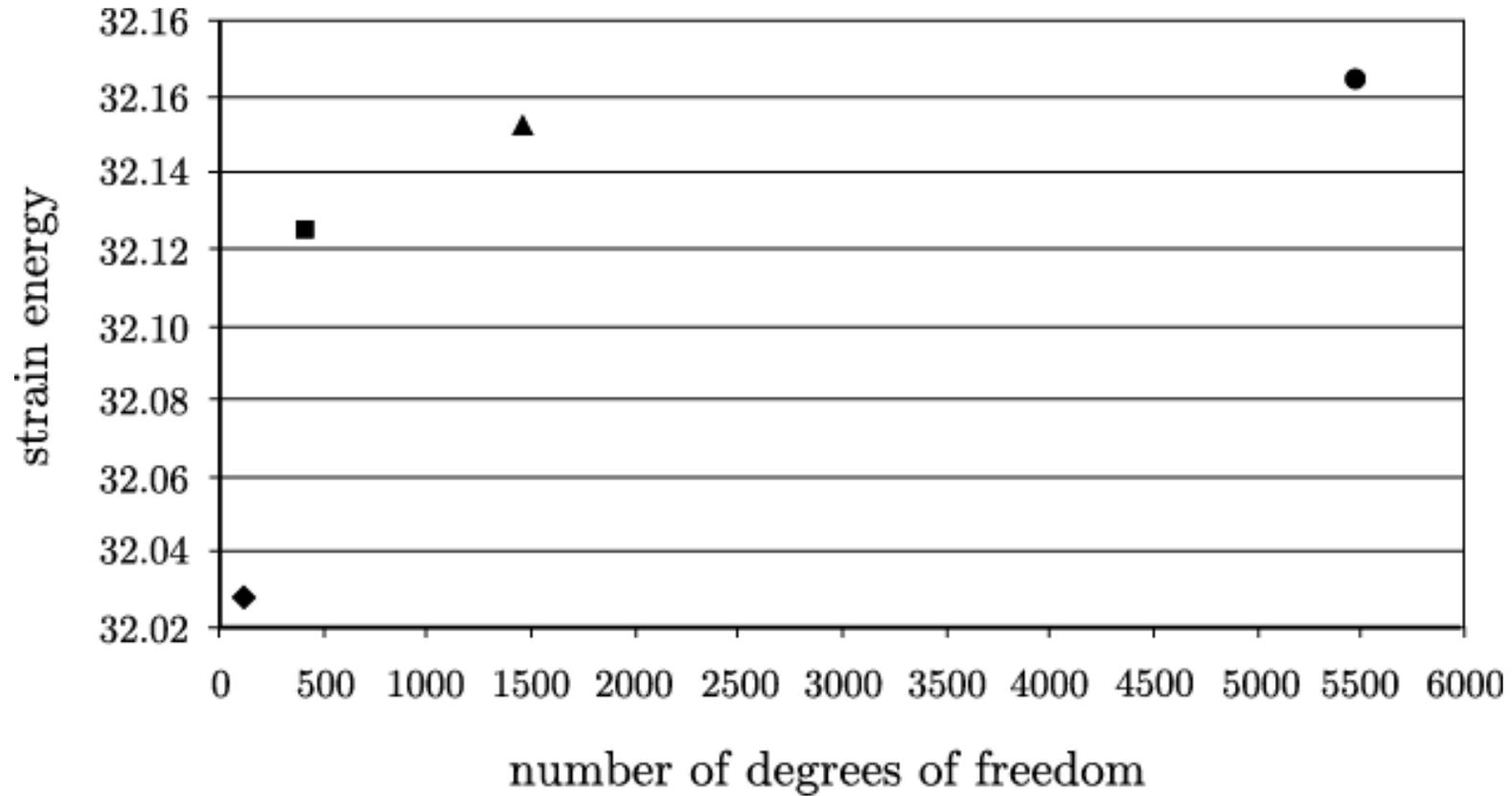


Figura 9

# Comparativo de modelos

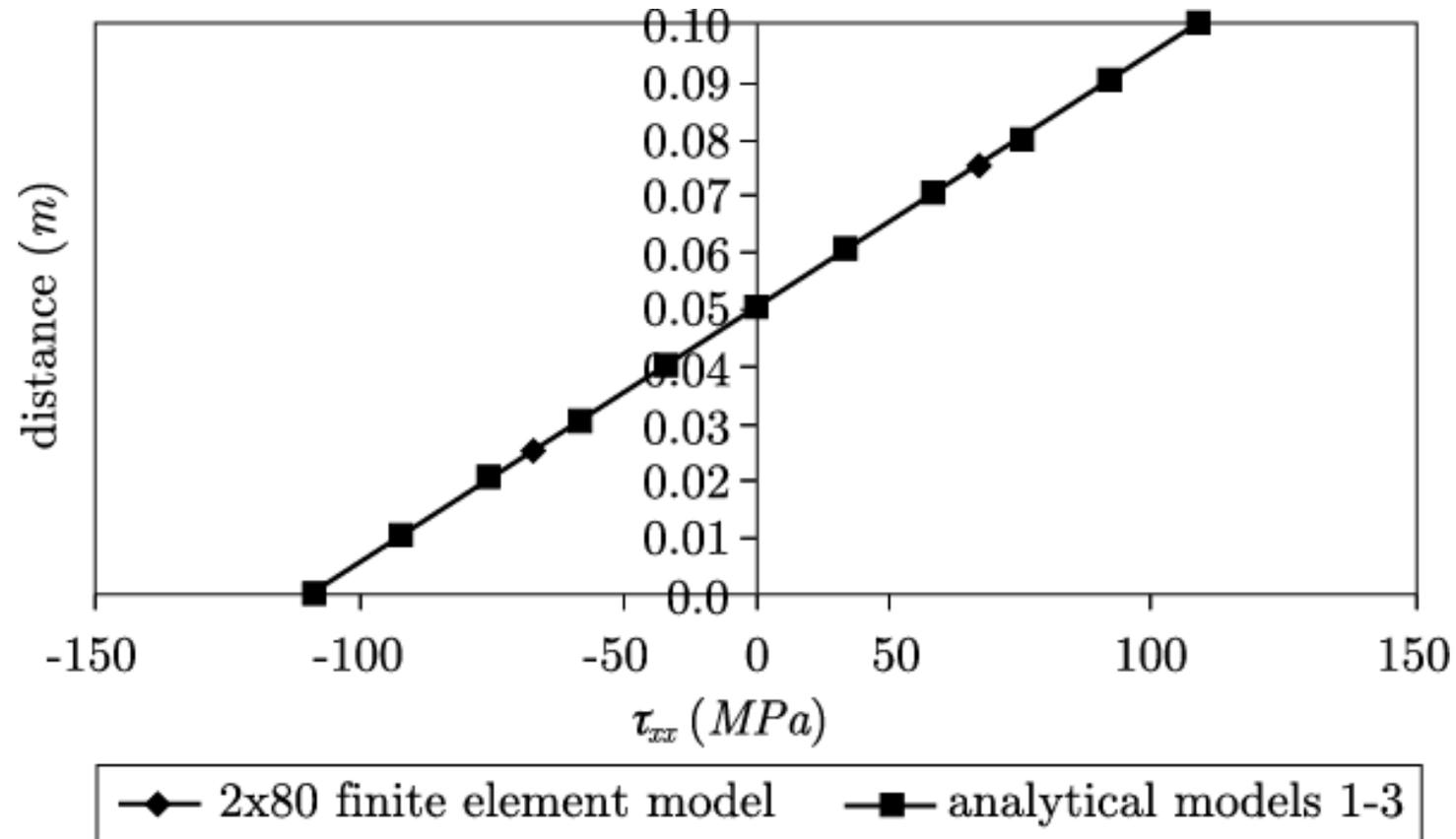


Figura 10 –  $x = \ell/2$

# Comparativo de modelos

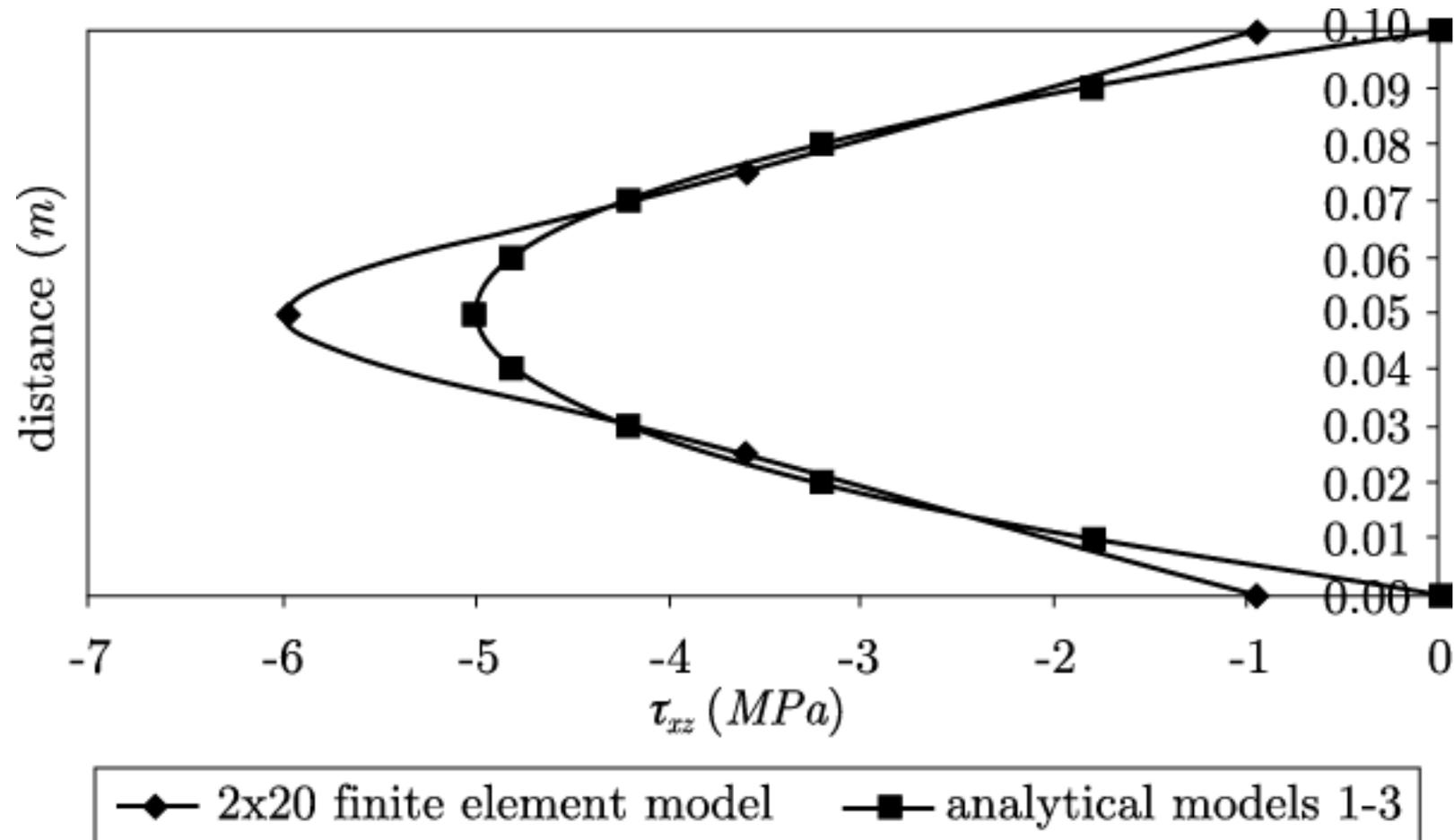


Figura 11 –  $x = \ell/2$

# Comparativo de modelos

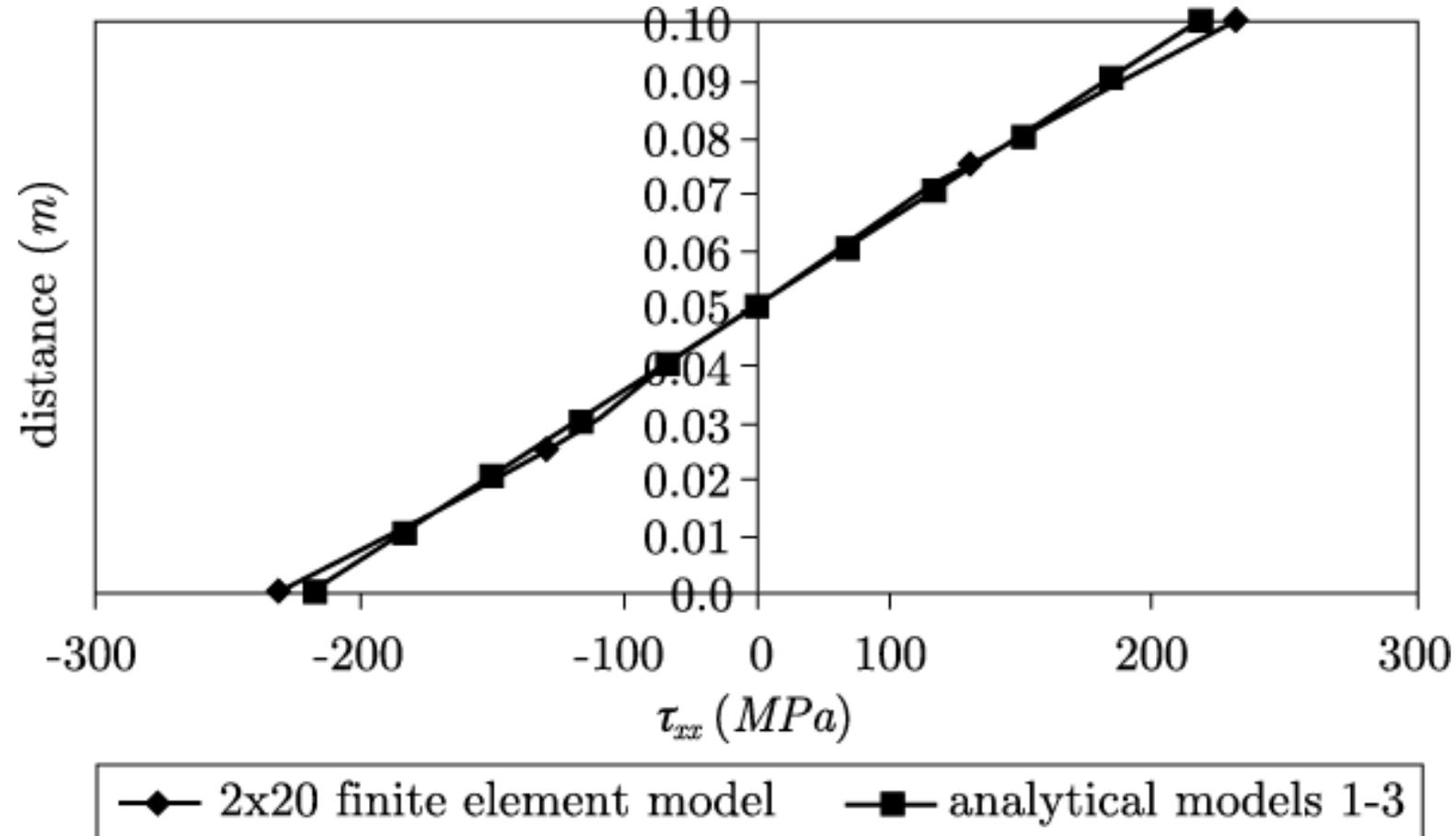


Figura 12 -  $x = 0$

# Comparativo de modelos

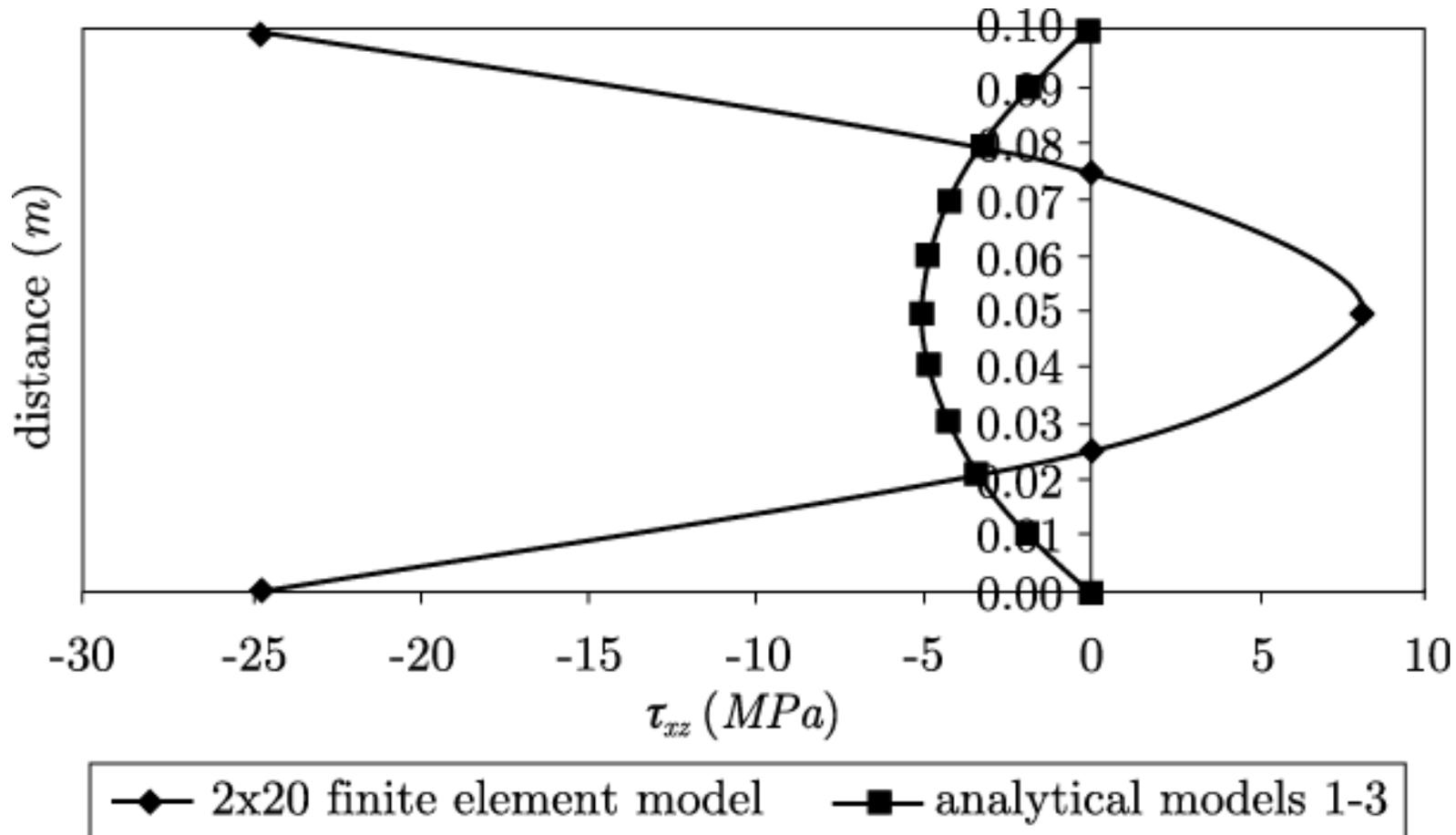


Figura 13 -  $x = 0$

# Comparativo de modelos

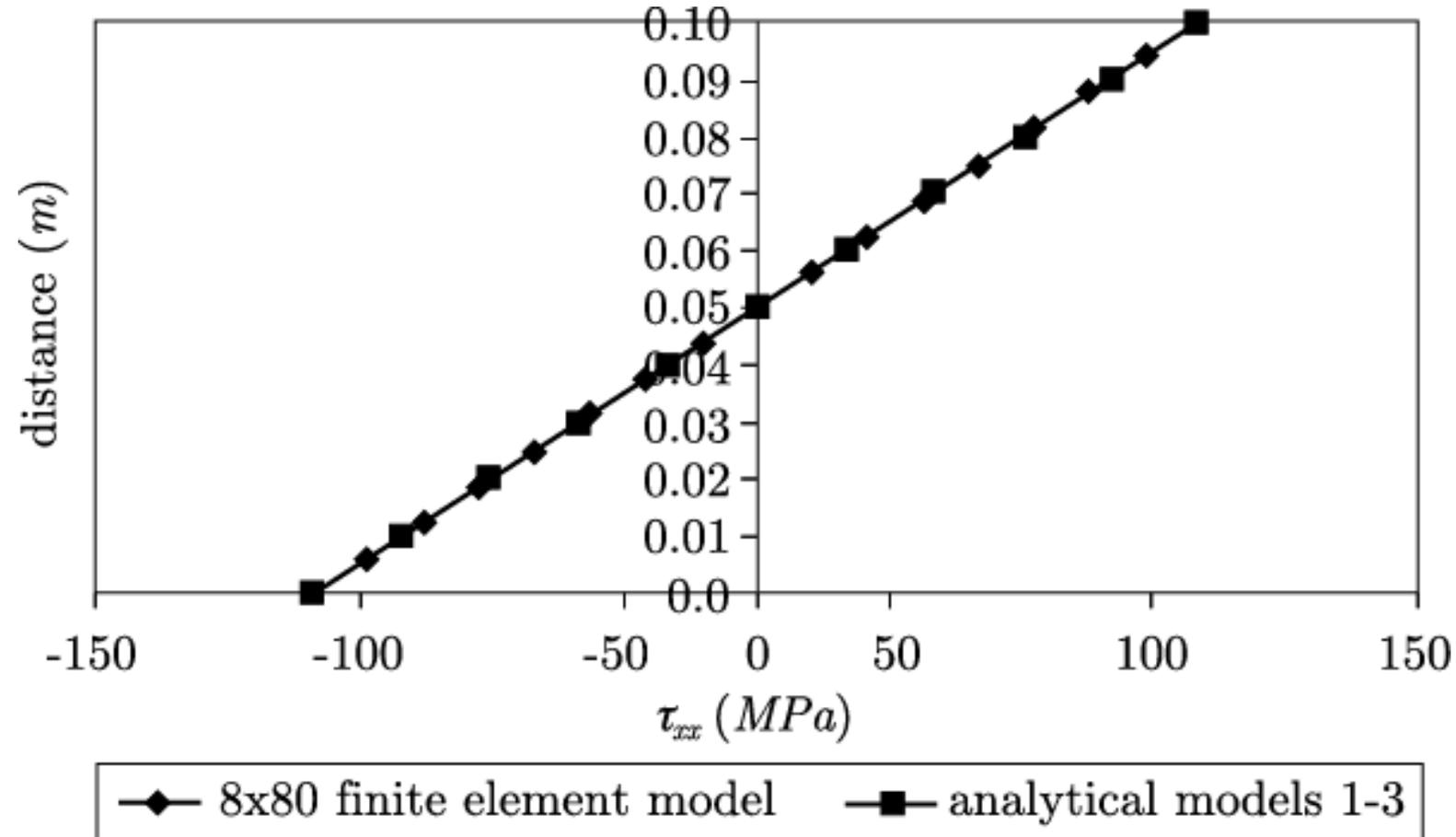


Figura 14 -  $x = \ell/2$

# Comparativo de modelos

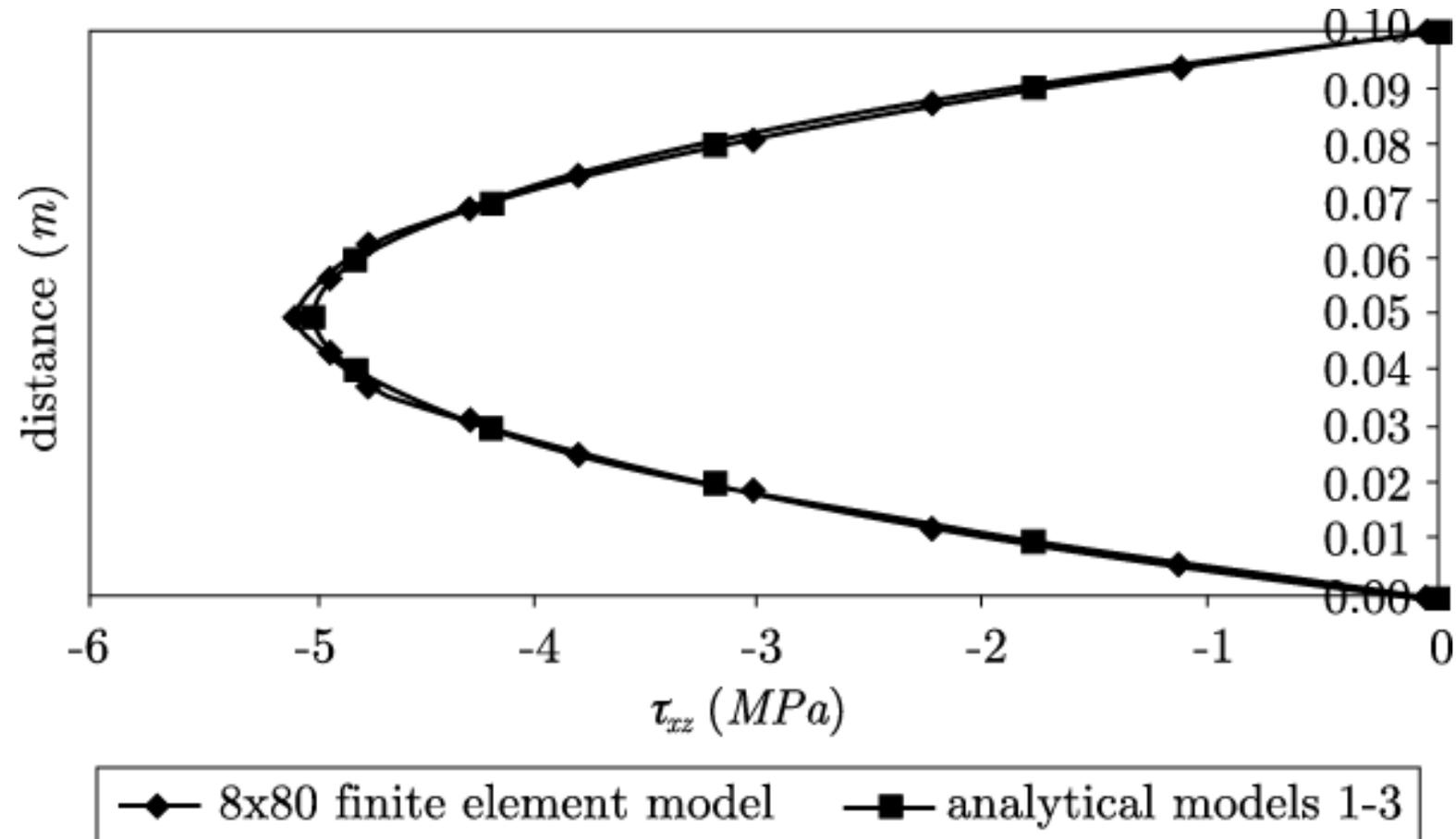


Figura 15 -  $x = \ell/2$

# Comparativo de modelos

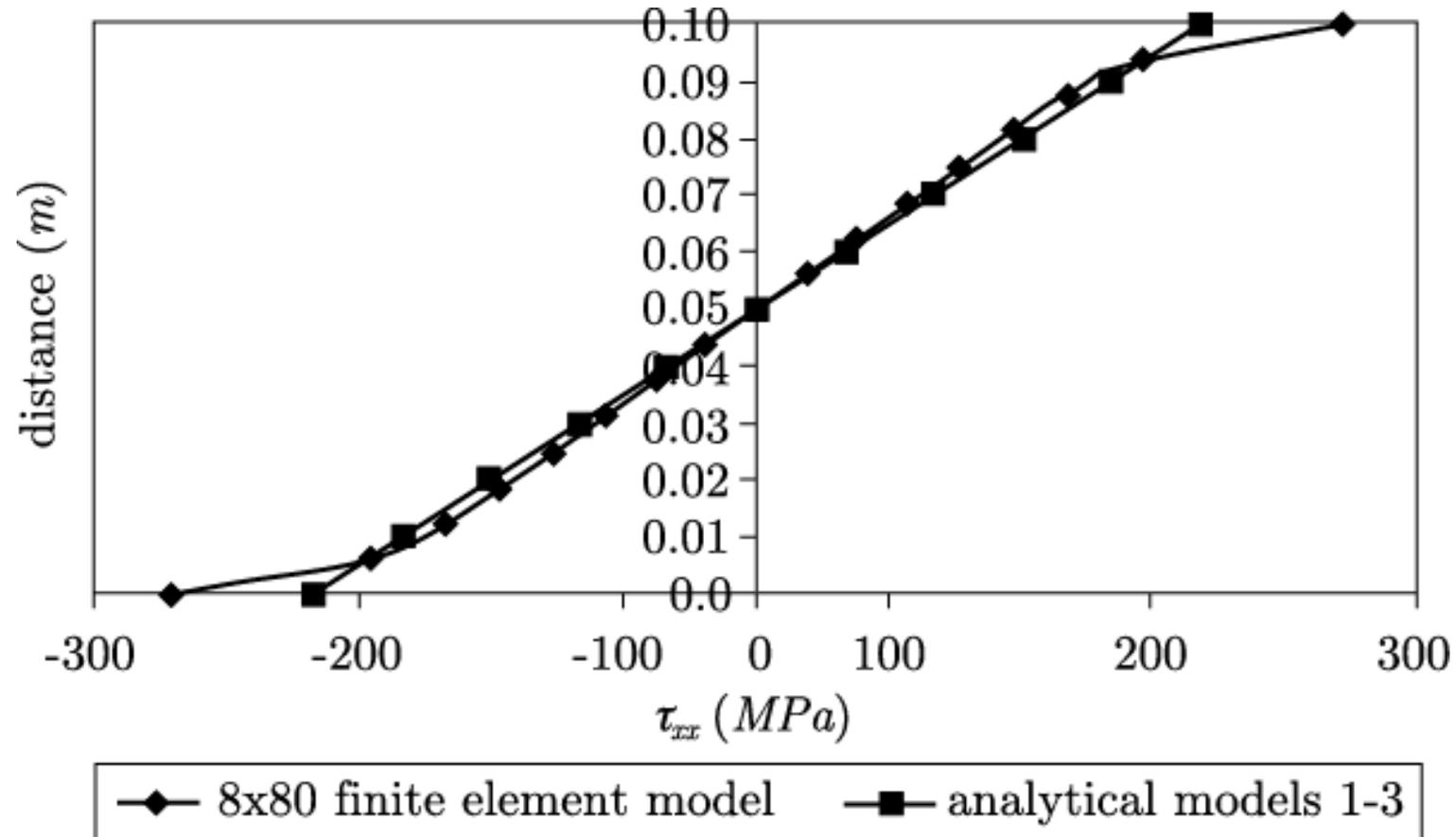


Figura 16 -  $x = 0$

# Comparativo de modelos

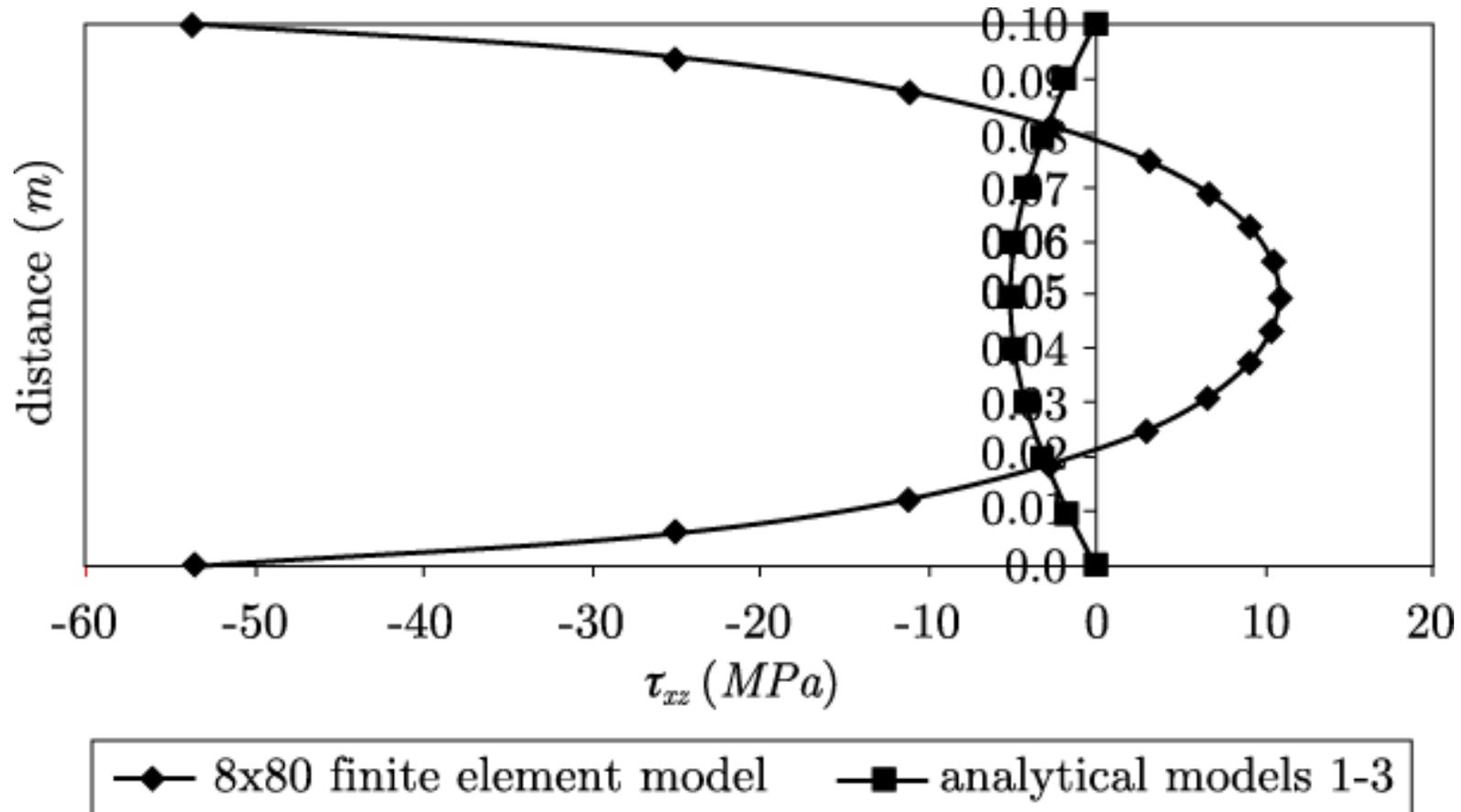


Figura 17 -  $x = 0$

# Comparativo de modelos

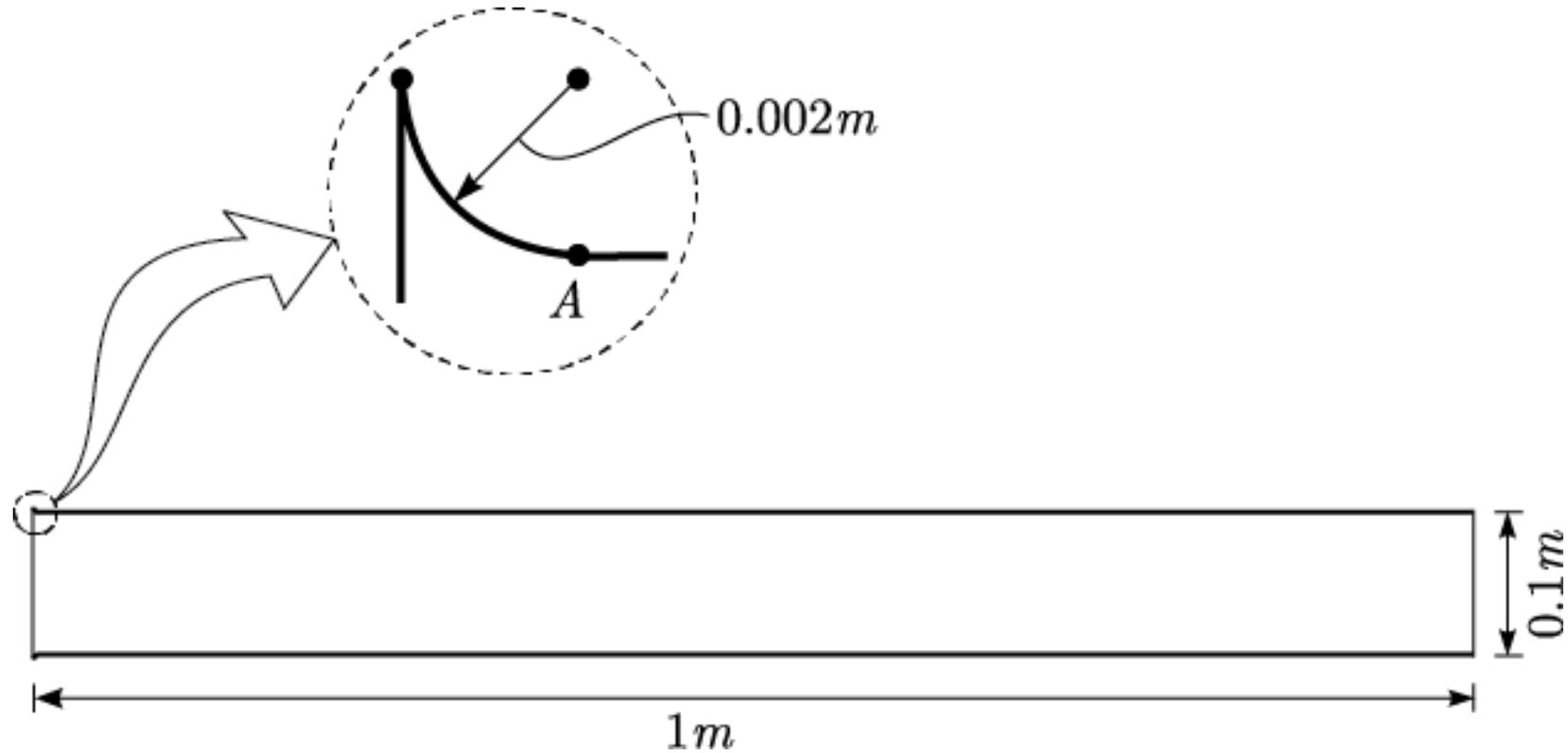


Figura 18

# Comparativo de modelos

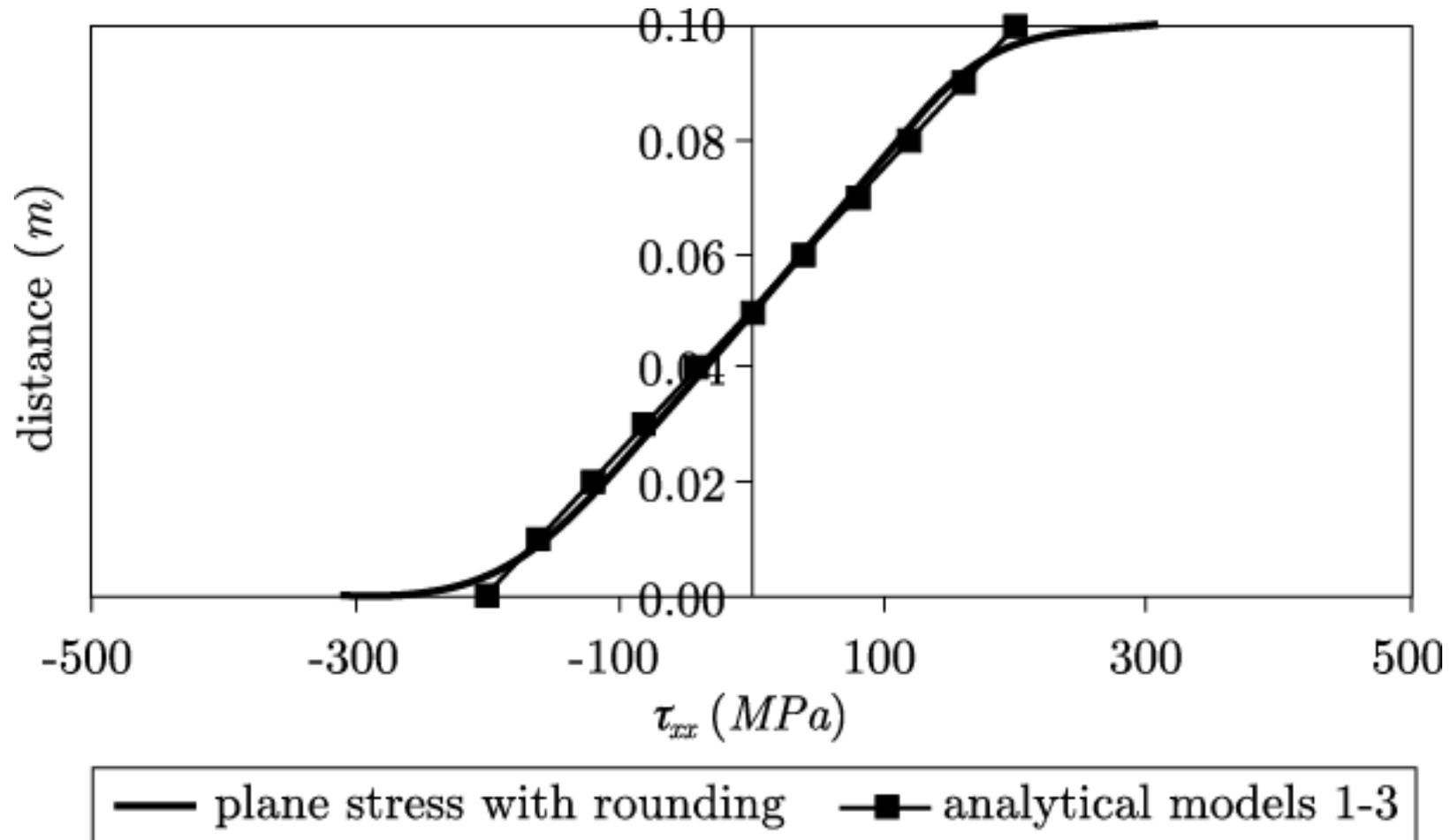


Figura 19 -  $x = 0$

# Comparativo de modelos

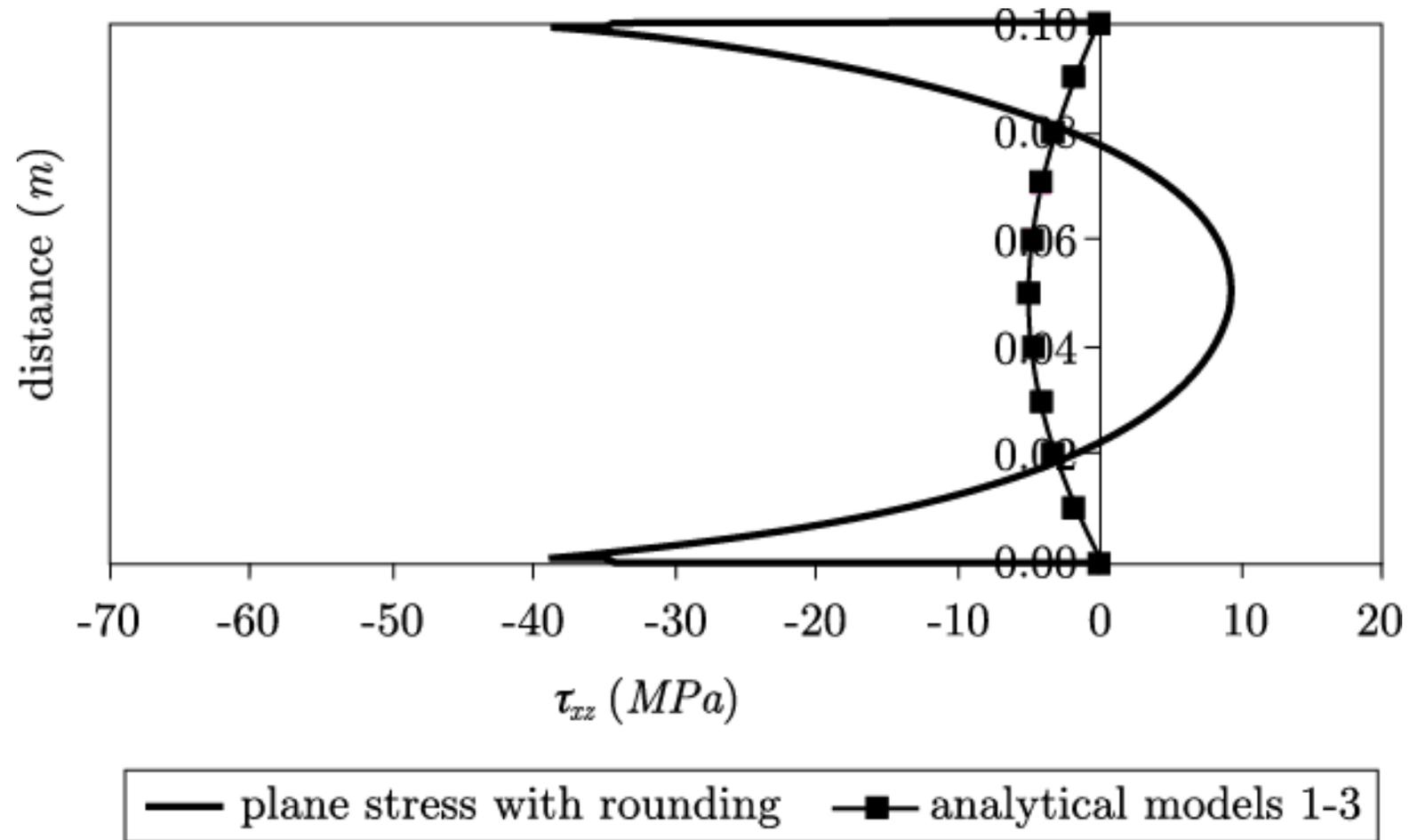


Figura 20 -  $x = 0$

# Comparativo de modelos

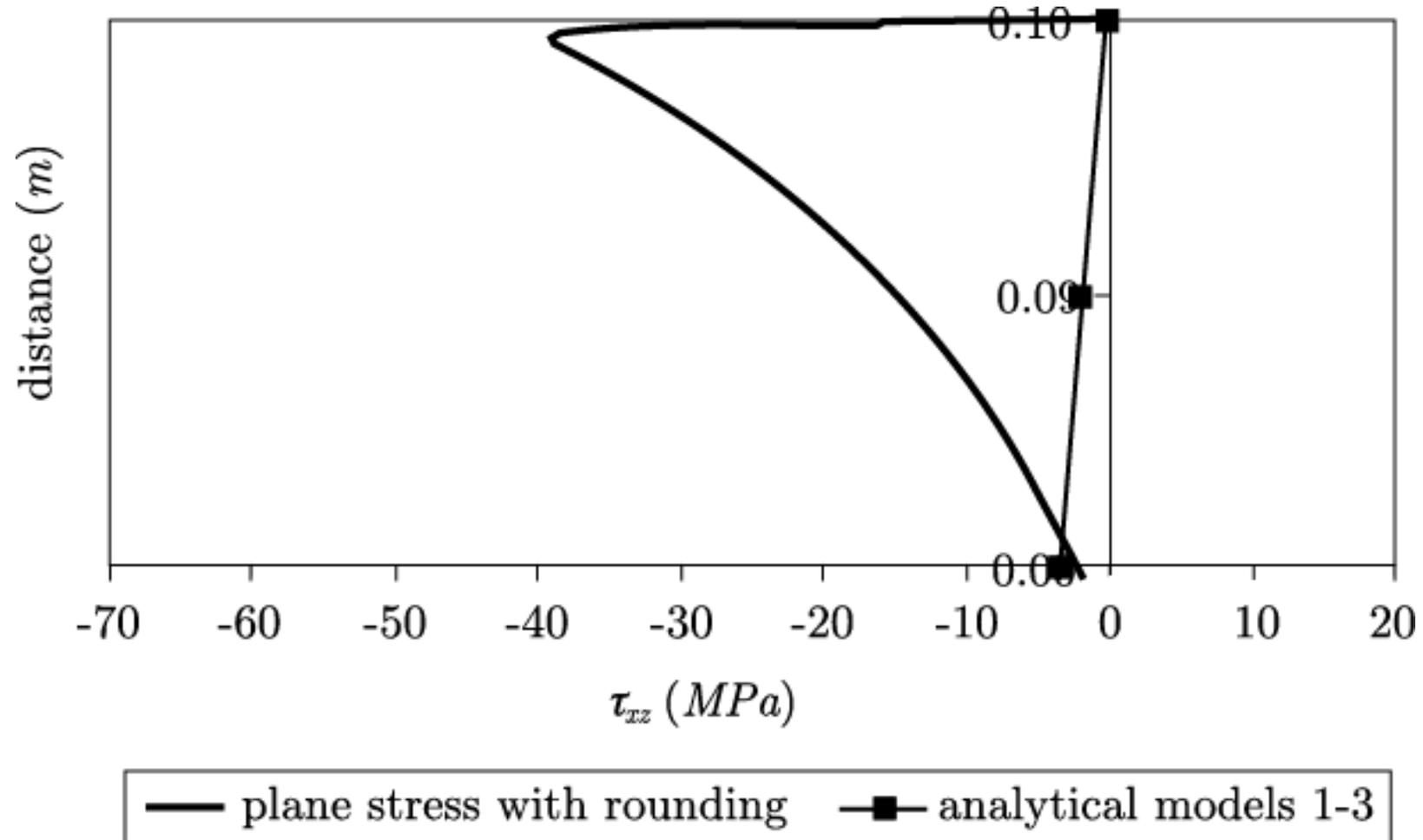


Figura 21 -  $x = 0$