Instantons, Tunneling and Metastability

Quantum Mechanics II Seminar – Profª Renata Funchal - 2020/2

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30 November 2020

Outline

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Introduction · Motivation and Applications

Instantons → Classical non-trivial solutions with finite action to equations of motion in Euclidean space-time.
(imaginary-time)

They are non-perturbative process: cannot be seen in any order of perturbation theory!

↳ dependence $\sim e^{-\frac{S}{\hbar}}[1+\ \mathcal{O}(\hbar)]$ $\Rightarrow$ essential singularity at $\hbar = 0$

Several applications:

↳ In one-dimensional QM: semiclassical (SC) description of tunneling processes

↳ In QFT
- QCD instantons shape the ground state of strong interactions
- e.g: Yang-Mills instantons display geometrical, topological and quantum effects that have fundamental impact on the spectrum of nonabelian gauge theories.
- Appear in many field theories, from scalar QFTs to supersymmetric Yang-Mills and string theory

↳ In particle physics they have impact on both weak-interaction and hard QCD processes, such as deep inelastic scattering.

↳ In Cosmology they describe the "decay of the false vacuum".

Introduction · SCA and WKB method

Semi-classical limit: $\hbar \to 0$

The WKB method, named after Wenzel, Kramers and Brioullin, is a "semiclassical calculation" in QM in which the wave function is assumed an exponential function with typical wavelength λ small in comparison to the spatial variations of the potential

$$\lambda = \frac{2\pi\hbar}{p} \to 0 \qquad \text{and} \qquad \psi(x) = e^{i\Phi(x)/\hbar}$$

Replacing in Schrodinger equation:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - E\right]\psi(x) = 0 \quad \Rightarrow \quad \frac{1}{2m}\Phi'^2(x) - \frac{i\hbar}{2m}\Phi''(x) = E - V(x)$$

and expanding

$$\Phi(x) = \Phi_0(x) + \hbar\Phi_1(x) + \hbar^2\Phi_2(x) + \ldots,$$

Collecting $\mathcal{O}(\hbar^0)$, we obtain

$$\Phi_0(x) = \pm\int^x dx' p(x') \qquad \text{with} \qquad p(x) \equiv \sqrt{2m[E - V(x)]}$$

momentum at constant potential

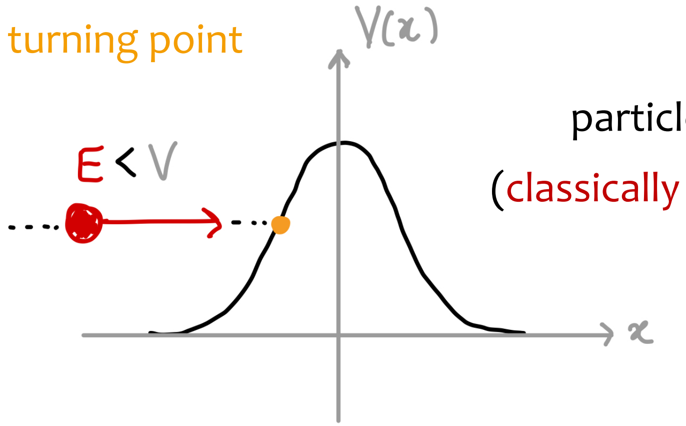
Introduction · SCA and WKB method

The approximation is good when $V(x)$ varies slowly compared to $\psi(x)$

$$\psi_0\,(x) = e^{i/\hbar \int^x dx' \sqrt{2m[E-V(x')]}}$$

However, we are interest in tunneling process…

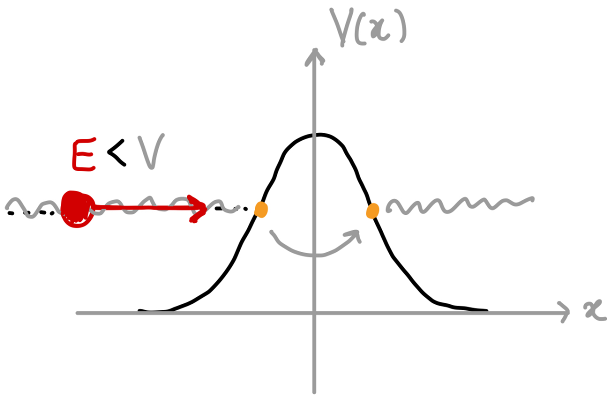
Classical Mechanics



particle is reflected

(classically forbidden region)

Quantum Mechanics



non-vanishing probability of tunneling!

barrier penetration

But what if we want to describe the barrier penetration with a classical trajectory?

$$\psi_{0,tunnel}\,(x) = e^{-1/\hbar \int_{x_l}^{x_u} dx' \sqrt{2m[V(x')-E]}}$$

This is possible if we go to imaginary time!

Formally, this is identical to the replacement $\quad t \to -i\tau \quad \Rightarrow \quad E \to iE, \quad V \to iV$

Euclidean Path Integral · Imaginary time

We will work on one-dimensional non-relativistic QM

First, let's remember the propagator at real time:

matrix element of
time evolution operator

$$\left\langle x_f \left| e^{-iHT/\hbar} \right| x_i \right\rangle$$

H is the generator of time-translations

for time-independent Hamiltonian H

This element represent the probability amplitude for the particle to propagate from

$$x_i \quad \text{at} \quad t = -\frac{T}{2} \quad \Rightarrow \quad x_f \quad \text{at} \quad t = \frac{T}{2}$$

The path integral representation is

$$\left\langle x_f \left| e^{-iHT/\hbar} \right| x_i \right\rangle = \mathcal{N} \int D[x] e^{i\frac{S[x]}{\hbar}} = \mathcal{N} \underbrace{\int D[x]}_{\text{sum over all paths}}\underbrace{{}_{\{x(-T/2)=x_i \,|\, x(T/2)=x_f\}}}_{\text{boundary conditions}} e^{\frac{i}{\hbar}\int_{-T/2}^{T/2} dt \mathcal{L}(x,\dot{x})}$$

where the classical action $S[x]$ of the path $x(t)$ is:

$$S\left[x\right] = \int_{-T/2}^{T/2} dt \mathcal{L}\left(x, \dot{x}\right) = \int_{-T/2}^{T/2} dt \left\{\frac{m}{2}\dot{x}^2\left(t\right) - V\left[x\left(t\right)\right]\right\}$$

Euclidean Path Integral · Imaginary time

The measure $\mathcal{D}[x]$ is defined as:

$$\int \mathcal{D}[x] = \lim_{\substack{t\to 0\\ N\to\infty}} \left(\frac{m}{2\pi i\hbar t}\right)^{N/2} \int_{-\infty}^{\infty} dx_{N-1} \cdots \int_{-\infty}^{\infty} dx_1$$

with $N\,t = \frac{T}{2} + \frac{T}{2} = T$

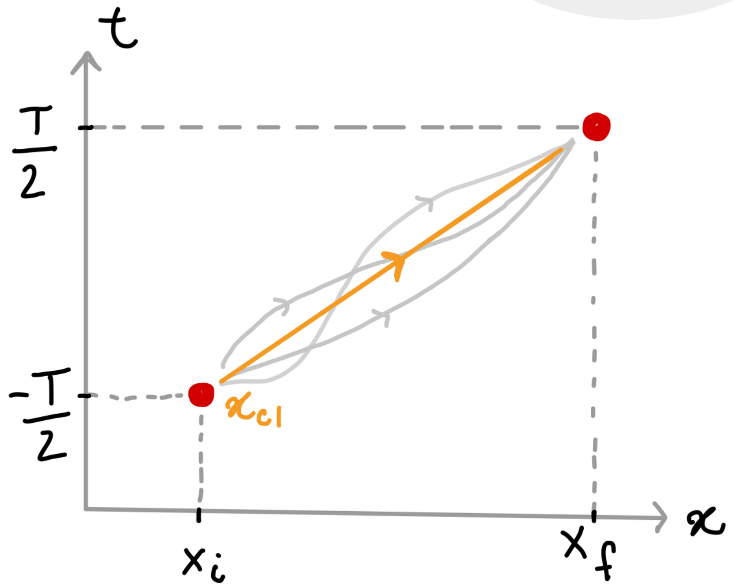
SC limit: Classical action is much larger than $\hbar$ → path integral dominated by the paths in the vicinity of the stationary point(s) of the action

Stationary phase approximation

$$\boxed{\frac{-\delta}{\delta x(t)} S[x] = m\ddot{x} + \frac{\partial V}{\partial x} = 0}$$

with boundary conditions

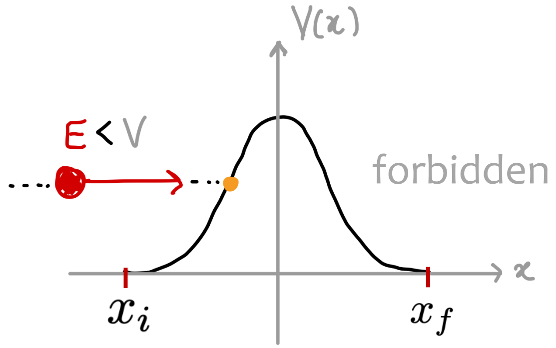
$$x_{cl}\left(-\frac{T}{2}\right) = x_i, \qquad x_{cl}\left(\frac{T}{2}\right) = x_f$$



action is stationary under variation of the critical/classical path

Coherence region → infinitely many neighboring paths add coherently!

Euclidean Path Integral · Imaginary time



However, the stationary-phase approximation cannot describe tunneling processes...

$S[x]$ has no extrema with tunneling boundary conditions!

Solution: analytic continuation to imaginary times

$$\left\langle x_f \left| e^{-iHT/\hbar} \right| x_i \right\rangle \xleftrightarrow[T_E \,=\, \text{Euclidean time}]{t \to -i\tau \qquad (\Rightarrow T \to -iT_E)} Z\left(x_f, x_i\right) = \left\langle x_f \left| e^{-HT_E/\hbar} \right| x_i \right\rangle$$

Minkowsi time	t
Euclidean time	τ

- This procedure is also called Wick Rotation
- Note that the imaginary-time evolution operator is NOT unitary ⟶ does not conserve probability
- Statistical Mechanics approach:

partition function $Z(\beta) = \mathrm{tr}\, \mathrm{e}^{-\beta H(\beta)} \Rightarrow \mathrm{e}^{-\beta H} \xleftrightarrow[\text{SC limit: } \beta \to \infty]{\beta \mapsto \tau/\hbar} \mathrm{e}^{-\tau H/\hbar}$

periodic trajectories

Euclidean Path Integral · Energy levels

We are interested in the low-lying energy levels in the SC limit, in particular the ground state (GS)

$$Z\left(x_f, x_i\right) = \sum_n e^{-E_n T/\hbar} \left\langle x_f | n \right\rangle \left\langle n | x_i \right\rangle$$

Energy spectrum: $H\left|n\right\rangle = E_n \left|n\right\rangle\,, \qquad 1 = \sum_n \left|n\right\rangle \left\langle n\right|$

For large Euclidean time T , the GS energy dominates:

$$Z\left(x_f, x_i\right) \to e^{-E_0 T/\hbar} \left\langle x_f | 0 \right\rangle \left\langle 0 | x_i \right\rangle \quad \Rightarrow \quad E_0 = -\hbar \lim_{T\to\infty} \frac{1}{T} \ln Z\left(x_f, x_i\right)$$

Statistical Mechanics interpretation: the SC limit $\beta \to \infty$ represents the low temperature limit $\longrightarrow$ $\mathrm{e}^{-\beta H}$ projects onto the GS

$$E_0 = -\lim_{\beta\to\infty} \frac{1}{\beta} \log\, Z(\beta)$$

Therefore, to calculate the GS energy (and wave function) we just need to take the $T \to \infty$ limit of the imaginary-time matrix element.

Euclidean Path Integral · Saddle point approximation

Path integral formulation in imaginary time:

$$Z(x_f, x_i) = \left\langle x_f \left| e^{-HT_E/\hbar} \right| x_i \right\rangle \quad \Rightarrow \quad Z(x_f, x_i) = \mathcal{N} \int D[x] e^{-\frac{S_E[x]}{\hbar}}$$

$t \to -i\,\tau$

$dt \to -i\, d\tau$

$$= \mathcal{N} \int D[x]_{\{x(-T/2)=x_i \,|\, x(T/2)=x_f\}} e^{-\frac{1}{\hbar}\int_{-T/2}^{T/2} d\tau \mathcal{L}_E(x,\dot{x})}$$

We want to define the measure more formally. Let's expand $x(\tau)$ into a complete, orthonormal set of real functions $\tilde{x}_n(\tau)$ around a fixed path $\bar{x}(\tau)$

$$x(\tau) = \bar{x}(\tau) + \underbrace{\eta(\tau)}_{\text{variation}} \qquad \text{with} \qquad \eta(\tau) = \sum_{n=0}^{\infty} c_n \tilde{x}_n(\tau)$$

Orthogonality and completeness relation

$$\int_{-T/2}^{T/2} d\tau \tilde{x}_n(\tau) \tilde{x}_m(\tau) = \delta_{mn}, \qquad \sum_n \tilde{x}_n(\tau) \tilde{x}_n(\tau') = \delta(\tau - \tau')$$

Boundary conditions

$$\bar{x}(\pm T/2) = x_f / x_i \qquad \text{and} \qquad \tilde{x}_n(\pm T/2) = 0$$

With this parametrization we can write

$$\boxed{D[x] = D[\eta] = \prod_n \frac{dc_n}{\sqrt{2\pi\hbar}}}$$

convenience factor

Euclidean Path Integral · Saddle point approximation

Now, let's obtain the explicit form of the Euclidean Lagrangian

$$iS = i\int_{-T/2}^{T/2} dt\left(\frac{m}{2}\dot{x}^2 - V[x]\right) \rightarrow i\int_{-\frac{T}{2}e^{-i\pi/2}}^{\frac{T}{2}e^{-i\pi/2}} dt\left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V[x]\right] \equiv -S_E$$

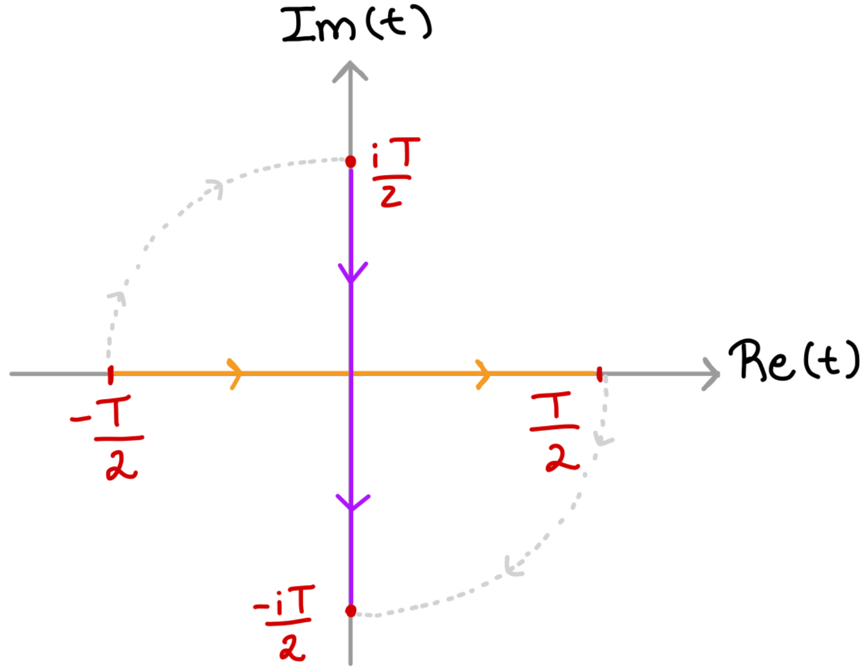
$t \rightarrow -i\tau$

$$\int_{-T/2}^{T/2} dt \rightarrow \int_{+i\frac{T}{2}}^{-i\frac{T}{2}} dt \rightarrow \int_{-T/2}^{T/2} (-i\,d\tau)$$

substituting $t \rightarrow -i\,\tau \quad \Rightarrow \quad \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = i\,\frac{dx}{d\tau} \quad \Rightarrow \quad \left(\frac{dx}{dt}\right)^2 = -\left(\frac{dx}{d\tau}\right)^2 \equiv -\dot{x}^2$

$$-S_E = i(-i)\int_{-T/2}^{T/2} d\tau\left(-\frac{m}{2}\dot{x}^2 - V[x]\right) \equiv -\int_{-T/2}^{T/2} d\tau\mathcal{L}_E[x]$$

$$\therefore \quad \boxed{\mathcal{L}_E[x] = \frac{m}{2}\dot{x}^2 + V(x)}$$



Euclidean Path Integral · Saddle point approximation

$$\mathcal{L}_E[x] = \frac{m}{2}\dot{x}^2 + V(x)$$

The potential changed sign!

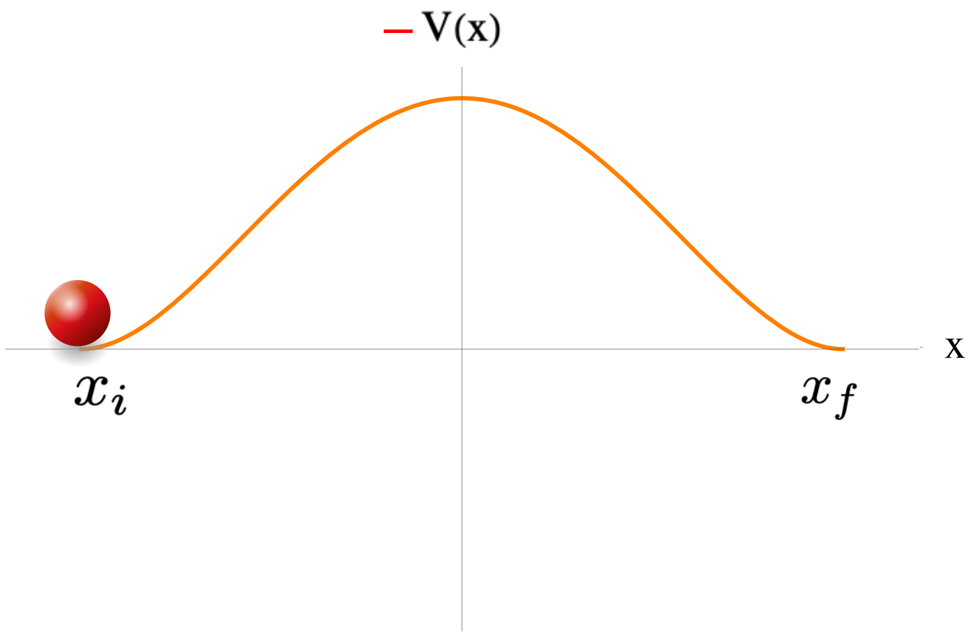
Saddle point approximation:

$$\frac{-\delta}{\delta x(t)} S[x] = m\ddot{x} + \frac{\partial V}{\partial x} = 0 \quad \longleftrightarrow \quad \frac{-\delta}{\delta x(\tau)} S_E[x] = m\ddot{x}_{cl} - V'(x_{cl}) = 0$$

real time — imaginary time

Consequence: now we have solutions to the imaginary-time equation of motion with tunneling boundary conditions!

$$x\left(-\frac{T}{2}\right) = x_i, \qquad x\left(\frac{T}{2}\right) = x_f$$



These solutions carry a conserved quantum number:

Euclidean energy

$$E_E = \frac{m}{2}\dot{x}^2 - V(x)$$

Euclidean Path Integral · Saddle point approximation

Finally, we can perform the saddle-point approximation explicitly.

Fluctuation around the classical path:

$$x(\tau) = x_{cl}(\tau) + \eta(\tau)$$

SC limit: the only nonvanishing contributions com from a neighborhood of x_{cl}

(least suppressed from Boltzmann weight $\exp(-S_E/\hbar)$)

Expanding the action to order $\mathcal{O}(\eta^2)$

$$S_E[x] = S_E[x_{cl}] + \frac{1}{2}\int d\tau \int d\tau' \eta(\tau) \frac{\delta^2 S_E[x_{cl}]}{\delta x(\tau)\,\delta x(\tau')}\eta(\tau') + O(\eta^3)$$

$$= S_E[x_{cl}] + \frac{1}{2}\int_{-T/2}^{T/2} d\tau \eta(\tau)\,\hat{F}(x_{cl})\,\eta(\tau) + O(\eta^3)\,,$$

where we defined the operator

$$\frac{\delta^2 S_E[x]}{\delta x(\tau)\,\delta x(\tau')} = \left[-m\frac{d^2}{d\tau^2} + \frac{d^2 V(x_{cl})}{dx^2}\right]\delta(\tau-\tau') \equiv \hat{F}(x_{cl})\,\delta(\tau-\tau')$$

For deduction see (2.32)

This operator drives the dynamics of the fluctuation around the classical path.

Euclidean Path Integral · Saddle point approximation

Now we expand the fluctuations into the basis of real eigenfunctions of $\hat{F}$

$$\eta(\tau) = \sum_n c_n \tilde{x}_n(\tau) \qquad \text{with} \qquad \hat{F}(x_{cl})\,\tilde{x}_n(\tau) = \lambda_n \tilde{x}_n(\tau)$$

boundary conditions
$\tilde{x}_n(\pm T/2) = 0$

We can then write the action as

$$S_E[x] = S_E[x_{cl}] + \frac{1}{2}\sum_n \lambda_n c_n^2 + O(\eta^3)$$

For deduction see (2.35)

Using the definition of measure

$$D[x] = D[\eta] = \prod_n \frac{dc_n}{\sqrt{2\pi\hbar}} \quad \Rightarrow \quad Z(x_f, x_i) = \mathcal{N} e^{-\frac{S_E[x_{cl}]}{\hbar}} \prod_n \int_{-\infty}^{\infty} dc_n \frac{e^{-\frac{1}{2\hbar}\lambda_n c_n^2}}{\sqrt{2\pi\hbar}}$$

performing the Gaussian integrals:

$$Z(x_f, x_i) = \mathcal{N} e^{-\frac{S_E[x_{cl}]}{\hbar}} \prod_n \lambda_n^{-1/2} \equiv \mathcal{N} e^{-\frac{S_E[x_{cl}]}{\hbar}} \underbrace{\left(\det \hat{F}[x_{cl}]\right)^{-1/2}}_{\text{functional determinant}}$$

Instanton Solution · The Double Well Potential

We will first study tunneling process between degenerate potential minima

↳ Classically, we have 2 degenerate energy levels, located at the two degenerate minima: parity symmetry is spontaneously broken.

But we know that the GS can't be degenerate!

Tunneling is responsible for the splitting between these levels.

↳ The GS wavefunction must corresponds to the symmetric combination of the two perturbative vacua.

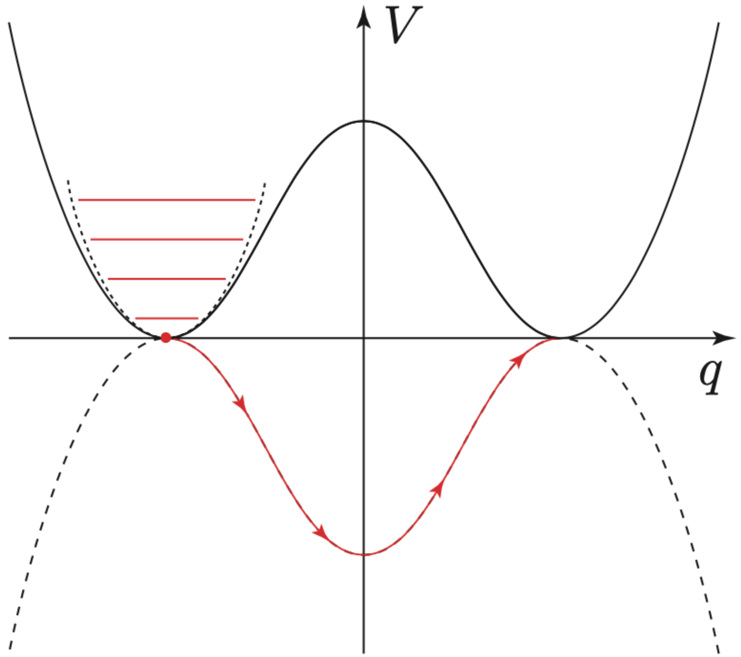
$$[P, H] = 0$$

$$[P\psi_{n,\pm}](q) = \psi_{n,\pm}(-q) = \overbrace{\pm\psi_{n,\pm}(q)}^{\text{even/odd eigenfunctions}}.$$

Let's consider the potential

$$V(x) = \frac{\alpha^2 m}{2x_0^2}\left(x^2 - x_0^2\right)^2$$

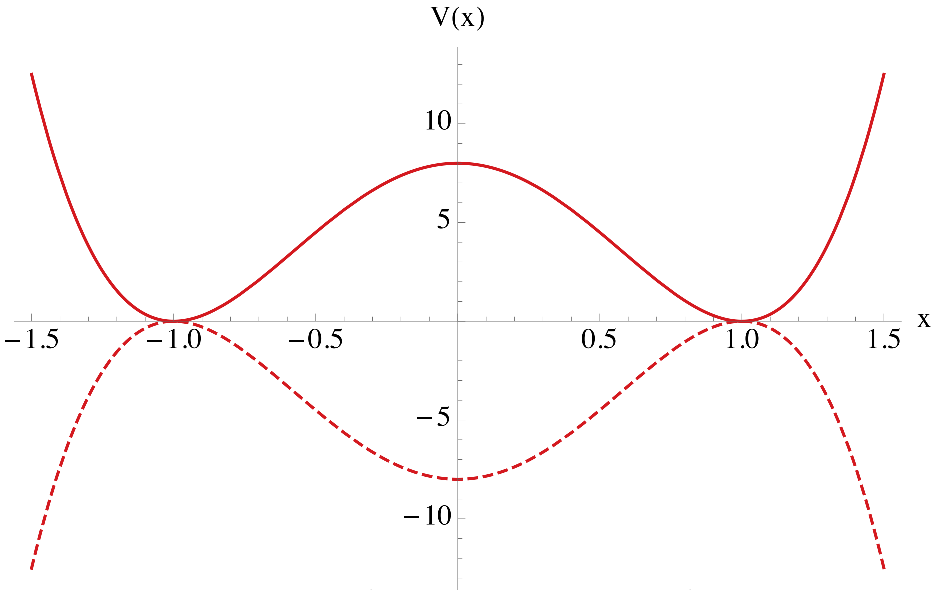
Instantons have the ability to lift perturbation theory degeneracies!



Extracted from Ref. [7].

Instanton Solution · The Double Well Potential

potential: $V(x) = \frac{\alpha^2 m}{2x_0^2}\left(x^2 - x_0^2\right)^2$



$m = 1\,;\alpha = 4\,;x_0 = 1$

Saddle point solutions:

trivial static solutions
$$\begin{cases} 1) & x_{cl}(\tau) = x_0 \\ 2) & x_{cl}(\tau) = -x_0 \end{cases}$$

They do not contribute to tunneling. But they contribute to $Z(x_0, x_0)$ or $Z(-x_0, -x_0)$

The other saddle point is time-dependent and correspond to the tunneling solution that interpolates between the minima:

tunneling boundary conditions
$$x_{cl}(-T/2) = +x_0,$$
$$x_{cl}(T/2) = -x_0,$$
(or in the opposite direction)

To obtain an analytic solution we start with the Euclidean energy $E_E = 0$ → correspond to $\lim T \to \infty$, because vanishing initial velocity requires infinite time.

Instanton Solution · The Double Well Potential

$$E_E = \frac{m}{2}\dot{x}^2 - V = 0 \quad \Rightarrow \quad \dot{x} = \mp\sqrt{\frac{2V}{m}} \quad \Rightarrow \quad \mp\sqrt{\frac{m}{2}}\frac{dx}{\sqrt{V(x)}} = d\tau,$$

If we integrate, we obtain:

$$\mp\sqrt{\frac{m}{2}}\int_{x_{cl}(\tau_0)}^{x_{cl}(\tau)}\frac{dx}{\sqrt{V(x)}} = \mp\frac{x_0}{\alpha}\int_{x_{cl}(\tau_0)}^{x_{cl}(\tau)}\frac{dx}{x_0^2 - x^2} = \mp\frac{x_0}{\alpha}\frac{1}{x_0}\,\mathrm{arctan}\,h\left(\frac{x}{x_0}\right)\Bigg|_{x_{cl}(\tau_0)}^{x_{cl}(\tau)} = \int_{\tau_0}^{\tau} d\tau = \tau - \tau_0.$$

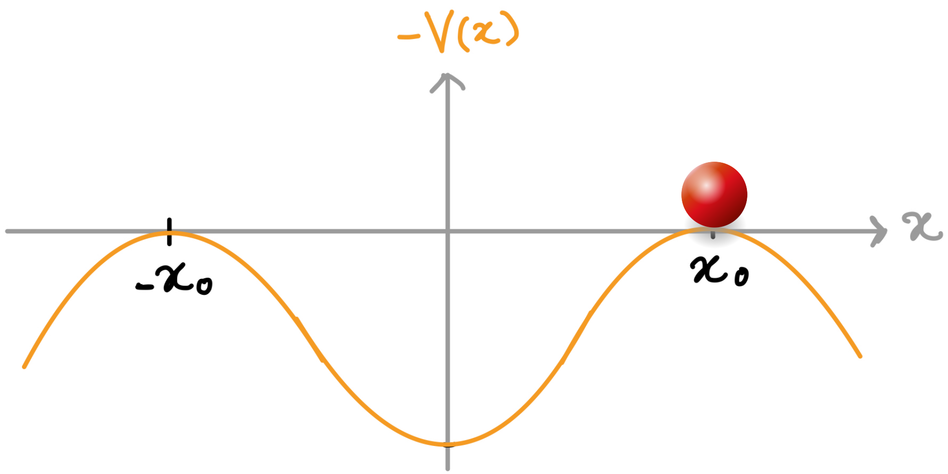
integration constant

We choose τ_0 to be the "center" of the solution by requiring $x_{cl}(\tau_0) = 0$

$$\boxed{3) \qquad x_{cl}(\tau) \equiv x_{I/\bar{I}}(\tau) = \mp x_0 \tanh\alpha(\tau - \tau_0)}$$

$-$ instanton solution

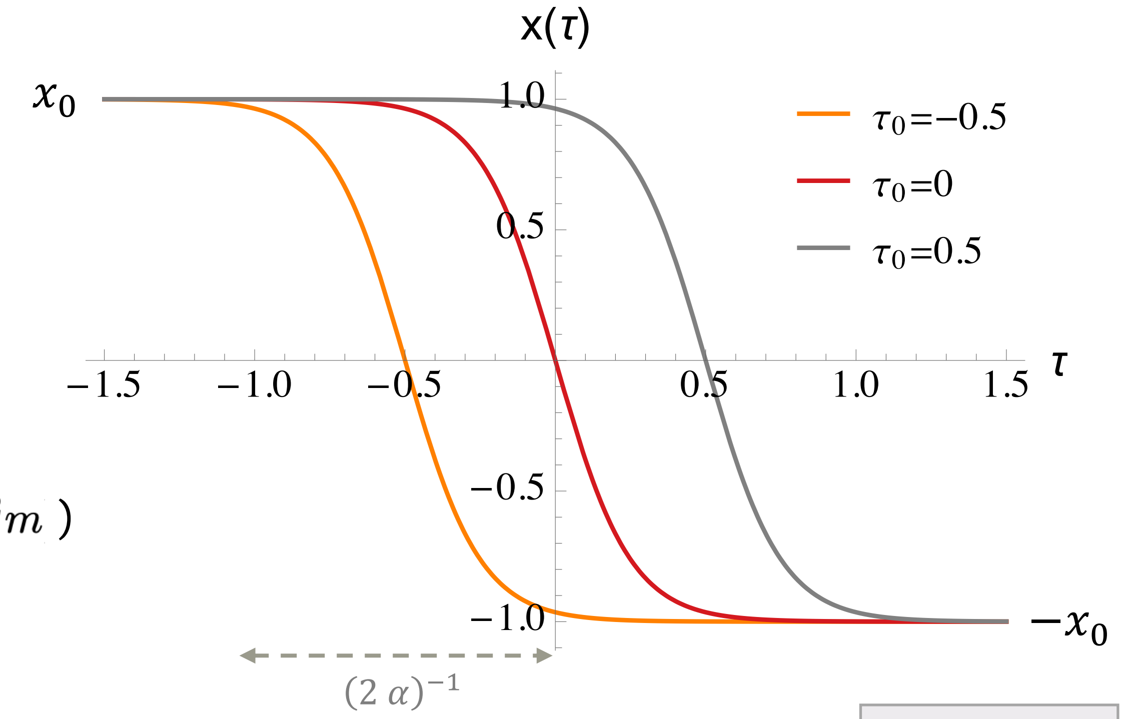
$+$ anti-instanton solution



Instanton Solution · The Double Well Potential

$$x_{cl}(\tau) \equiv x_{I/\bar{I}}(\tau) = \mp x_0 \tanh \alpha(\tau - \tau_0)$$

Note that the tunneling transition happens very fast, almost instantaneous → name "instanton"



from $E_E = 0$ $\Rightarrow$ $$\frac{dx_I}{d\tau} = -\sqrt{\frac{2}{m} V(x_I)}$$

At large τ we expand the potential around $-x_0$ (using $V''(-x_0) = 4\alpha^2 m$)

$$\frac{dx_I}{d\tau} \simeq -2\alpha\left[x_I - (-x_0)\right]$$

$\Rightarrow$ $\Delta x_I(\tau_0 + \tau) \to \Delta x_I(\tau_0)\, e^{-2\alpha\tau}$ where we defined the deviation $\Delta x_I(\tau) \equiv x_I(\tau) + x_0$

For deduction see (2.54)-(2.57)

The characteristic time scale $\tilde{\tau} = \dfrac{1}{2\alpha}$ of the decay becomes arbitrarily small for large α.

↳ "abruptness" of the transition increases with the coupling parameter, related to the height of the potential barrier.

Euclidean action of the instanton solution

$$S_E[x_I] = m\int_{-T/2}^{T/2} d\tau\, \dot{x}_I^2 = m\int_{x_0}^{-x_0} dx_I\, \dot{x}_I = -\frac{\alpha m}{x_0}\int_{x_0}^{-x_0} dx_I\left(x_0^2 - x_I^2\right) = \frac{4}{3}\alpha m x_0^2$$

Instanton Solution · The Zero Mode

From the e.o.m: $m\ddot{x}_{cl} - V'(x_{cl}) = 0 \underset{\text{time derivative}}{\Rightarrow} \left[m\partial_\tau^2 - V''(x_I)\right]\dot{x}_I = -\hat{F}\dot{x}_I = 0 \Rightarrow \hat{F}\tilde{x}_0(\tau) = 0$

eigenvalue $\lambda_0 = 0$

Zero mode!

The zero mode eigenfunction, correctly normalized, is:

$$\tilde{x}_0(\tau) = \sqrt{\frac{m}{S_I}}\dot{x}_I = -\frac{\sqrt{3\alpha}}{2}\frac{1}{\cosh^2(\alpha(\tau-\tau_0))}$$

Interpretation: unlike the constant solutions, the instanton solution is well localized in imaginary time. Due to the time-translation invariance of the Hamiltonian, they form a one-parameter family of degenerate saddle points related by continuous time-translations, whose members are parametrized by their time center τ_0.

collective coordinate

↳ we need to sum over the contributions from all τ_0 !

We also know that the spectrum of a Schrodinger operator has the property that the GS has no nodes, the first excited state has one node, etc.

↳ In the double-well potential, $\dot{x}_I$ never vanishes ⟶ must correspond to the GS and all other eigenvalues are positive!

Therefore, the fluctuation determinant is well behaved (Gaussian) for all λ_n $(n \neq 0)$, but we still need to deal with the c_0 integration:

$$\int_{-\infty}^{\infty}\frac{dc_0 e^{-\frac{1}{2\hbar}\lambda_0 c_0^2}}{\sqrt{2\pi\hbar}} = \int_{-\infty}^{\infty}\frac{dc_0}{\sqrt{2\pi\hbar}},$$

Instanton Solution · The Zero Mode

However, this kind of divergence is exactly what we expect from integrating over the infinite set of saddle points!

We can interchange the integration $dc_0 \longrightarrow d\tau_0$ by comparing the deviations dx from a given path $x(\tau)$:

$$\text{small time translations } \tau_0 \to \tau_0 + d\tau_0 \quad \Rightarrow \quad dx = \frac{dx_I}{d\tau_0} d\tau_0 = \dot{x}_I d\tau_0, \quad \text{and}$$

$$\text{small coefficient shifts } c_0 \to c_0 + dc_0 \quad \Rightarrow \quad dx = \frac{dx}{dc_0} dc_0 = \tilde{x}_0 dc_0 = \sqrt{\frac{m}{S_I}} \dot{x}_I dc_0.$$

$$\Rightarrow \quad dc_0 = \sqrt{\frac{S_I}{m}} d\tau_0.$$

Therefore:

$$\int_{-\infty}^{\infty} \frac{dc_0}{\sqrt{2\pi\hbar}} = \int_{-T/2}^{T/2} \sqrt{\frac{S_E(x_I)}{2\,\pi\hbar m}}\, d\tau_0 = \sqrt{\frac{S_E(x_I)}{2\,\pi\hbar m}}\; T$$

and we obtain

$$\boxed{Z_I(-x_0, x_0) = \mathcal{N} e^{-\frac{S_E[x_I]}{\hbar}} \sqrt{\frac{S_E[x_I]}{2\pi\hbar m}}\, T \left(\det \hat{F}[x_I]'\right)^{-1/2}}$$

determinant without the zero mode

in the $\lim T \to \infty$: this divergence is not disturbing since it will be cancelled by other infinities.

Instanton Solution · Functional Determinant

Finally, we need to compute:

$$\det \hat{F}\left[x_I\right]' = \det\left[-m\frac{d^2}{d\tau^2}+V''\left(x_{cl}\right)\right]'$$

For deduction see eq. (2.70)-(2.85)

To this purpose we will use the result of the harmonic oscillator functional determinant:

$$\hat{F}\left[x_{ho}\right] = -\frac{d^2}{d\tau^2}+\omega^2 \quad \Rightarrow \quad Z_{ho}\left(0,0\right) = \mathcal{N}\left(\det \hat{F}\left[x_{ho}\right]\right)^{-1/2} = \sqrt{\frac{m\hbar\omega}{2\pi}}\left[\sinh\left(\omega T\right)\right]^{-1/2} \xrightarrow[\lim T \to \infty]{} \sqrt{\frac{m\hbar\omega}{\pi}}e^{-\omega T/2}$$

from which we recover, using $E_0 = -\hbar \lim_{T\to\infty}\frac{1}{T}\ln Z\left(x_f, x_i\right) \quad \Rightarrow \quad E_{0,ho} = \frac{1}{2}\hbar\omega.$

To compute Z_I we will write the normalization as $\mathcal{N} = \dfrac{Z_{ho}\left(0,0\right)}{\left(\det \hat{F}\left[x_{ho}\right]\right)^{-1/2}}$

$$\boxed{\Rightarrow \quad Z_I\left(-x_0, x_0\right) = Z_{ho}\left(0,0\right)e^{-\frac{S_E\left[x_I\right]}{\hbar}}\sqrt{\frac{S_E\left[x_I\right]}{2\pi\hbar m}}\,\omega T\left\{\frac{\det \hat{F}\left[x_I\right]'}{\omega^{-2}\det \hat{F}\left[x_{ho}\right]}\right\}^{-1/2}}$$

Instanton Solution · Functional Determinant

To calculate det $\hat{F}\,[x_I]'$, we have to deal with the spectrum of the operator

$$\hat{F}\,[x_I] = -m\frac{d^2}{d\tau^2} + V''\,(x_I)$$

where

$$V''\,(x_I) = \frac{2\alpha^2 m}{x_0^2}\left(3x_I^2 - x_0^2\right) = 2\alpha^2 m\left[3\tanh^2\alpha\,(\tau-\tau_0) - 1\right] = 2\alpha^2 m\left[2 - \frac{3}{\cosh^2\alpha\,(\tau-\tau_0)}\right]$$

So, we have to solve

$$\Rightarrow \left[-\frac{d^2}{d\tau^2} \overbrace{+\,4\alpha^2 - \frac{6\alpha^2}{\cosh^2(\alpha\tau)}}^{\tilde{V}(\tau)}\right]\tilde{x}_\lambda\,(\tau) = \lambda\tilde{x}_\lambda\,(\tau)$$

boundary conditions
$\tilde{x}_n\,(\pm T/2) = 0$

Because this eq. is of "Schrodinger type", we know that the eigenvalue spectrum will be discrete for the "bound" states with $\lambda < 4\,\alpha^2$ and continuous when $\lambda > 4\,\alpha^2$

$$\lim_{\tau\to\infty}\tilde{V}(\tau) = 4\,\alpha^2$$

We can solve this equation analytically, first we substitute:

$$\xi = \tanh\,(\alpha\tau) \Rightarrow \left[\frac{d}{d\xi}\left(1-\xi^2\right)\frac{d}{d\xi} - \frac{\epsilon^2}{1-\xi^2} + 6\right]\tilde{x}_\lambda\,(\xi) = 0 \quad \text{where} \quad \epsilon^2 = 4 - \frac{\lambda}{\alpha^2}$$

Instanton Solution · Functional Determinant

The solutions are

$$\tilde{x}_\lambda(\xi) = c_1 P_2^\epsilon(\xi) + c_2 Q_2^\epsilon(\xi)$$

associated Legendre functions of the first and second kind

discrete levels: $\epsilon^2 > 0$ ("bound states")

$$\begin{cases} \lambda_0 = 0, & \text{zero mode} \Rightarrow \text{associated with } P_2^2(\xi) = 3\left(1-\xi^2\right) = 3/\cosh^2(\alpha\tau) \propto \dot{x}_I \\ \lambda_1 = 3\alpha^2. \end{cases}$$

continuum states: $\epsilon^2 < 0$: the associated Legendre functions are no longer convenient...

re-writing $\tilde{x}_\lambda(\xi) = (1-\xi^2)^{\epsilon/2} w(\xi)$ + $u = \dfrac{1-\xi}{2}$ $\Rightarrow$ $(\cdots)$

see eq. (2.99) – (2.102)

$$\Rightarrow \quad \left[u(1-u)\frac{d^2}{du^2} + (\epsilon+1)(1-2u)\frac{d}{du} - (\epsilon - s)(\epsilon + \underbrace{s}_{s=2} + 1)\right] w(u) = 0$$

hypergeometric differential equation

solution: $w(u) = c_3\, {}_2F_1\left[\dfrac{ik}{\alpha} - s, s + \dfrac{ik}{\alpha} + 1, 1 + \dfrac{ik}{\alpha}, u\right] + c_4 u^{-ik/\alpha}\, {}_2F_1\left[-s, s+1, 1 - \dfrac{ik}{\alpha}, u\right]$

where $\epsilon = ik/\alpha$

Instanton Solution · Functional Determinant

For $s = 2$:

$$w(\xi) = c_3 \left(\frac{1+\xi}{2}\right)^{-ik/\alpha} \left[k^2 - 3ik\alpha\xi + 2\alpha^2(2+3\xi)\right] + c_4 \left(\frac{1-\xi}{2}\right)^{-ik/\alpha} \left[1 + \frac{3\alpha(1-\xi)(-ik+\alpha(1+\xi))}{(k+i\alpha)(k+2i\alpha)}\right]$$

$$\Rightarrow \quad \tilde{x}_\lambda(\tau) = \tilde{c}_3 e^{-ik\tau}\left[k^2 - 3ik\alpha\tanh(\alpha\tau) + 2\alpha^2(2+3\tanh(\alpha\tau))\right]$$

see eq. (2.104) – (2.113)

$$+ \tilde{c}_4 e^{ik\tau}\left[1 + \frac{3\alpha(1-\tanh(\alpha\tau))(-ik+\alpha(1+\tanh(\alpha\tau)))}{(k+i\alpha)(k+2i\alpha)}\right]$$

Now we use the fact that the eigenvalue eq. is a local equation, so we can obtain the eigenvalues in the asymptotic region $|\tau| \rightarrow \infty$, where the potential induced by the instanton field vanishes:

$$\left[\frac{d^2}{d\tau^2} + k^2\right]\tilde{x}_\lambda(\tau) = 0. \qquad \text{where} \qquad k^2 \equiv \lambda - 4\alpha^2 \;\; (\geq 0)$$

solution: “plane waves”

Elastic scattering $\Rightarrow$ only effect of the potential can be a k-dependent phase shift

$$\begin{cases} \tilde{x}_\lambda(\tau) \propto e^{ik\tau + i\delta_k} & \text{for } \tau \rightarrow -\infty, \\ \tilde{x}_\lambda(\tau) \propto e^{ik\tau} & \text{for } \tau \rightarrow +\infty. \end{cases}$$

Instanton Solution · Functional Determinant

Therefore:

$\lim \tau \to -\infty$

$$\tilde{x}_\lambda(\tau) \to \tilde{c}_3 e^{-ik\tau}\left[k^2 + 3ik\alpha - 2\alpha^2\right] + \tilde{c}_4 e^{ik\tau}\left[\frac{(1+ik/\alpha)(2+ik/\alpha)}{(1-ik/\alpha)(2-ik/\alpha)}\right]$$

where we can read the phase shifts

$$\delta_k = -i\ln\left[\left(\frac{1+ik/\alpha}{1-ik/\alpha}\right)\left(\frac{2+ik/\alpha}{2-ik/\alpha}\right)\right]$$

Now, to obtain the relation of the phase shifts with the eigenvalues k, we write: $\tilde{x}_{gen,\lambda}(\tau) = A\tilde{x}_\lambda(\tau) + B\tilde{x}_\lambda(-\tau)$

imposing the boundary conditions: $A\tilde{x}_\lambda\left(\frac{T}{2}\right) + B\tilde{x}_\lambda\left(-\frac{T}{2}\right) = A\tilde{x}_\lambda\left(-\frac{T}{2}\right) + B\tilde{x}_\lambda\left(-\frac{T}{2}\right) = 0.$

see eq. (2.120) – (2.123)

$$\Rightarrow \quad \frac{\tilde{x}_\lambda\left(-\frac{T}{2}\right)}{\tilde{x}_\lambda\left(\frac{T}{2}\right)} = e^{-ikT-i\delta_k} = \pm 1 \quad \Rightarrow \quad \boxed{k_n = \frac{n\pi - \delta_k}{T}}$$

Due to the boundary conditions the k_n are discrete for finite T and become continuous in the limit $T \to \infty$.

Instanton Solution · Functional Determinant

Now, finally, we are able to calculate the functional determinant!

(Remember $k^2 \equiv \lambda - 4\alpha^2$)

discrete eigenvalue = $3\alpha^2$

$$\frac{\det \hat{F}\left[x_I\right]'}{\omega^{-2}\det \hat{F}\left[x_{ho}\right]} = \frac{\lambda_1}{\lambda_{ho,2}}\frac{\Pi_{n=1}\left(k_n^2+4\alpha^2\right)}{\Pi_{n=3}\left(k_{ho,n}^2+\omega^2\right)} = \frac{3}{4}\frac{\Pi_{n=1}\left(k_n^2+4\alpha^2\right)}{\Pi_{n=3}\left(k_{ho,n}^2+4\alpha^2\right)}, \qquad (\omega = 2\alpha)$$

and $\lambda_{ho,n} = \Bigg(\overbrace{\frac{n\pi}{T}}^{k_{ho,n}}\Bigg)^2 + \omega^2$. To perform the calculation, we write:

see eq. (2.126) – (2.136)

$$\frac{\Pi_{n=1}\left(k_n^2+4\alpha^2\right)}{\Pi_{n=1}\left(k_{ho,n}^2+4\alpha^2\right)} = \exp\sum_{n=1}^{\infty}\ln\left[\frac{k_n^2+4\alpha^2}{k_{ho,n}^2+4\alpha^2}\right] = (\cdots) = \exp\left(-\frac{1}{\pi}\int_0^\infty dk\,\frac{2\delta_k k}{k^2+4\alpha^2}\right)$$

where

$$\int_0^\infty dk\,\frac{2k\delta_k}{k^2+4\alpha^2} = \pi\ln 9 \quad \Rightarrow \quad \frac{\det \hat{F}\left[x_I\right]'}{\omega^{-2}\det \hat{F}\left[x_{ho}\right]} = \frac{1}{12}$$

Replacing in Z_I:

$$\boxed{Z_I\left(-x_0,x_0\right) = \sqrt{\frac{m\hbar\omega}{\pi}}e^{-\omega T/2}\omega T\sqrt{\frac{6S_E\left[x_I\right]}{\pi\hbar m}}e^{-\frac{S_E\left[x_I\right]}{\hbar}}}$$

propagator of the double-well tunneling problem to $\mathcal{O}(\hbar)$ in the SCA around a single instanton.

Multi-Instantons · Dilute Instanton Gas

Beyond the single-instanton solutions, there are additional saddle points which also contribute to SC tunneling amplitude for large T.

↳ approximate solutions of the stationary eq. involving further anti–instanton/instanton pairs: "multi-instanton solutions"

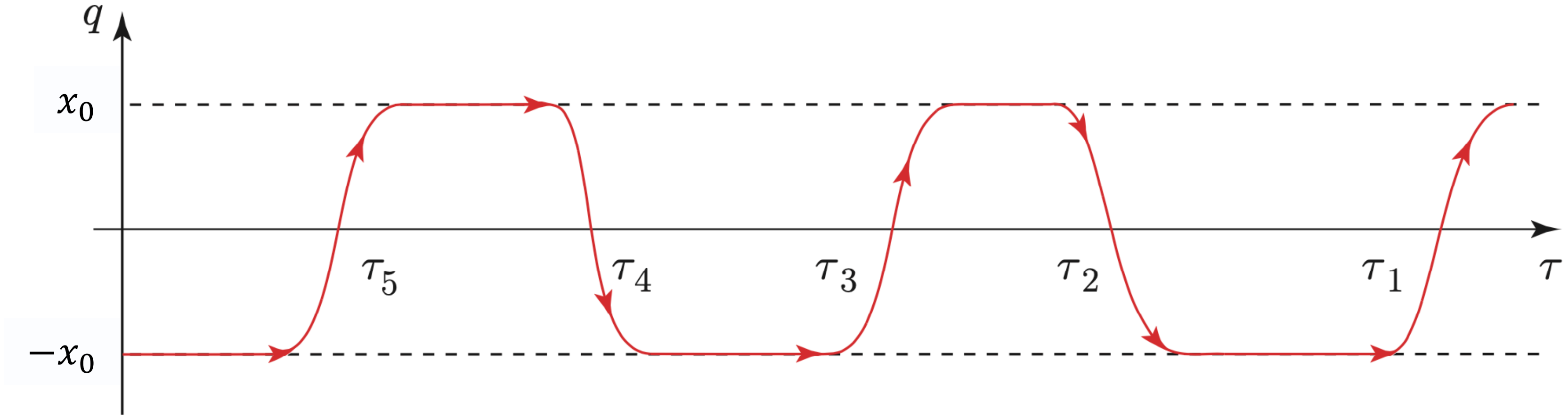
Instantons are well localized ⟶ deviates only in the interval $\Delta\tau = 1/\,(2\alpha)$ appreciably from x_0 or $-x_0$

very small overlap between neighboring instantons and anti-instantons

So we can write the multi-(anti-)-instanton solutions as a chain of N alternating instantons and anti-instantons, sufficiently far separated in time by the interval

$$\bar{\Delta}_\tau = \frac{T}{N} \gg \frac{1}{2\alpha}. \quad \Rightarrow \quad \text{solution:} \quad x_N(\tau) = \sum_{k=1}^{N} x_{I,\bar{I}}(\tau - \tau_{0,k})$$

N must be odd to satisfy boundary conditions



N tunneling processes, back and forth between both minima of the potential.

DILUTE INSTANTON GAS APPROXIMATION

(DIGA)

Extracted from Ref. [7].

Multi-Instantons · Dilute Instanton Gas

Due to the "diluteness", the (anti-)instantons have too little overlap to interact and we can write:

$$S_E\left[x_N+\eta\right] \simeq S_E\left[x_0+\eta_0\right]+\sum_{k=1}^{N} S_E\left[x_I+\eta_k\right]$$

fluctuations around the constant $x(\tau) = \pm x_0$

fluctuations around the single (anti-) instantons

$$\Rightarrow \quad Z_N\left(-x_0, x_0\right) \simeq \mathcal{N} \int D[\eta_0] e^{-S_E[x_0+\eta_0]/\hbar} \times \prod_{k=1}^{N} \mathcal{N} \int D[\eta_k] e^{-S_E[x_I+\eta_k]/\hbar}$$

$$= Z_0\left(x_0, x_0\right)\left[Z_I\left(-x_0, x_0\right)\right]^N .$$

N Zero modes: we write

$$Z_I = Z_I^{\prime} \sqrt{\frac{S_I}{2\pi\hbar m}} \int d\tau_0 \equiv \tilde{Z}_I \int d\tau_0$$

we need to integrate in each time center, but respecting the temporal ordering

$$\int_{-T/2}^{T/2} d\tau_{0,1} \int_{\tau_{0,1}}^{T/2} d\tau_{0,2} \ldots \int_{\tau_{0,N-1}}^{T/2} d\tau_{0,N} = \frac{T^N}{N!}$$

Instantons behave like identical particles!

Multi-Instantons · Dilute Instanton Gas

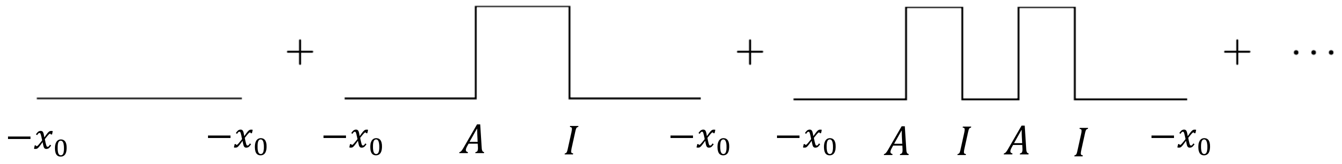
Therefore

$$\Rightarrow \quad Z_N \simeq Z_0 \frac{\left(\tilde{Z}_I T\right)^N}{N!} \quad \begin{cases} Z_0\left(\pm x_0, \pm x_0\right) = \mathcal{N}\left(\det\left[-\partial_\tau^2 + \omega^2\right]\right)^{-1/2} \to \left(\dfrac{m\hbar\omega}{\pi}\right)^{1/2} e^{-\omega T/2}, \\ \tilde{Z}_I = 2\alpha\sqrt{\dfrac{6 S_E\left[x_I\right]}{\pi\hbar m}} e^{-\frac{S_E[x_I]}{\hbar}} = 4\sqrt{\dfrac{2\alpha^3 x_0^2}{\pi\hbar}} e^{-\frac{4}{3}\alpha m x_0^2/\hbar}. \end{cases}$$

To collect all the multi-instantons contributions we must sum over all odd N

$$Z_{DIGA}\left(x_0, -x_0\right) = Z_0 \sum_{N \ \text{odd}} \frac{\left(\tilde{Z}_I T\right)^N}{N!} = \frac{Z_0}{2}\left\{e^{\tilde{Z}_I T} - e^{-\tilde{Z}_I T}\right\} = Z_0 \sinh\left(\tilde{Z}_I T\right)$$

Analogously, we can compute $Z_{DIGA}\left(-x_0, -x_0\right)$ summing even numbers of instantons:



Combining the results:

$$Z_{DIGA}\left(\pm x_0, -x_0\right) = \frac{1}{2}\left(\frac{\hbar\omega}{\pi}\right)^{1/2}\left\{e^{-\left(\omega/2 - \tilde{Z}_I\right)T} \mp e^{-\left(\omega/2 + \tilde{Z}_I\right)T}\right\}$$

Finally, to obtain the two lowest energy levels of the system, we can compare:

$\pm x_0$ $-x_0$

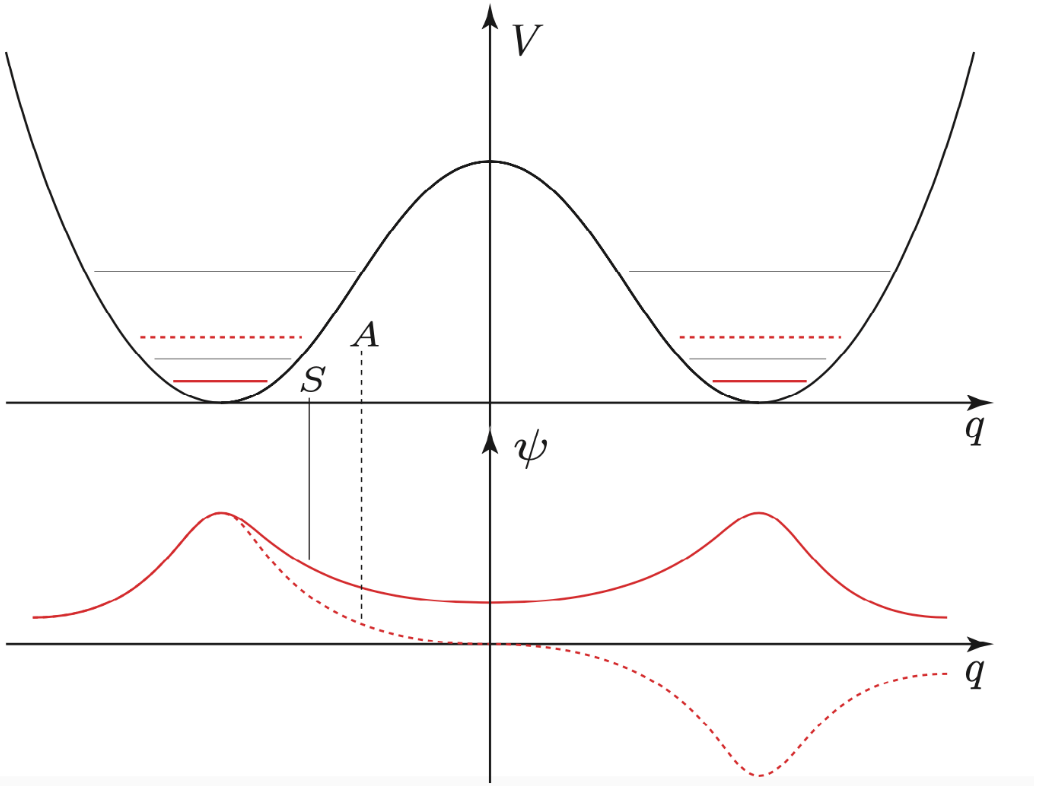
$$Z_{DIGA}(\pm x_0, -x_0) = \frac{1}{2}\left(\frac{\hbar\omega}{\pi}\right)^{1/2}\left\{e^{-\left(\omega/2-\tilde{Z}_I\right)T} \mp e^{-\left(\omega/2+\tilde{Z}_I\right)T}\right\} \longleftrightarrow Z(x_f, x_i) = \sum_n e^{-E_nT/\hbar}\langle x_f|n\rangle\langle n|x_i\rangle$$

$$\begin{cases} E_0 = \dfrac{\hbar\omega}{2} - \hbar\tilde{Z}_I, \\ E_1 = \dfrac{\hbar\omega}{2} + \hbar\tilde{Z}_I. \end{cases} \quad \Rightarrow \quad \Delta E \sim e^{-\frac{S_E[x_I]}{\hbar}}$$

the effect of tunneling is to split the degenerate GS energies!

and

$$\begin{cases} |0\rangle = \dfrac{1}{\sqrt{2}}\left\{|x_0\rangle + |-x_0\rangle\right\}, & \text{symmetric} \\ |1\rangle = \dfrac{1}{\sqrt{2}}\left\{|x_0\rangle - |-x_0\rangle\right\}. & \text{anti-symmetric} \end{cases}$$



The artificially broken parity in the absence of tunneling is restored.

Extracted from Ref. [7].

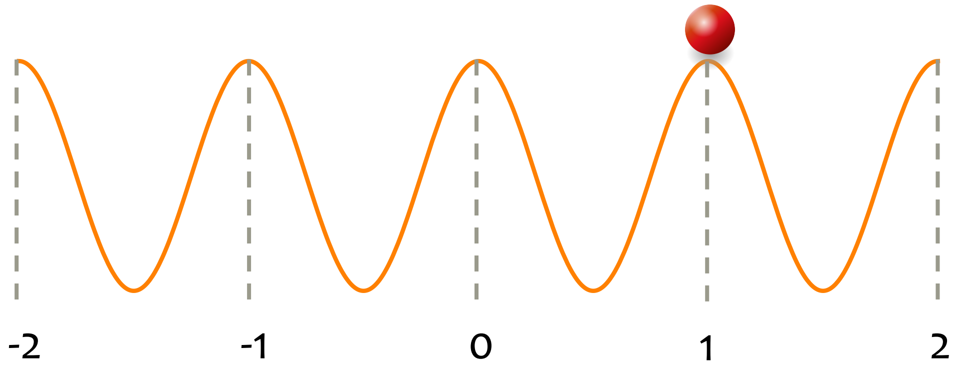
Multi-Instantons · Periodic Potential

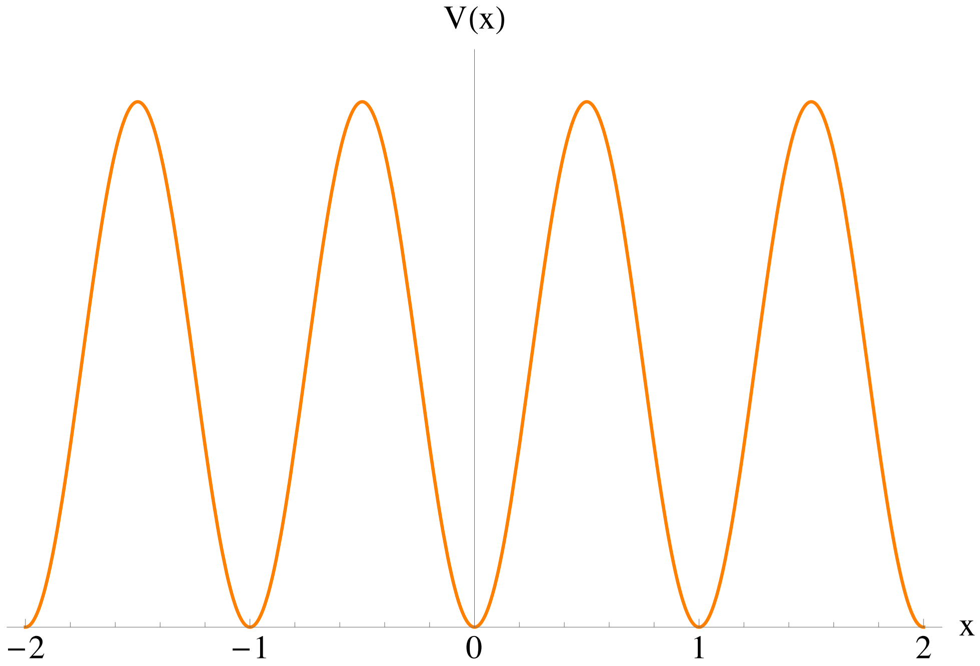
Let's consider now the periodic potential with degenerate minima:

- Resembles the QCD vacuum situation.
- Condensed matter physics: electrons in crystal lattices

The main difference is that now instantons and anti-instantons can arbitrarily follow each other:

$$\begin{cases} \text{Instantons:} \quad x = j \longrightarrow x = j - 1 \\ \text{Anti-instantons:} \quad x = j \longrightarrow x = j + 1 \end{cases}$$





Boundary conditions

$$x\left(-\frac{T}{2}\right) = x_{0,n_i}, \qquad x\left(\frac{T}{2}\right) = x_{0,n_f}$$

where

$$N_{\bar{I}} - N_I = n_f - n_i$$

Multi-Instantons · Periodic Potential

Then, we write the SC propagator as:

$$\delta_{ab} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(a-b)}$$

$$Z_{per}\left(x_{n_f}, x_{n_i}\right) \simeq Z_0 \sum_{N_I=0}^{\infty} \sum_{N_{\bar{I}}=0}^{\infty} \frac{\left(\tilde{Z}_I T\right)^{N+N_{\bar{I}}}}{N! N_{\bar{I}}!} \delta_{N_{\bar{I}}-N_I-(n_f-n_i)}$$

see eq. (2.163) – (2.169)

$$\Rightarrow \quad Z_{per}\left(x_{n_f}, x_{n_i}\right) \simeq Z_0 \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(n_f-n_i)} e^{\tilde{Z}_I T e^{-i\theta} + \tilde{Z}_I T e^{i\theta}} = \left(\frac{m\hbar\omega}{\pi}\right)^{1/2} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(n_f-n_i)} e^{-\left(\omega/2 - 2\tilde{Z}_I \cos\theta\right)T}.$$

Now, we can obtain the low-lying energy levels from the $T \to \infty$ limit:

$$E_0(\theta) = \frac{\hbar\omega}{2} - 2\hbar\tilde{Z}_I \cos\theta$$

continuous "band" of energies parametrized by θ

and the eigenstates are the Bloch waves:

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \left(\frac{\hbar\omega}{\pi}\right)^{1/4} \sum_n e^{in\theta} |n\rangle$$

state localized at the n-th minimum of the potential

Decay of a Meta-stable State · Bounce Solution

In general, a meta-stable state arises due to the existence of a local minimum of the potential, which is not the global minimum.

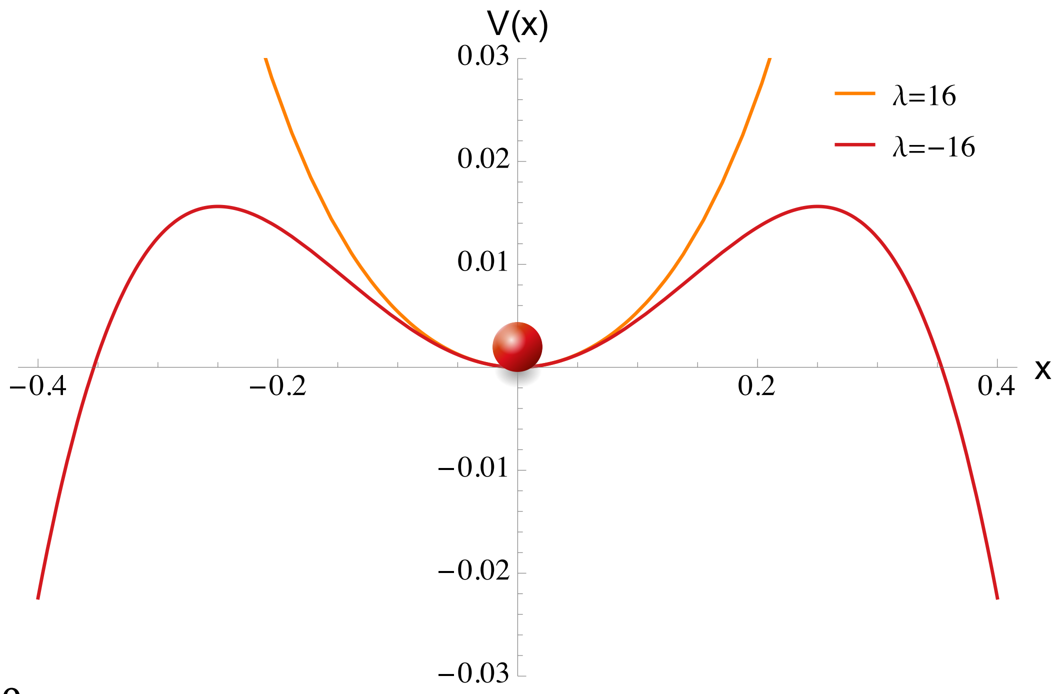
Example: consider the quartic anharmonic oscillator potential

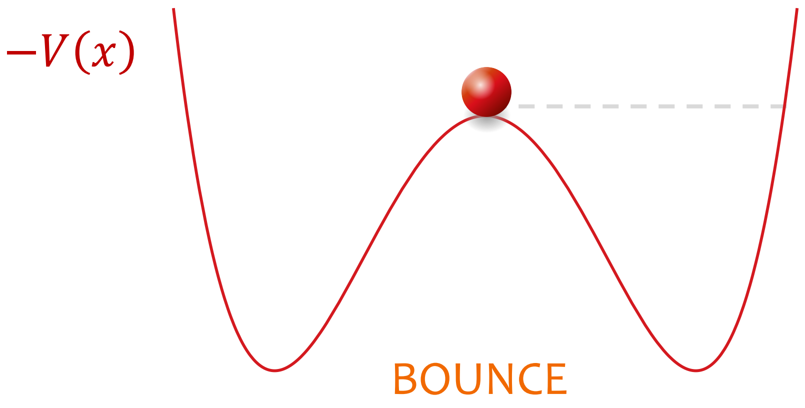
$$V(x) = \frac{1}{2}x^2 + \frac{\lambda}{4}x^4$$

For negative coupling $\lambda < 0$, the Hamiltonian is no longer bounded from below, and then

$$Z \propto e^{-S_E} = \exp\left[-\int \left(\frac{m}{2}\dot{x}^2 + V(x)\right) d\tau\right] \quad \text{diverges}$$

To solve this problem we analytic continuate $E(\lambda)$ from $\lambda > 0$ to $\lambda < 0$.





A particle in the GS at the bottom of the local unstable minimum will decay by tunneling through the barrier.

We want to obtain the mean lifetime of the particle

imaginary part of the GS energy!

$$E_0 \rightarrow E_0 + i\Gamma/2$$

We want the saddle point solution with boundary conditions: $x\left(\pm\frac{T}{2}\right) = 0$

Decay of a Meta-stable State · Bounce Solution

saddle-point equation

$$\frac{-\delta}{\delta x(\tau)} S_E[x] = m\ddot{x}_{cl} - V'(x_{cl}) = 0 \xrightarrow{\lambda < 0}$$

Let's first consider constant solutions: $\dot{x} = 0$

$$-V'(x) = -(x - |\lambda|\, x^3) = 0$$

$$\Rightarrow \quad x(1 - |\lambda|\, x^2) = 0 \quad \begin{cases} x(\tau) = 0 \\ x(\tau) = \pm\sqrt{\dfrac{1}{-\lambda}} \end{cases}$$

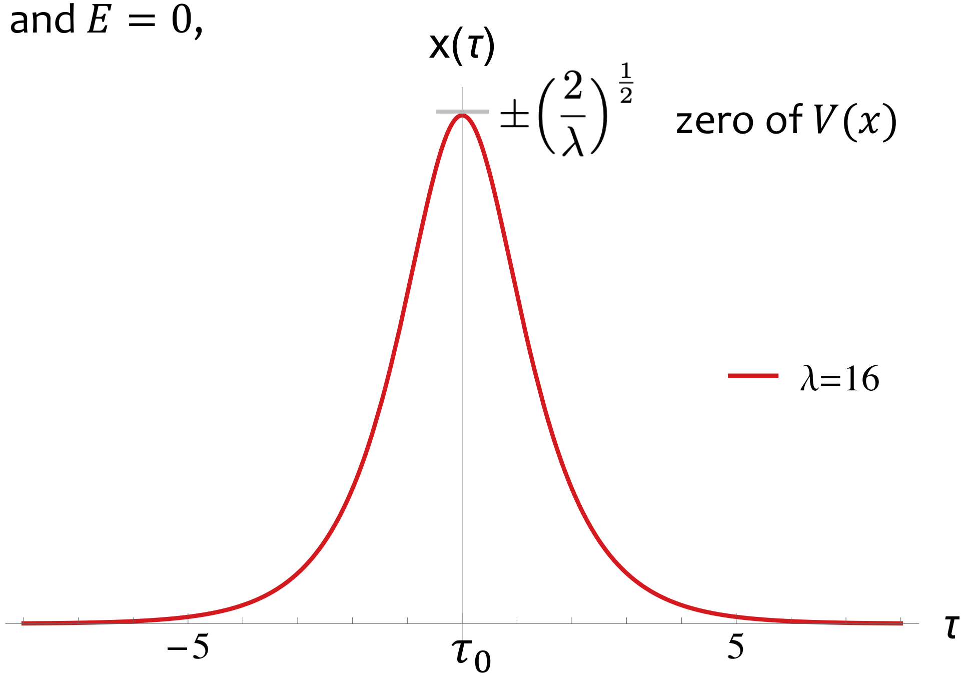
But what about the boundary conditions $x\left(\pm\frac{T}{2}\right) = 0$?? These paths do not appear, but they "nearly" do.

We are interest in the non-constant solutions. Taking the $\lim T \to \infty$, and $E = 0$, i.e. particle arrives to the turning point with zero energy, we obtain:

$$\boxed{x(\tau) = \pm\left(\frac{2}{\lambda}\right)^{\frac{1}{2}} \frac{1}{\cosh(\tau - \tau_0)}}$$

bounces

(where $\lambda \to -\lambda > 0$ and $m = 0$)



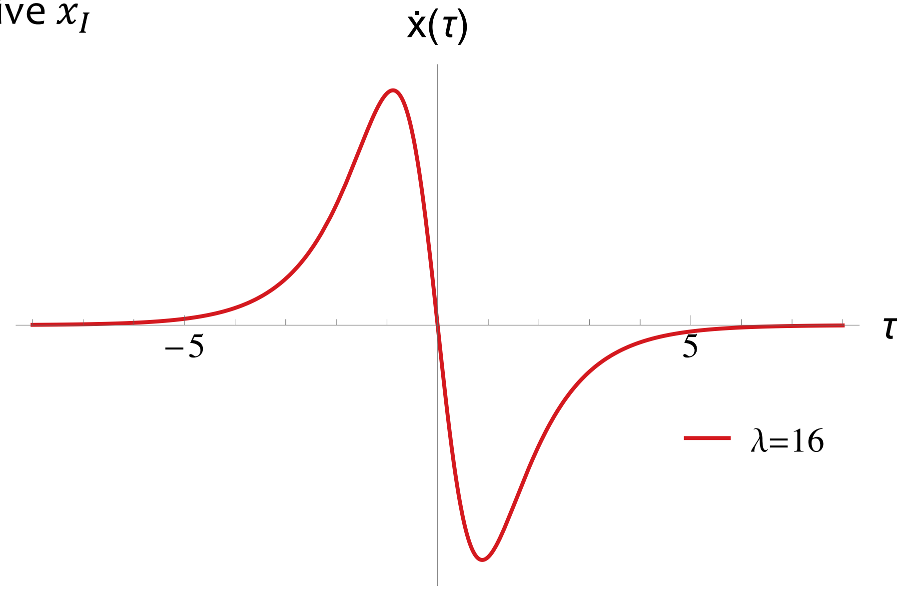
Decay of a Meta-stable State · Negative Mode

Now, similarly to what we did with the double well we want to examine the spectrum of the operator:

$$\hat{F}\left[x_I\right] = -m\frac{d^2}{d\tau^2} + V''\left(x_I\right) \quad \text{with the eigenvalue eq.} \quad \hat{F}\left(x_{cl}\right)\tilde{x}_n\left(\tau\right) = \lambda_n\tilde{x}_n\left(\tau\right) \quad \text{where} \quad V(x) = \frac{1}{2}x^2 + \frac{\lambda}{4}x^4$$

Zero mode: The mode with $\lambda = 0$ is proportional to the imaginary time derivative $\dot{x}_I$

However, note that now $\dot{x}_I$ posses one node, this means that it must correspond to the first excited state!



So, we have a negative mode that corresponds to the GS!!

The negative mode will be responsible to the imaginary part contribution!

Negative mode: using that

$$\hat{F} = -\frac{d^2}{d\tau^2} + 1 - \frac{6}{\cosh^2(\tau - \tau_0)} \quad \Rightarrow \quad \tilde{x}_0 = \frac{1}{\cosh^2(\tau)} \quad \text{and} \quad \hat{F}\ \tilde{x}_0 = -3\ \tilde{x}_0$$

Decay of a Meta-stable State · Negative Mode

However the integral for this mode is not Gaussian anymore…

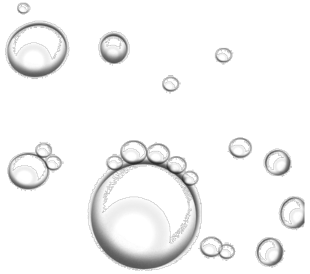
$$Z \propto \prod_n \int_{-\infty}^{\infty} dc_n \frac{e^{-\frac{1}{2\hbar}\lambda_n c_n^2}}{\sqrt{2\pi\hbar}} \xrightarrow{\text{negative mode}} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dc_0 \, e^{\frac{3}{2\hbar}c_0^2}$$

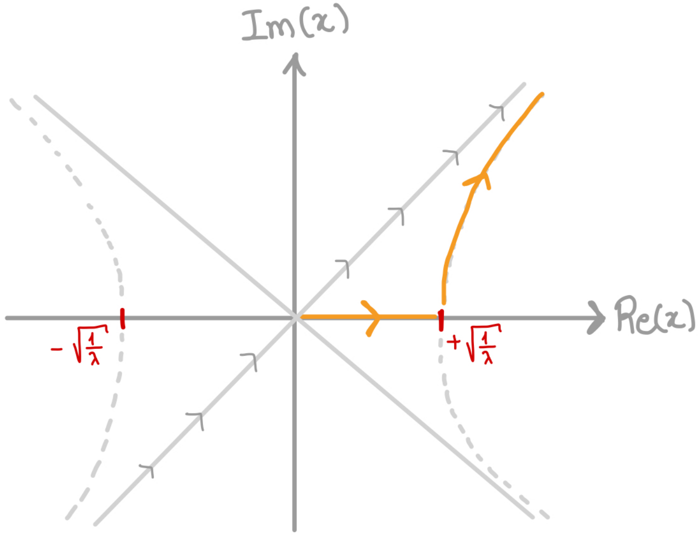
divergent!

To perform this integral we will use analytic continuation using the steepest descent method.

$$(\cdots) \quad \Rightarrow \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dc_0 \, e^{\frac{3}{2\hbar}c_0^2} = \pm\, i\, \frac{1}{2} \sqrt{\frac{1}{3}}$$

λ_0





Finally, combining all the pieces, we obtain:

$$Z(0,0) = \sqrt{\frac{\hbar\omega}{\pi}} e^{-\omega T/2} exp(T\sqrt{\frac{S_b}{2\pi\hbar}} e^{-S_b/\hbar} K) = e^{-T(E_0 + i\Gamma/2)/\hbar} A.C\{| < E_0 | 0 > |^2 + ...\}$$

where

$$K = Re[K] + i\frac{1}{2}\frac{1}{\sqrt{3}} \left[\frac{det'\left(-\frac{d^2}{d\tau^2} + V''\right)}{det\left(-\frac{d^2}{d\tau^2} + \omega^2\right)} \right]^{-1/2} \Rightarrow \boxed{\Gamma = \hbar\sqrt{\frac{S_b}{2\pi\hbar}} e^{-S_b/\hbar} \frac{1}{\sqrt{3}} \left[\frac{det'\left(-\frac{d^2}{d\tau^2} + V''\right)}{det\left(-\frac{d^2}{d\tau^2} + \omega^2\right)} \right]^{-1/2}}$$

Conclusion

- Instantons are localized, non-perturbative processes with several applications in many areas of physics.
- In QM, we can use instantons to describe tunneling/decay phenomena in the SC approximation if we analytic continue to imaginary/Euclidean time.

  ↳ The formal effect is to change the potential sign, so we can interpret the instanton solution as a classical particle moving in the inverted potential!

- We can extract the low-lying energy levels from the path integral/partition function in the SC approximation by taking the limit $T \to \infty$.
- For potentials with degenerate minima (double-well), the effect of tunneling is to split the degenerate energy levels. In this case we can calculate the instanton solution analytically.
- The spectrum of the fluctuation operator reveals the appearance of a **zero mode**, related to the translational invariance. The instanton solutions then form a one-parameter family parametrized by a collective coordinate.
- Due to their localized behavior we can consider multi-instantons solutions ⟶ Dilute Instanton Gas Approximation
- For the decay of an unstable state, we call the instanton solution "the bounce". Beyond the zero mode, a negative mode appear, responsible for the imaginary part of the partition function, where we extract the decay width.

References

[1] *H. Forkel, "A primer on instantons in QCD", arXiv:hep-ph/0009136v2 (2002);*

[2] J. Zinn-Justin, "*Barrier Penetration and Instantons*"

[3] *L.S. Shulman, "Techniques and Applications of Path Integrals", Dover (2005)*, Cap. 29

[4] Manu Paranjape, "*The Theory and Applications of Instanton Calculations*", Cambridge University Press (2017)

[5] Marcos Marino, "*Instantons and large N*", Cambridge University Press (2015), Cap. 1 e Cap 2

[6] Riccardo Rattazzi , "*The Path Integral approach to Quantum Mechanics*" (2011), Cap. 4

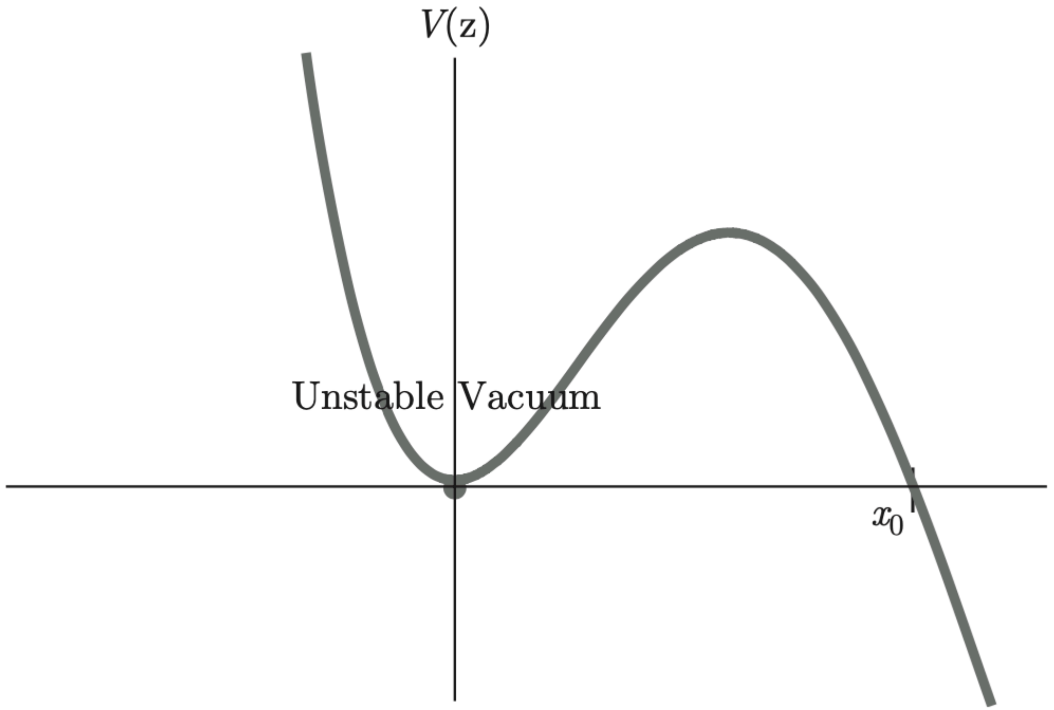
[7] A. Altland and B. Simons, "*Condensed Matter Field Theory*", Cambridge University Press (2010), Cap. 3

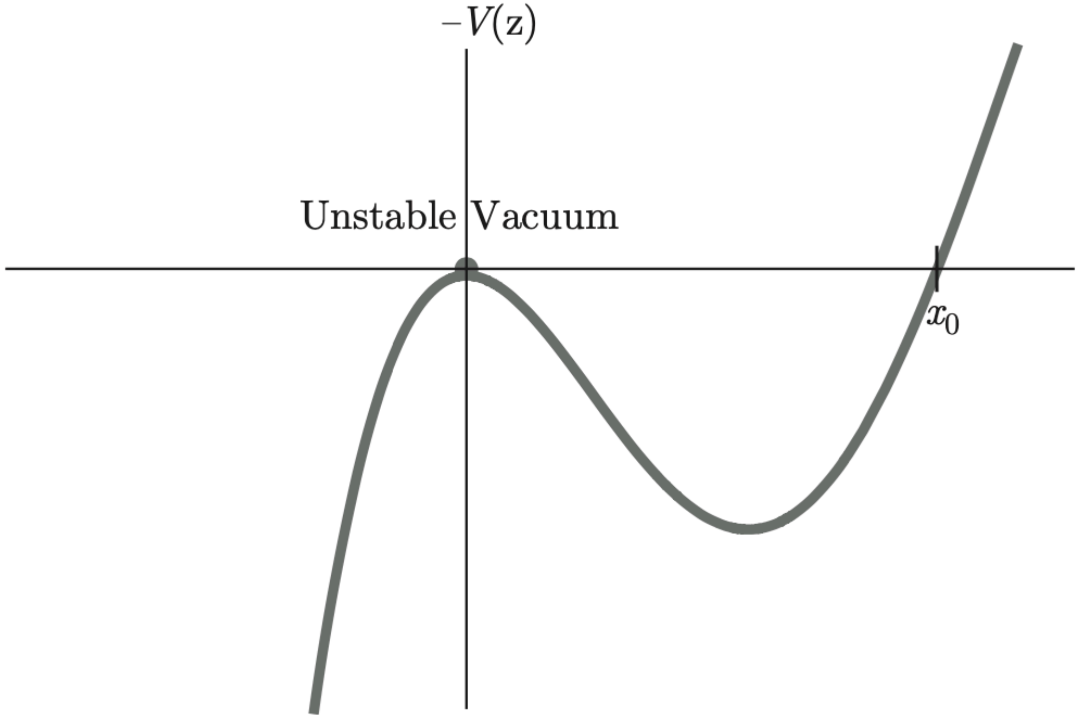
[8] *S. Coleman, "The uses of instantons", Proc. of the 1977 School of Subnuclear Physics (1977)*

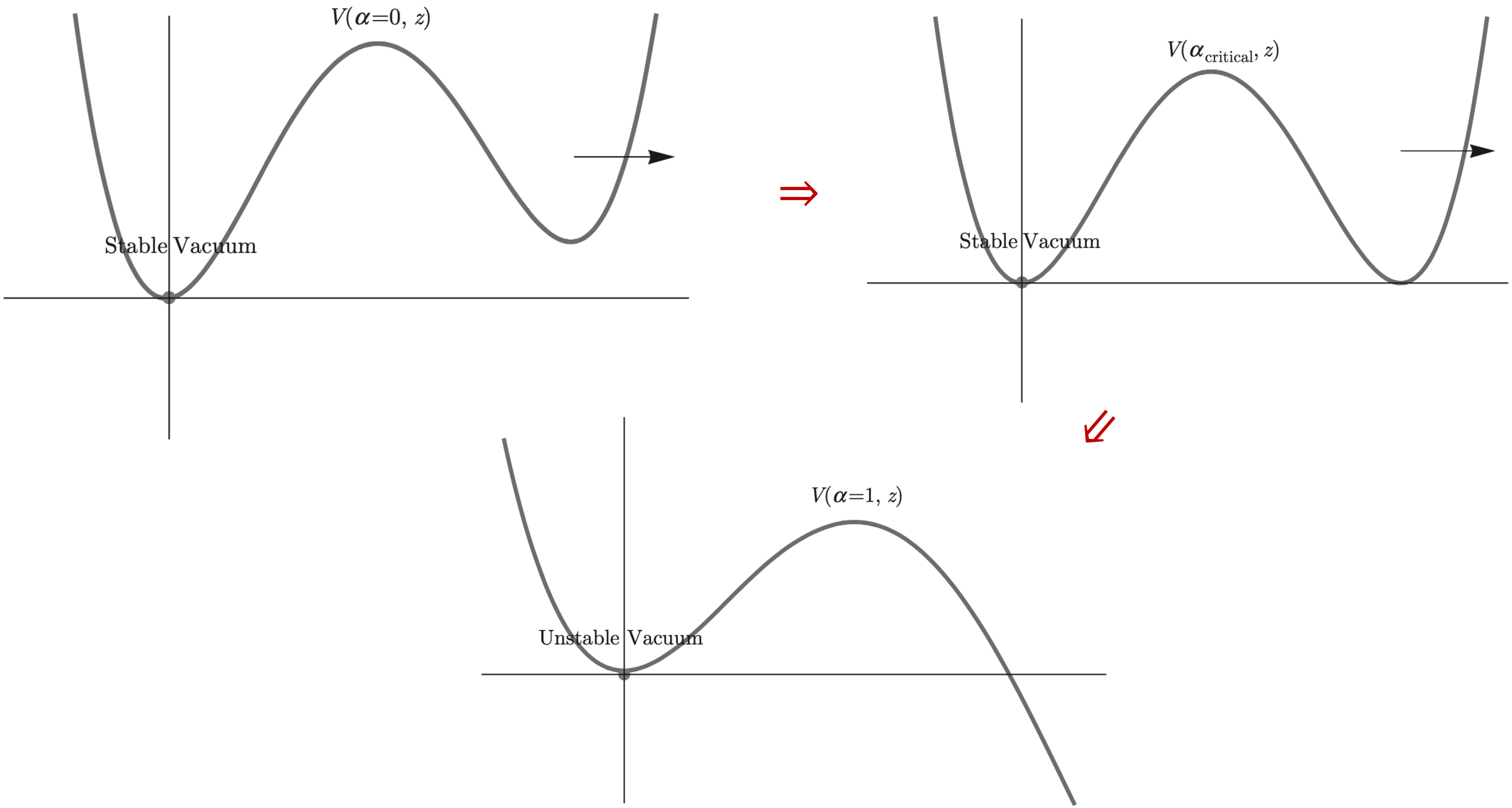
Thank you for your kind attention!

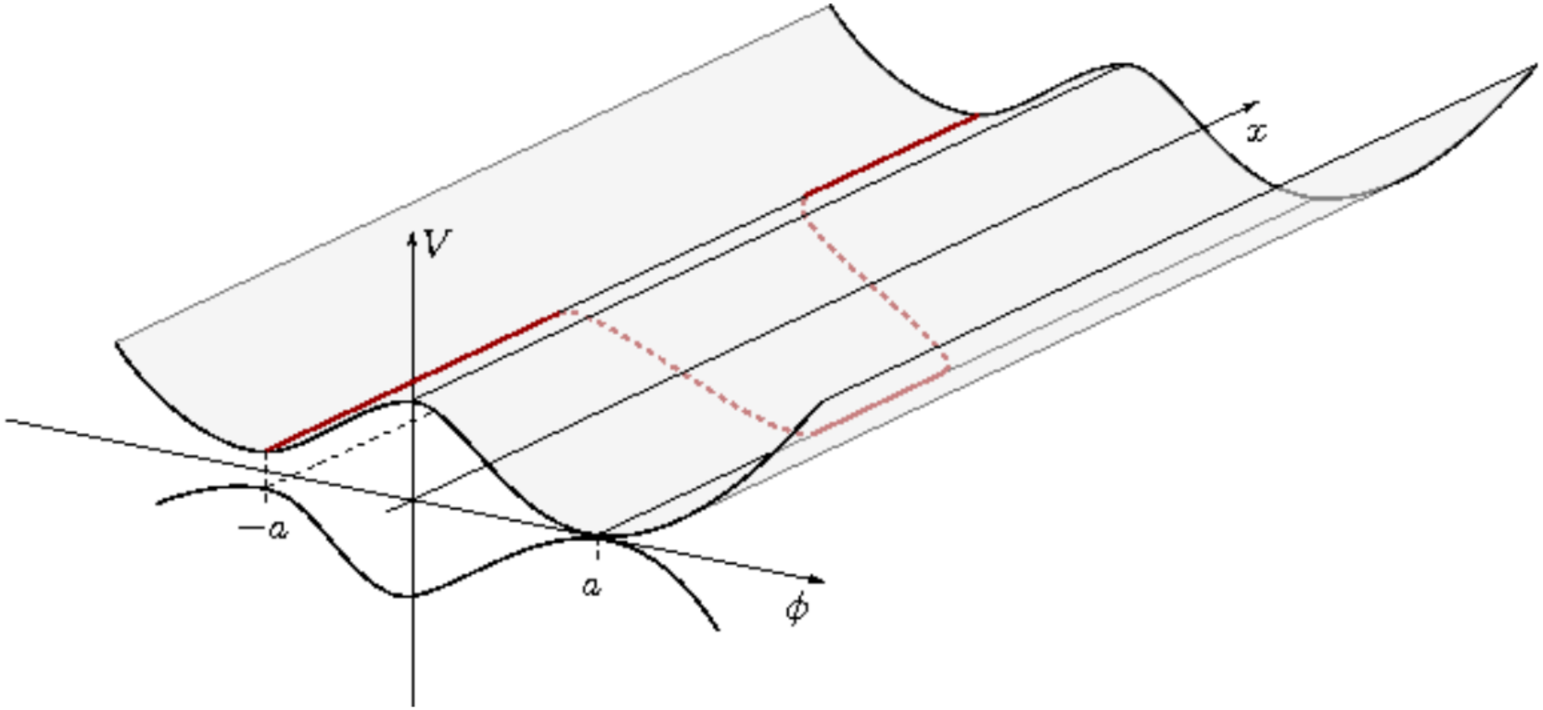
BACKUP

Decay of a Meta-stable State






$V(\alpha{=}0,\ z)$
Stable Vacuum
$V(\alpha_{\text{critical}},\ z)$
Stable Vacuum
$V(\alpha{=}1,\ z)$
Unstable Vacuum



V
x
$-a$
a
ϕ