Analytic Properties of the Scattering Amplitude - Antonio Pedrosa

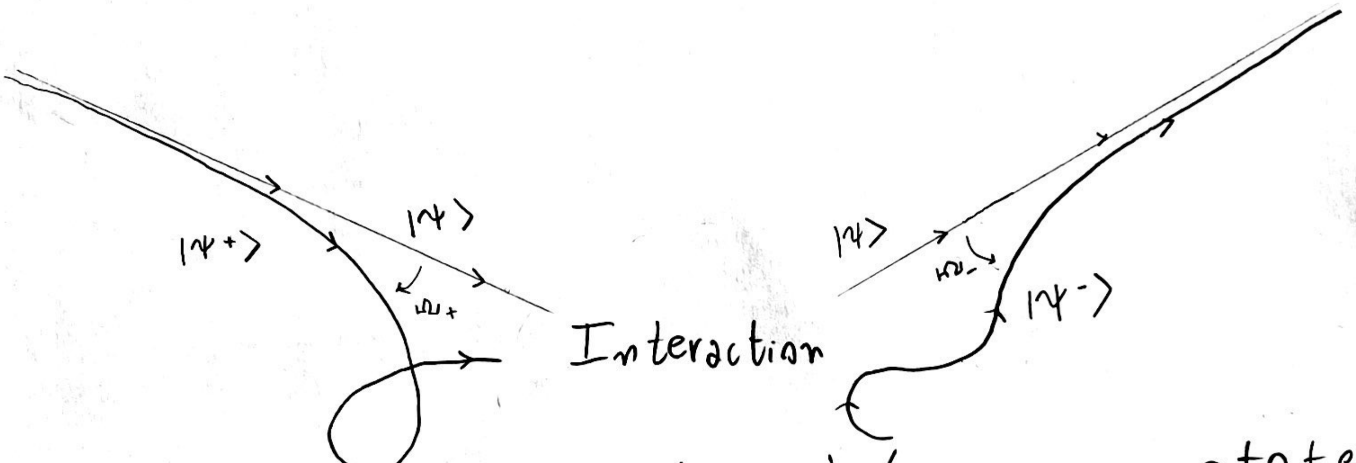
1. Scattering operator and partial-wave expansion

1.1 Møller wave operators, scattering operator

In scattering theory we're concerned with the study of asymptotic states. If the system is described by $H = H_0 + V$, where H_0 is the free hamiltonian (that may be channel dependent, but in the simplest elastic two body case is just the relative motion kinetic energy) and V is an interaction, we may define the general evolution operator $U(t) = e^{iHt}$ (using $\hbar = 1$) and the free (asymptotic) evolution operator as $U_0(t) = e^{iH_0t}$. If a state $|\Psi\rangle = |\Psi(t=0)\rangle$ that is in the interaction region at $t=0$ originated from a asymptotic free state $|\Psi_{in}\rangle$, we have

$$\lim_{t\to-\infty} U(t)|\Psi\rangle = \lim_{t\to-\infty} U_0(t)|\Psi_{in}\rangle \quad , \quad \text{and}$$

So we define the Møller operator as the limit $\Omega_+ := \lim_{t \to -\infty} U^\dagger(t)\, U_0(t)$, that "evolves" an asymptotic state into the interaction region: $|\psi\rangle = \Omega_+ |\psi_{in}\rangle \equiv |\psi^+\rangle$. And similarly $\Omega_- := \lim_{t \to +\infty} U^\dagger(t)\, U_0(t)$, that gives the actual state $\Omega_- |\psi\rangle = |\psi^-\rangle$ at $t=0$ that would evolve to $|\psi\rangle$.



(In asymptote) (Actual state at $t=0$) (Out asymptote)

$$|\psi_{in}\rangle \xrightarrow{\Omega_+} |\psi\rangle \xleftarrow[\xrightarrow{\Omega_-^\dagger}]{\Omega_-} |\psi_{out}\rangle$$

($\Omega_\pm$ are isometric). And finally we define the scattering operator as $S|\psi_{in}\rangle = |\psi_{out}\rangle$, $S = \Omega_-^\dagger \Omega_+$, that's unitary by construction.

If the interaction is, for exemple, spherically symmetrical and satisfy the usual assumptions:

(i) As $r \to \infty$ $V(r)$ falls faster than r^{-3}.

(ii) As $r \to 0$ $V(r)$ is less singular than r^{-2}.

(iii) $V(r)$ is continuous except for a finite number of finite discontinuities.

then the limits defining $\Omega_\pm$ exist.

It's the momentum state $\Omega_\pm |\vec{k}\rangle = |\vec{k}\pm\rangle$ defined this way that satisfy the Lippmann-Schwinger equation

$$|\vec{k}\pm\rangle = |\vec{k}\rangle + \frac{1}{E - H_0 \pm i\varepsilon} V |\vec{k}\pm\rangle, \qquad (1.1)$$

and with the transition operator $T|\vec{k}\rangle = V|\vec{k}+\rangle$ we've showed that

$$\langle\vec{k}'|S|\vec{k}\rangle = \delta(\vec{k}' - \vec{k}) - 2\pi i\, \delta(E' - E)\langle\vec{k}'|T|\vec{k}\rangle \qquad (1.2)$$

And the scattering amplitude

$$f(\vec{k}'|\vec{k}) = -(2\pi)^2 m \langle\vec{k}'|T|\vec{k}\rangle. \qquad (1.3)$$

1.2 Paritial-Wave Expansion

Assuming that the interaction is rotational invariant, S is a scalar operator. To better exploit this fact we use the angular momentum base $|E l m\rangle$, so the Wigner-Eckart theorem gives:

$$\langle E'l'm'|S|E l m\rangle = \delta(E'-E)\,\delta_{l'l}\,\delta_{m'm}\,S_l(E) \quad (1.4)$$

and since S is unitary $|S_l(E)| = 1$, so we can define $S_l(E) = e^{2i\delta_l(E)}$ (this is not always the case if other channels are open at that energy).

Since we can expand the state

$$\langle \vec{x}|E l m\rangle = \sqrt{\frac{2m\kappa}{\pi}}\, j_l(\kappa r)\, Y_{lm}(\hat{x})$$ and

$$\langle \vec{x}|\vec{\kappa}\rangle = \sqrt{\frac{2}{\pi}} \sum_{l,m} j_l(\kappa r)\, Y_{lm}(\hat{x})\, Y_{lm}(\hat{\kappa}),$$ we have

$$\langle \vec{\kappa}'|E l m\rangle = \int d^3x\, \langle \vec{\kappa}'|\vec{x}\rangle\langle \vec{x}|E l m\rangle =$$

$$= \int d^3x \left(\sqrt{\frac{2}{\pi}} \sum_{l',m'} j_{l'}^*(\kappa' r)\, Y_{l'm'}^*(\hat{x})\, Y_{l'm'}(\hat{\kappa}')\right) \times$$

$$\times \left(\sqrt{\frac{2m\kappa}{\pi}}\, j_l(\kappa r)\, Y_{lm}(\hat{x})\right)$$

(where $\varkappa = \sqrt{2mE}$)

$$= \frac{2}{\pi}\sqrt{m\varkappa} \sum_{l',m'} Y_{l'm'}(\hat{\varkappa}) \underbrace{\int dr\, r^2 j_{l'}(\varkappa' r) j_l(\varkappa r)}_{\frac{\pi}{2\varkappa^2}\delta(\varkappa'-\varkappa)} \underbrace{\int d\hat{x}\, Y_{l'm'}(\hat{x}) Y_{lm}(\hat{x})}_{\delta_{l'l}\delta_{m'm}}$$

$$= \frac{\sqrt{m\varkappa}}{\varkappa^2} Y_{lm}(\hat{\varkappa})\, \delta(\varkappa' - \varkappa), \quad \text{let } E' = \sqrt{2m\varkappa'},$$

since $\delta(ax) = \frac{1}{|a|}\delta(x)$ and $\delta(x^2-a^2) = \frac{\delta(x+a)+\delta(x-a)}{2|a|}$,

$$= 2\sqrt{\frac{m}{\varkappa}}\, Y_{lm}(\hat{\varkappa}) \frac{1}{2m} \delta(E'-E) = \sqrt{m\varkappa}\, Y_{lm}(\hat{\varkappa})\, \delta(E'-E),$$

so $$\langle \vec{\varkappa}' | E l m \rangle = \sqrt{m\varkappa}\, Y_{lm}(\hat{\varkappa})\, \delta(E'-E) \qquad (1.5).$$

Combining (1.2), (1.3) and $\langle \vec{\varkappa}' | \vec{\varkappa} \rangle = \delta(\vec{\varkappa}' - \vec{\varkappa})$,

we see that $\langle \vec{\varkappa}' | (S-1) | \vec{\varkappa} \rangle = \frac{i}{2\pi m} \delta(E'-E)\, f(\vec{\varkappa}' | \vec{\varkappa})$

but $$\langle \vec{\varkappa}' | (S-1) | \vec{\varkappa} \rangle = \int dE \sum_{l,m} \langle \vec{\varkappa}' | (S-1) | E l m \rangle \langle E l m | \vec{\varkappa} \rangle$$

with (1.4), (1.5) $$= \frac{1}{m\varkappa} \sum_{l,m} (S_l(E) - 1)\, Y_{lm}(\hat{\varkappa})\, Y_{lm}(\hat{\varkappa}'),$$

and with (1.2), (1.3) and the addition formula for spherical harmonics we obtain

$$f(\vec{\varkappa}' | \vec{\varkappa}) = \frac{1}{2\varkappa i} \sum_{l \geq 0} (2l+1)(S_l(E) - 1)\, P_l(\cos\theta) \qquad (1.6)$$

$$\equiv \sum_{l \geq 0} (2l+1)\, f_l(E)\, P_l(\cos\theta)$$

where $\cos\theta = \hat{k}\cdot\hat{k}'$, and we've defined the partial wave amplitude as

$$f_l(E) = \frac{1 - S_l(E)}{2ki} = \frac{e^{2i\delta_l(E)} - 1}{2ki} = \frac{e^{i\delta_l(E)}\sin\delta_l(E)}{k} \qquad (1.7)$$

2. Radial equation solutions

2.1 Free solutions

Seeking free solutions to a central potential with the form $\Psi(\vec{r}) = \sum_{l,m} C_{lm}\, y_l(r)\, Y_{lm}(\hat{r})$ will lead us to the spherical Bessel $j_l(kr)$ and Neumann $n_l(kr)$ functions or the spherical Hankel $h_l^{\pm}(kr) = n_l(kr) \pm i\, j_l(kr)$ functions. A different approach is to propose solutions with the form $\frac{y_l(r)}{r} Y_{lm}(\hat{r})$, leading us to the free radial equation

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + p^2\right] y(r) = 0 \qquad (2.1).$$

We can now introduce the Riccati counterparts of the functions previously mentioned. A clear advantage of, say, the Riccati-Bessel function, is that it is entire.

	Riccati-Bessel	Riccati-Neumann	Riccati-Hankel
Definition	$\hat{j}_\ell = z\, j_\ell(z) = z^{\ell+1}\sum_{n\geq 0} \frac{(-z/2)^n}{(2\ell+2n+1)!!\, n!}$	$\hat{\eta}_\ell(z) = z\,\eta_\ell(z) = z^{-\ell}\sum_{n\geq 0} \frac{(-z/2)^n (2\ell-2n-1)!!}{n!}$	$\hat{h}^{\pm}_\ell = \hat{\eta}_\ell \pm i \hat{j}_\ell$
Behavior as $\lvert z\rvert \to 0$	$\frac{z^{\ell+1}}{(2\ell+1)!!}(1+O(z^2))$	$z^{-\ell}(2\ell-1)!!\,(1+O(z^2))$	$z^{-\ell}(2\ell-1)!!\,(1+O(z^2))$
Behavior as $\lvert z\rvert \to \infty$	$\sin(z-\frac{1}{2}\ell\pi)(1+O(z^{-1}))$	$\cos(z-\frac{1}{2}\ell\pi)(1+O(z^{-1}))$	$e^{\pm i(z-\frac{1}{2}\ell\pi)}(1+O(z^{-1}))$

Both $\{\hat{j}_\ell, \hat{\eta}_\ell\}$ and $\{\hat{h}^{\pm}_\ell\}$ are linearly independent sets of solutions to (2.1), that span the space of solutions.

It will be handy to establish some bounds to those functions. Both $\hat{j}_\ell$ and $\hat{\eta}_\ell$, because of the trigonometric behaviour as $\lvert z\rvert \to \infty$, have a growth dominated by $e^{\lvert \mathrm{Im}\, z\rvert}$.

As $\lvert z\rvert \to 0$, $\lvert \hat{j}_\ell\rvert \sim \lvert z\rvert^{\ell+1} \sim \left(\frac{\lvert z\rvert}{1+\lvert z\rvert}\right)^{\ell+1}$, and since $\left(\frac{\lvert z\rvert}{1+\lvert z\rvert}\right)^{\ell+1} \sim 1$ as $\lvert z\rvert \to \infty$. Since $\hat{j}_\ell$ is continuous and hence bounded by a constant in the finite interval between those extremes, we have:

$$\lvert \hat{j}_\ell(z)\rvert \leq C_j \left(\frac{\lvert z\rvert}{1+\lvert z\rvert}\right)^{\ell+1} e^{\lvert \mathrm{Im}\, z\rvert} \qquad (2.2)$$

By similar arguments,

$$|\hat{\eta}_\ell(z)| \leqslant C_\eta \left(\frac{|z|}{1+|z|}\right)^{-\ell} e^{|\operatorname{Im} z|} \qquad (2.3)$$

The growth of the Riccati-Hankel functions near the origin is dominated by the singular nature of $\eta_\ell \sim z^{-\ell}$. As $|z| \to \infty$, since $|e^{\pm z}| = |e^{i \operatorname{Re} z}|\,|e^{\mp \operatorname{Im} z}| = e^{\mp \operatorname{Im} z}$, we have $|\hat{h}^\pm(z)| \sim e^{\pm \operatorname{Im} z}$, so

$$|\hat{h}^\pm(z)| \leq C_{h^\pm} \left(\frac{|z|}{1+|z|}\right)^{-\ell} e^{\mp \operatorname{Im} z} \qquad (2.4).$$

2.2 Normalized solution

In class, we've explored stationary solutions of the form

$$\psi_\kappa(\vec{r}) \xrightarrow{r\to\infty} (2\pi)^{-3/2} \left(e^{i\vec{\kappa}\cdot\vec{r}} + \frac{e^{i\kappa r}}{r} f(\vec{\kappa}'|\vec{\kappa})\right), \qquad (2.5)$$

constructed as $\psi_\kappa(\vec{r}) = \sum_{\ell,m} \sqrt{\frac{2\ell+1}{2\pi^2}}\, i^\ell R_\ell(\kappa,r)\, Y_{\ell m}(\hat{r})$

where $r\kappa R_\ell(\kappa,r)$ solves

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + p^2 - (\lambda)U(r)\right] y(r) = 0 \qquad (2.6)$$

with boundary conditions defined by (2.5). Those requirements can be combined in an integral equation of the form

$$R_\ell(\kappa,r) = j_\ell(\kappa r) + \int_0^\infty dr'\, r'^2\, G_\ell^\kappa(r,r')\, U(r')\, R(\kappa,r')$$

where $G_\ell^{\varkappa}(r,r') = -i\varkappa\, j_\ell(\varkappa r_<)\, h^+_\ell(\varkappa r_>)$ is the appropriete Green function, and $U(r) = 2m V(r)$. $\Psi_\varkappa(\vec{r})$ can also be expanded as $\sum_{\ell,m} \sqrt{\frac{2\ell+1}{2\pi^2}}\, i^\ell \frac{\psi_{\ell,\varkappa}(r)}{r\varkappa} Y_{\ell m}(\hat{r})$,

now with $\psi_{\ell,\varkappa}(r) = \hat{j}_\ell(\varkappa r) + \lambda \int_0^\infty dr'\, \hat{G}_\ell^{\varkappa}(r,r')\, U(r')\, \psi_{\varkappa,\ell}(\vec{r}')$ (2.7)

and $\hat{G}_\ell^{\varkappa}(r,r') = -\frac{i}{\varkappa}\, \hat{j}_\ell(\varkappa r)\, \hat{h}^+_\ell(\varkappa r)$. I've introduced in (2.7) a strengh parameter $\lambda V(r)$ to the potential, and notice that $\psi_{\ell,\varkappa}$ solve (2.7). In fact by construction we have $\langle \vec{r} | \Omega_+ | E\ell m\rangle = \text{const} \times \frac{\psi_{\ell,\varkappa}(\vec{r})}{\varkappa r} Y_{\ell m}(\hat{r})$, and since $\langle E'\ell' m' + | E\ell m +\rangle = \langle E'\ell' m' | \Omega_+^\dagger \Omega_+ | E\ell m\rangle = \langle E'\ell' m' | E\ell m\rangle = \delta(E'-E)\,\delta_{\ell'\ell}\,\delta_{m'm}$, $\langle \vec{r} | E\ell m\rangle = \text{const} \times \frac{\hat{j}_\ell(\varkappa r)}{\varkappa r} Y_{\ell m}(\hat{r})$,

the normalization of $\psi_{\ell,\varkappa}(\vec{r})$ mimic that of $\hat{j}_\ell(\varkappa r)$. If we align $\hat{\varkappa}$ and $\hat{z}$,

$$\Psi_\varkappa(\vec{r}) = \langle \vec{r} | \vec{\varkappa}+\rangle = (2\pi)^{-3/2} \frac{1}{\varkappa r} \sum_{\ell\geq 0} (2\ell+1)\, i^\ell\, \psi_{\ell,\varkappa}(\vec{r})\, P_\ell(\hat{r}\cdot\hat{\varkappa}) \longrightarrow$$

$$\xrightarrow{r\to\infty} (2\pi)^{-3/2}\left(e^{i\vec{\varkappa}\cdot\vec{x}} + f(\varkappa\hat{r}|\vec{\varkappa})\frac{e^{i\varkappa r}}{r}\right) =$$

$$= (2\pi)^{-3/2} \frac{1}{\varkappa r} \sum_{\ell\geq 0} (2\ell+1)\left(i^\ell \hat{j}_\ell(\varkappa r) + \varkappa f_\ell(\varkappa) e^{i\varkappa r}\right) P_\ell(\hat{r}\cdot\hat{\varkappa})$$

using (1.6) and $e^{i\vec{\varkappa}\cdot\vec{r}} = \sum_{\ell\geq 0} (2\ell+1)\, i^\ell\, j_\ell(\varkappa r)\, P_\ell(\hat{r}\cdot\hat{\varkappa})$

Since $i^{\ell} = e^{\ell \log i} = e^{i\ell\pi/2}$, comparing we obtain

$$\psi_{\ell,\kappa}(r) \xrightarrow{r\to\infty} \hat{j}_\ell(\kappa r) + \kappa f_\ell(\kappa) e^{i(\kappa r - \frac{1}{2}\ell\pi)} \quad \text{or}$$

$$\longrightarrow \hat{j}_\ell(\kappa r) + \kappa f_\ell(\kappa)\hat{h}^+_\ell(\kappa r)$$

$$\longrightarrow \frac{i}{2}\left(\hat{h}^-_\ell(\kappa r) - S_\ell(\kappa)\hat{h}^+_\ell(\kappa r)\right),$$

using the $r\to\infty$ form of the Riccati-Hankel functions and $\kappa f_\ell = (S_\ell - 1)/2i$, $\hat{j} = (\hat{h}^+ - \hat{h}^-)/2i$.

2.3 Regular solution

The initial conditions defining $\psi_{\ell,\kappa}(r)$ are $\psi_{\ell,\kappa}(0) = 0 = \text{const.}\,\hat{j}_\ell(\kappa r)$, $\psi'_{\ell,\kappa}(0) = \text{const.}\,\hat{j}'_\ell(\kappa r)$, or $\psi_{\ell,\kappa}(r) \xrightarrow{r\to 0} \text{const.}\,\hat{j}_\ell(\kappa r)$. We now define a new solution $\phi_{\ell,\kappa}(r)$ proportional to $\psi_{\ell,\kappa}(r)$, but now with $\phi_{\ell,\kappa}(r) \xrightarrow{r\to 0} \hat{j}_\ell(\kappa r)$. The boundary conditions are contained in the following integral (Volterra) equation:

$$\phi_{\ell,\kappa}(r) = \hat{j}_\ell(\kappa r) + \lambda \int_0^r dr'\, g^\kappa_\ell(r,r')\, U(r')\, \phi_{\ell,\kappa}(r'), \quad (2.8)$$

where $g^\ell_\kappa(r,r') = \frac{1}{\kappa}\left(\hat{j}_\ell(\kappa r)\hat{\eta}_\ell(\kappa r') - \hat{\eta}_\ell(\kappa r)\hat{j}_\ell(\kappa r')\right)$.

Let $\frac{d}{dr} = {}'$, since $g' = \frac{1}{\kappa}(\hat{j}'\hat{\eta} - \hat{\eta}'\hat{j}) = \frac{1}{\kappa}\kappa = 1$

and $g'' = \frac{1}{\kappa}(\hat{j}''\hat{\eta} - \hat{\eta}''\hat{j}) = 0$, because both

$\hat{j}$ and $\hat{n}_\ell$ solve (2.1). Now (dropping the indexes ℓ, κ) since $g(r,r) = 0$,

$$\phi''(r) = \hat{j}''(r) + \lambda \int_0^r dr' g''(r,r') U(r') \phi(r') + g(r,r) U(r) \phi(r) +$$

$$+ g'(r,r) U(r) \phi(r)$$

Again, $\hat{j}, \hat{n}$ solve (2.1) so $\hat{j}'' = \left(\frac{\ell(\ell+1)}{r^2} - \kappa^2\right)\hat{j}$ etc, so $g''(r,r') = \left(\frac{\ell(\ell+1)}{r^2} - \kappa^2\right) g(r,r')$, hence

$$\phi''(r) = \left(\frac{\ell(\ell+1)}{r^2} - \kappa^2\right)\hat{j} + \lambda\left(\frac{\ell(\ell+1)}{r^2} - \kappa^2\right)\int_0^r dr' g(r,r') U(r') \phi(r') +$$

$$+ U(r)\phi(r),$$ since $\phi(r) = \hat{j}(r) + \lambda \int_0^r dr' g(r,r') U(r') \phi(r')$,

$$\phi''(r) = \left(\frac{\ell(\ell+1)}{r^2} - \kappa^2 + U(r)\right)\phi(r),$$ so in fact a solution to (2.8) is a solution to (2.6) with $\phi_{\ell,\kappa}(r) \xrightarrow{r \to 0} \hat{j}_\ell(\kappa r)$, as is clear by the Voltera form of the equation.

<u>Theorem 2.1</u>: (2.8) can be solved by iteration with any <u>complex</u> κ, λ, and the solution $\phi_{\ell,\kappa,\lambda}(r) = \sum_{n \geq 0} \lambda^n \phi_{\ell,\kappa}^{(n)}(r)$ with

$$\phi_{\ell,\kappa}^{(0)}(r) = \hat{j}_\ell(\kappa r), \quad \phi_{\kappa,\ell}^{(n+1)}(r) = \int_0^r dr' g_\kappa^\ell(r,r') U(r') \phi_{\ell,\kappa}^{(n)}(r)$$

is an entire function of κ and λ.

<u>Proof</u>: If every $\phi^{(n)}$ is bounded, since $\hat{j}_\ell(\kappa r)$ is entire in κ, so will be $\phi^{(n+1)}(r) = \int_0^r dr'\, g(r,r')\, U(r')\, \phi^{(n)}(r')$. Using the bounds of (2.2) and (2.3) we conclude that

$$|g(r,r')| \leq \frac{C_g}{|\kappa|}\left(\frac{|\kappa| r_>}{1+|\kappa| r_>}\right)^{\ell+1}\left(\frac{|\kappa| r_<}{1+|\kappa| r_<}\right)^{-\ell} e^{|\mathrm{Im}\,\kappa|(r_>-r_<)}.$$

Since $\phi^{(n)}(r) = \int_0^r dr_n \int_0^{r_n} dr_{n-1} \cdots \int_0^{r_2} dr_1\, g(r,r_n) U(r_n)\, g(r_n, r_{n-1}) \cdots U(r_1)\, \hat{j}(\kappa r_1)$,

and $r \geq r_n \geq r_{n-1} \geq \cdots \geq r_2 \geq r_1$, using our bounds

$$|\phi^{(n)}(r)| \leq C_g^n \left(\frac{|\kappa| r}{1+|\kappa| r}\right)^{\ell+1} e^{|\mathrm{Im}\,\kappa| r} \frac{1}{|\kappa|^n} \int_0^r dr_n \int_0^{r_n} dr_{n-1} \cdots \int_0^{r_2} dr_1 \left(\frac{|\kappa| r_n}{1+|\kappa| r_n}\right)^{-\ell} e^{-|\mathrm{Im}\,\kappa| r_n} \quad (*)$$

$$|U(r_n)| \left(\frac{|\kappa| r_n}{1+|\kappa| r_n}\right)^{\ell+1} \left(\frac{|\kappa| r_{n-1}}{1+|\kappa| r_{n-1}}\right)^{-\ell} e^{|\mathrm{Im}\,\kappa|(r_n - r_{n-1})} |U(r_{n-1})| \times$$

$$\times \left(\frac{|\kappa| r_2}{1+|\kappa| r_2}\right)^{\ell+1} \left(\frac{|\kappa| r_1}{1+|\kappa| r_1}\right)^{-\ell} e^{|\mathrm{Im}\,\kappa|(r_2-r_1)} |U(r_1)|\, C_j \left(\frac{|\kappa| r_1}{1+|\kappa| r_1}\right)^{\ell+1} e^{|\mathrm{Im}\,\kappa| r_1}$$

$$= C_g^n C_j \left(\frac{|\kappa| r}{1+|\kappa| r}\right)^{\ell+1} e^{|\mathrm{Im}\,\kappa| r} \frac{1}{\cancel{|\kappa|^n}} \int_0^r dr_n \int_0^{r_n} dr_{n-1} \cdots \int_0^{r_2} dr_1 \prod_{i=1}^{n} \left(\frac{\cancel{|\kappa|} r_i}{1+|\kappa| r_i}\right) |U(r_i)|$$

The integral is invariant under the exchange $r_i \leftrightarrow r_j$, and it's performed in the "triangular" region $r \geq r_n \geq \cdots \geq r_1$, that is $1/n!$ of the volume of $0 \leq r_1, \ldots, r_n \leq r$, whe the integrand is certanly bounded.

so we have

$$\int_0^r dr_n \int_0^{r_n} dr_{n-1} \cdots \int_0^{r_2} dr_1 \prod_{i=1}^{n} \frac{r_i}{1+|k|r_i} |U(r_i)| \leq \int_0^r dr_n \int_0^{r_n} dr_{n-1} \cdots \int_0^{r_2} dr_1 \prod_{i=1}^{n} r_i |U(r_i)|$$

$$\leq c' \frac{1}{n!} \left(\int_0^r dr' \, r' |U(r')| \right)^n = c' \frac{\alpha^n}{n!} , \quad \text{since} \quad V(r) = O(r^{-2+\varepsilon})$$

as $r \to 0$ the integral is convergent.

Now $|\phi^{(n)}(r)| \leq \left(\frac{|k|r}{1+|k|r} \right)^{\ell+1} e^{|\mathrm{Im}\, k| r} c' \frac{(c_g \alpha)^n}{n!}$, and

$$|\phi(r)| = \left| \sum_{n \geq 0} \lambda^n \phi^{(n)}(r) \right| \leq \sum_{n \geq 0} |\lambda^n| |\phi^{(n)}(r)| \leq$$

$$\leq \left(\frac{|k|r}{1+|k|r} \right)^{\ell+1} e^{|\mathrm{Im}\, k| r} c' \sum_{n \geq 0} \frac{(\lambda c_g \alpha)^n}{n!} =$$

$$= \gamma \left(\frac{|k|r}{1+|k|r} \right)^{\ell+1} e^{|\mathrm{Im}\, k| r} , \quad \text{where } \gamma = c' e^{\lambda c_g \alpha} .$$

We conclude that $\phi^{(n)}_{\ell,k}(r)$ is entire on k, and that $\phi_{k,\ell,\lambda}(r) = \sum_{n \geq 0} \lambda^n \phi^{(n)}_{\ell,k}(r)$ is an absolutely convergent series of analytic function, so $\phi_{k,\ell,\lambda}(r)$ is entire on k and λ, with $|\phi_{k,\ell}(r)| \leq \gamma_\ell \left(\frac{|k|r}{1+|k|r} \right)^{\ell+1} e^{|\mathrm{Im}\, k| r}$.

Now we show that, in fact, $\phi_{\ell,k}$ is a solution to (2.8). Since the series is absolutely convergent, the order of sum and integration can be exchanged: Since $\phi^{(0)}_{\ell,\varkappa}(r) = \hat{j}_\ell(\varkappa r)$,

$$\phi(r) = \hat{j}(\varkappa r) + \lambda \int_0^r dr'\, g(r,r')\, U(r')\, \phi(r')$$

$$= \phi^{(0)}(r) + \sum_{n\geq 0} \lambda^{n+1} \underline{\int_0^r dr'\, g(r,r')\, U(r')\, \phi^{(n)}(r')},$$

we see the definition of $\phi^{(n+1)}$, hence

$$\phi(r) = \phi^{(0)}(r) + \sum_{n\geq 0} \lambda^{n+1} \phi^{(n+1)}(r) = \sum_{n\geq 0} \lambda^{n} \phi^{(n)}(r),$$

as expected. $\blacksquare$

We highlight that the only physical λ are those on the real line, and the only physical $\varkappa$ are those on the positive real line.

With the definitions of $\hat{j}$, $\hat{n}$ and $\hat{h}^{\pm}$, we conclude that $\hat{j}_\ell(-z) = (-1)^{\ell+1}\hat{j}_\ell(z)$, $\hat{n}_\ell(-z) = (-1)^{\ell}\hat{n}_\ell(z)$ so $g_\ell^{-\varkappa}(r,r') = g_\ell^{\varkappa}(r,r')$. So by definition we have $\phi_{\ell,-\varkappa}(r) = (-1)^{\ell+1}\phi_{\ell,\varkappa}(r)$.

If we consider physical λ and $\varkappa$, since $\phi_{\ell,\varkappa}(r) \to \hat{j}_\ell(\varkappa r)$ is real and (2.6) is a real equation, so is $\phi_{\ell,\varkappa}(r)$.

2.4 Jost function

Since $\phi_{\ell,\kappa}(r)$ and $\psi_{\ell,\kappa}(r)$ satisfy proportional initial conditions, they are proportional. Let $\phi_{\ell,\kappa}(r) = \omega_\ell(\kappa)\psi_{\ell,\kappa}(r)$, where we define the Jost function $\omega_\ell(\kappa)$. When we consider the strength parameter of the potential, we may write $\phi_{\ell,\kappa,\lambda}(r) = \omega_\ell(\kappa,\lambda)\psi_{\ell,\kappa,\lambda}(r)$. Since $\phi_{\ell,\kappa}(r)$ is real and as $r \to \infty$ it becomes a solution to the free equation (2.1), and $\hat{h}^{\pm *} = \hat{h}^{\mp}$ span the solutions of (2.1),

$$\phi_{\ell,\kappa}(r) \xrightarrow{r\to\infty} \frac{i}{2}\left(c_{\ell,\kappa}\hat{h}_\ell^-(\kappa r) - c_{\ell,\kappa}^* \hat{h}_\ell^+(\kappa r)\right) \quad (\text{real})$$

$$\longrightarrow \omega_\ell(\kappa)\frac{i}{2}\left(\hat{h}_\ell^-(\kappa r) - S_\ell(p)\,\hat{h}^+(\kappa r)\right)$$

(asymptotic form of $\psi_{\ell,\kappa}(r)$), we conclude that $c_{\ell,p} = \omega_\ell(\kappa)$, $\phi_{\ell,\kappa}(r) \xrightarrow{r\to\infty} \frac{i}{2}\left(\omega_\ell(\kappa)\hat{h}_\ell^+(\kappa r) - \omega_\ell(\kappa)^*\hat{h}_\ell^-(\kappa r)\right)$, so $S_\ell(p) = \dfrac{\omega_\ell(\kappa)^*}{\omega_\ell(\kappa)} = e^{2i\delta_\ell(\kappa)}$, so we have $\omega_\ell(\kappa) = |\omega_\ell(\kappa)|\, e^{-i\delta_\ell(\kappa)}$. Since by definition $\hat{\jmath}_\ell = \frac{i}{2}(\hat{h}_\ell^- - \hat{h}_\ell^+)$ and $\hat{\eta}_\ell = \frac{1}{2}(\hat{h}_\ell^- + \hat{h}_\ell^+)$, we can write the Green function of (2.8) as

$$g_\ell^\kappa(r,r') = \frac{1}{2\kappa}\left(\hat{h}_\ell^-(\kappa r)\,\hat{h}_\ell^+(\kappa r') - \hat{h}_\ell^+(\kappa r)\,\hat{h}_\ell^-(\kappa r')\right),$$

$$\phi_{\ell,\kappa}(r) = \hat{j}_\ell(\kappa r) + \lambda\int_0^r dr'\, g_\ell^\kappa(r,r')\, U(r')\,\phi_{\ell,\kappa}(r') =$$

$$= \frac{i}{2}\left[\left(1+\frac{\lambda}{\kappa}\int_0^r dr'\,\hat{h}_\ell^+(\kappa r')\,U(r')\,\phi_{\ell,\kappa}(r')\right)\hat{h}_\ell^-(\kappa r) - \left(\cdots\right)^*\hat{h}_\ell^+(\kappa r)\right]$$

$$\xrightarrow{r\to\infty} \frac{i}{2}\left(\omega_\ell(\kappa)\,\hat{h}_\ell^-(\kappa r) - \omega_\ell(\kappa)^*\,\hat{h}_\ell^+(\kappa r)\right),$$ so we

conclude that

$$\omega_\ell(\kappa,\lambda) = 1 + \frac{\lambda}{\kappa}\int_0^\infty dr\,\hat{h}_\ell^+(\kappa r)\,U(r)\,\phi_{\ell,\kappa,\lambda}(r) \qquad (2.9).$$

It will be handy to stablish a bound to $\omega - 1$. Using the bounds already established for $\hat{h}^+$ and ϕ, we have:

$$|\omega_\ell(\kappa,\lambda)-1| \leq \text{const.}\left|\frac{\lambda}{\kappa}\right|\int_0^\infty dr\left(\frac{|\kappa| r}{1+|\kappa| r}\right)^{-\ell}\left(\frac{|\kappa| r}{1+|\kappa| r}\right)^{\ell+1}|U(r)|\,e^{(|\mathrm{Im}\,\kappa|-\mathrm{Im}\,\kappa)r}$$

$$\leq \text{const.}\,|\lambda|\int_0^\infty dr\; r\,|U(r)|\,e^{(|\mathrm{Im}\,\kappa|-\mathrm{Im}\,\kappa)r} \qquad (2.10),$$

so without further assumptions on V ω_ℓ is bounded on $\mathrm{Im}\,\kappa \geq 0$ and analytic on $\mathrm{Im}\,\kappa > 0$ (open region), since the exponential term diverge otherwise.

2.5 Hankel-like solutions.

We need to introduce yet another pair of solutions to (2.6). We now seek solutions of the form $\chi^{\pm}_{l,\kappa}(r)$, with $\chi^{\pm}_{l,\kappa}(r) \xrightarrow{r\to\infty} \hat{h}^{\pm}_{l}(\kappa r)$. Luckly, we can use the same Green function $g_l^{\kappa}(r,r')$, but now with

$$\chi^{\pm}_{l,\kappa}(r) = \hat{h}^{\pm}_{l}(\kappa r) - \lambda \int_r^{\infty} dr'\, g_l^{\kappa}(r,r')\, U(r')\, \chi^{\pm}_{l,\kappa}(r'). \quad (2.11)$$

It's clear that $\chi^{\pm} \longrightarrow \hat{h}^{\pm}$ as $r \longrightarrow \infty$, and this definition solves (2.6) by the same arguments given to ϕ. And just like ϕ, we have

<u>Theorem 2.2</u> Without further assumption on the potential, (2.11) can be solved by iteration with any <u>complex</u> λ and with $\kappa \in \{\mathrm{Im}\,\kappa \geq 0\}$ in case of χ^{+} or $\kappa \in \{\mathrm{Im}\,\kappa \leq 0\}$ in case of χ^{-}. The solution $\chi^{\pm} = \sum_{n\geq 0} \lambda^n \chi^{\pm(n)}$ ($\chi^{\pm(0)}(r) = \hat{h}^{\pm}(\kappa r)$, $\chi^{\pm(n+1)}_{l,\kappa,\lambda} = -\int_r^{\infty} dr'\, g_l^{\kappa}(r,r')\, U(r')\, \chi^{\pm(n)}_{l,\kappa,\lambda}(r')$) is entire in λ, analytic for $\kappa \in \{\mathrm{Im}\,\kappa > (<)\, 0\}$ and

continuous for $\kappa \in \{ \operatorname{Im} \kappa \geq (\leq) 0 \}$.

"<u>Proof</u>": The proof is analogous to that of theorem 2.1. The only difference is that in the cancelations of (*) we have to restrict ourselves to the upper (lower) half plane of κ, since $|\hat{h}_\ell^{\pm}(\kappa r)| \leq C_{h^\pm} \left(\frac{|\kappa| r}{1+|\kappa| r} \right)^{-\ell} e^{\pm \operatorname{Im} \kappa r}$.

2.6 The Wronskian

We introduce the Wronskian of a pair of functions $f(r)$ and $g(r)$ as the function $W(f,g)(r) = \det \begin{bmatrix} f(r) & g(r) \\ f'(r) & g'(r) \end{bmatrix} = f(r)g'(r) - f'(r)g(r)$.

<u>Claim</u>: if f, g are linearly dependent, $W(f,g) = 0$.

We see that if $f = cg$ for some $c \in \mathbb{C}$, $W(f,g) = fg' - f'g = cgg' - cg'g = 0$.

<u>Claim</u>: if f, g solve $\left(\frac{d^2}{dr^2} + F(r) \right) y(r) = 0$, then $W(f,g)$ is independent of r.

We see that $W(f,g)' = f'g' + fg'' - f''g - f'g' = fg'' - f''g = -Ffg + Ffg = 0$, so $W(f,g)$ is independent of r.

It's conviniemt now to calculate the Wronskian of some solutions to our radial equations (2.1) and (2.6): When we're dealing with solutions to the same equation, we can calculate the Wronskian is some limit since it will be independent of r.

$$W(\hat{\jmath}_\ell, \hat{n}_\ell) \overset{r\to\infty}{=} W\left(\sin\left(\kappa r - \tfrac{1}{2}\ell\pi\right), -\cos\left(\kappa r - \tfrac{1}{2}\ell\pi\right)\right) = -\kappa$$

$$W(\chi^+_{\ell,\kappa}, \chi^-_{\ell,\kappa}) \overset{r\to\infty}{=} W(\hat{h}^+_\ell, \hat{h}^-_\ell) \overset{r\to\infty}{=} W\left(e^{i(\kappa r - \frac{1}{2}\ell\pi)}, e^{-i(\kappa r - \frac{1}{2}\ell\pi)}\right) = -2i\kappa$$

$$W(\chi^+_{\ell,\kappa}, \phi_{\ell,\kappa}) \overset{r\to\infty}{=} W\left(\hat{h}^+_\ell, \tfrac{i}{2}\left(\omega_\ell(\kappa)\hat{h}^-_\ell - \omega_\ell(\kappa)^*\hat{h}^+_\ell\right)\right) = \kappa\,\omega_\ell(\kappa),$$

since $W(\hat{h}^+_\ell, \hat{h}^+_\ell) = 0$. We arrived at the important result

$$\omega_\ell(\kappa) = \frac{1}{\kappa} W(\chi^+_{\ell,\kappa}, \phi_{\ell,\kappa}). \qquad (2.12)$$

3. Analytic properties of the Jost function and of the S matrix elements

3.1 Continuation of $\omega_\ell(\kappa)$ to the lower half plane, "reasonable" potentials.

The analytic properties of depend on the convergence of the integral (2.10),

$$|\omega_\ell(k) - 1| \leqslant \text{const.} \int_0^\infty dr\, r\,|U(r)|\, e^{(|\mathrm{Im}\,k| - \mathrm{Im}\,k)r},$$

that certainly convergent for $k \in \mathrm{Im}\,k$. But with further assumptions on $V(r)$, we can make $\omega_\ell(k)$ analytic in larger regions: We consider the "resonable" potentials:

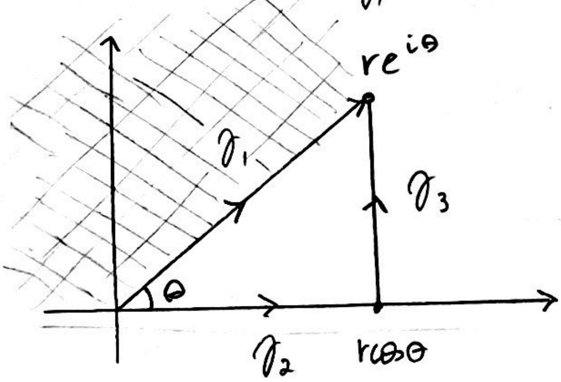
(i) $V(r) = 0$ for $r \geqslant a$, finite range potential: In this case, the integral of (2.10) is certanly convergent for all k because it runs from 0 to a (if thr other assumptions on V still hold). Then $\omega_\ell(k)$ is entire on k.

(ii) $V(r) \sim e^{-\mu r}$ with $\mu > 0$.
In this case, on the lower half plane the exponential part of the integral (2.10) will be $e^{-2\mathrm{Im}\,k\,r}$, combining with the potential we obtain $e^{-(\mu + 2\mathrm{Im}\,k)r}$, so the integral converges in the region $\mathrm{Im}\,k \geqslant -\mu/2$ of the lower half plane, and $\omega_\ell(k)$ is analytic for $\mathrm{Im}\,k > -\mu/2$.

(iii) $V(r)$ is analytic on $r>0$, and satisfy the usual assumptions on any ray $\{re^{i\theta};\ 0\le r<\infty\}$ on the half plane $\mathrm{Re}\, re^{i\theta}>0$: (A Yukawa potential is an exemple of this, since in the half plane $\mathrm{Re}\, re^{i\theta}<0$ it's divergent).

In this case we can see that since for a contour integral $\left|\int_\gamma f(z)\,dz\right| \le \max_{z\in\gamma} f(z)\, L_\gamma$ where γ is a finite path of length L_γ, and $\int_\gamma dz\, \hat{h}^+ U\phi$ is path independent,

$$\underbrace{\int_{[0,re^{i\theta}]} dz\, \hat{h}^+(\kappa z)\,U(z)\,\phi(z)}_{\gamma_1} = \underbrace{\int_0^r dr\, \hat{h}^+(\kappa r)\,U(r)\phi(r)}_{\gamma_2} + \underbrace{\int_{[r\cos\theta,\, re^{i\theta}]} dz\, \hat{h}^+(\kappa z)U(z)\phi(z)}_{\gamma_3}$$



as $r\to\infty$, $|U(z)|\sim |z|^{-3-\varepsilon} = r^{-3-\varepsilon}$,

$$\left|\int_{[r\cos\theta,\, re^{i\theta}]} dz\, \hat{h}^+(\kappa z)U(z)\phi(z)\right| \le c\left(\frac{|\kappa|r}{1+|\kappa|r}\right)^{-\ell} e^{\mathrm{Im}\,\kappa re^{i\theta}} \frac{1}{r^{3+\varepsilon}}$$

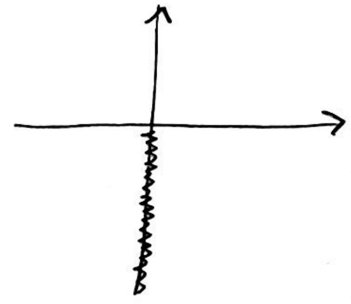
$$\left(\frac{|\kappa|r}{1+|\kappa|r}\right)^{\ell+1} e^{|\mathrm{Im}\,\kappa re^{i\theta}|} \xrightarrow{r\to\infty} 0 \quad \text{if} \quad \mathrm{Im}(\kappa e^{i\theta})>0,$$

the shaded region, and we have

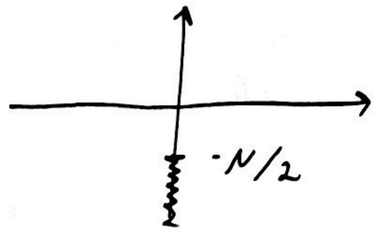
$$\int_0^{\infty e^{i\theta}} dr\, \hat{h}^+(\kappa r)\,U(r)\,\phi(r) = \int_0^\infty dr\, \hat{h}^+(\kappa r)\,U(r)\,\phi(r), \quad \text{and}$$

those are analytic functions of κ on $\mathrm{Im}(\kappa e^{i\theta})>0$.

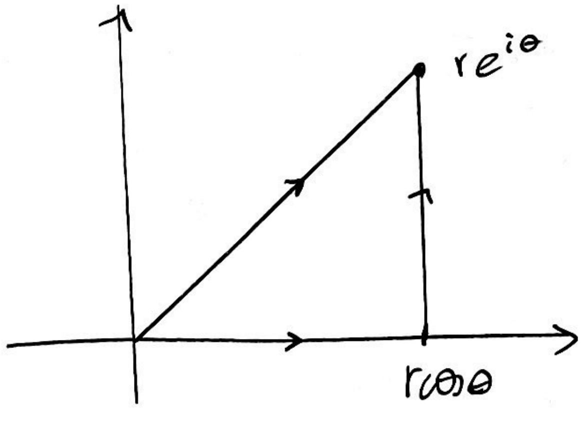
Since θ can take any value from $-\pi$ to π, $\omega_\ell(k)$ is analytic in this case for $k \in \mathbb{C} \setminus \{bi;\ b<0\}$:

$V(r)$ analytic $\Rightarrow$ $\omega_\ell(k)$ analytic on 

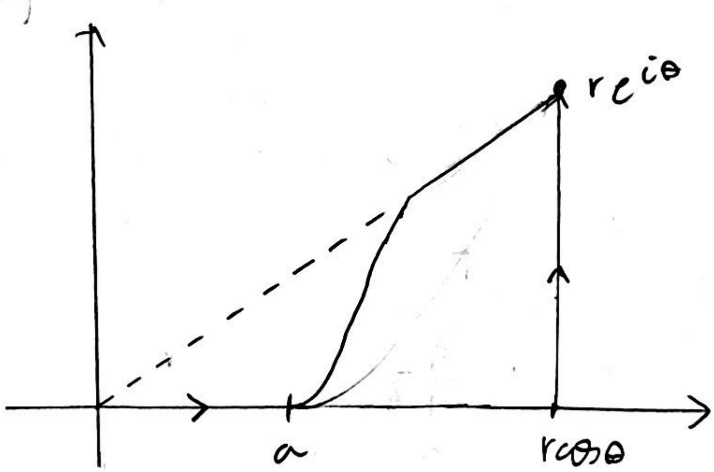
If $V(r) \sim e^{-Nr}$ also, we combine the two conclusions to state that $\omega_\ell(k)$ is analytic for

$k \in \mathbb{C} \setminus \{bi\ ;\ b< -N/2\}$ 

We can draw the same conclusion if $V(r)$ is analytic for $r>a$, all we have to do is use a different contour :



V analytic



V analytic for $r>a$

It's convenient to highlight that there isn't much physical meaning to the obstacles that prevent us from continuing $\omega_\ell(\varkappa)$ to the lower half plane: if, for instance, $V(r) = 0$ for $r > 10$ Km the continuation is possible, but in practice a scattering experiment has now way to tell if $V(r) \neq 0$ for r that large. We shall call a potential that allows the continuation of $\omega_\ell(\varkappa)$ to the lower half plane (possibly with the exclusion of some region where $\mathrm{Im}\,\varkappa < -\mu/2$) reasonable potentials.

3.2 S matrix elements as meromorphic functions.

We've seen in section 2.3 that $\phi_{\ell,-\varkappa}(r) = (-1)^{\ell+1}\phi_{\ell,\varkappa}(r)$, and since $\hat{h}^+_\ell(-\varkappa r) = \hat{h}^{-*}_\ell(\varkappa r) = (-1)^\ell \hat{h}^{+*}_\ell(\varkappa r)$, from 2.9 we have

$$\omega_\ell(-\varkappa) = 1 - \frac{1}{\varkappa}\int_0^\infty dr\, \hat{h}^+_\ell(-\varkappa r)\, U(r)\, \phi_{\ell,\varkappa}(r)$$

$$= 1^* + \frac{1}{\varkappa^*}\int_0^\infty dr\, \hat{h}^{+*}_\ell(\varkappa r)\, U(r)^*\, \phi_{\ell,\varkappa}(r)^* = \omega_\ell(\varkappa^*)^*$$

for physical $k = k^*$. By the Schwartz reflection principle, if an analytic function $f: G \to \mathbb{C}$ can be continued to G^* and $f(z) = f(z^*)^*$ in some set with a cluster point on G, then $f(z) = f(z^*)^*$ $\forall z \in G$. Since on the real line we have $\omega_\ell(k) = \omega_\ell(-k^*)^*$. Applying Schwartz principle for $k = iz$, $\omega_\ell(z) = \omega_\ell(z^*)^*$, so we conclude that $\omega_\ell(k) = \omega_\ell(-k^*)^*$ $\forall k$ in the region of analycity of ω_ℓ.

For a reasonable potential, we see that since for physical k we have that

$$S_\ell(k) = \frac{\omega_\ell(k)^*}{\omega_\ell(k)} = \frac{\omega_\ell(k^*)^*}{\omega_\ell(k)} = \frac{\omega_\ell(-k)}{\omega_\ell(k)}, \quad (3.1)$$

$S_\ell(k)$ is meromorphic in some region around the real axis (if, for exemple, $\omega_\ell(k)$ is analytic for $\operatorname{Im} k > -\mu/2$, $S_\ell(k)$ is meromorphic on the strip $-\mu/2 < \operatorname{Im} k < \mu/2$).

3.3 Bound states and zeros of ω_ℓ / poles of S_ℓ.

Suppose that $\omega_\ell(\varkappa_0) = 0$ for some $\varkappa_0$ with $\operatorname{Im} \varkappa_0 > 0$. If the potential is reasonable, this correspond to a pole of $S_\ell(\varkappa) = \dfrac{\omega_\ell(-\varkappa)}{\omega_\ell(\varkappa)}$ at $\varkappa = \varkappa_0$.

According to (2.12), $\varkappa\omega_\ell(\varkappa) = W(\chi^+_{\ell,\varkappa}, \phi_{\ell,\varkappa})$, so $W(\chi^+_{\ell,\varkappa_0}, \phi_{\ell,\varkappa_0}) = \varkappa_0\omega_\ell(\varkappa_0) = 0$, so using our discution about the Wronskian on section 2.6 we can state that $\chi^+_{\ell,\varkappa_0}(r)$ and $\phi_{\ell,\varkappa_0}(r)$ are proportional. Since $\chi^+_{\ell,\varkappa}(r) \xrightarrow{r\to\infty} \hat{h}^+_\ell(\varkappa r)$, we have $|\phi_{\ell,\varkappa_0}(r)| = |c\chi^+_{\ell,\varkappa_0}(r)| \xrightarrow{r\to\infty} |c h^+_\ell(\varkappa_0 r)| \leqslant$ $\leqslant \text{const} \cdot e^{-\operatorname{Im}\varkappa_0 r}$ according to (2.14), and since $\operatorname{Im} \varkappa_0 > 0$ and $\phi_{\ell,\varkappa_0}(r) \xrightarrow{r\to 0} \hat{\gamma}_\ell(\varkappa_0 r)$, $\dfrac{\phi_{\ell,\varkappa_0}(r)}{r\varkappa_0} Y_{0,m}(\hat{r})$ is a proper (normalizable) eigenfunction of the Hamiltonian with eigenvalue $E = \dfrac{\varkappa_0^2}{2m}$. Since H is hermitian, $\dfrac{\varkappa_0^2}{2m} \in \mathbb{R}$, but by assumption $\operatorname{Im} k_0 > 0$, so $k_0 = i\alpha$ and $E = \frac{k_0^2}{2m} = -\frac{\alpha^2}{2m}$, we've constructed a normalizable, localized state with negative energy: a bound state. On the other hand, if the Hamiltonian admits a bound state of energy $-\frac{\alpha^2}{2m}$, the Jost function has to vanish at $i\alpha$, because $\phi_{i\alpha, l}$ has to be normalizable.

So we've just prooved

<u>Theorem 3.1</u> : There is a one-to-one correspondence between zeros of $\omega_l(k_0) = 0$ in the positive imaginary axis and bound states of energy $E = \frac{-|k_0|^2}{2m}$. If the potential is reasonable, the correspondence also holds for poles of S_l, on the positive imaginary axis, since if $\omega_l(k_0) = 0$, $\omega_l(-k_0) \neq 0$, otherwise we would have $\phi_{l,k_0}(r) = \frac{i}{2}\left(\omega_l(k_0)\chi^-_{k_0,l}(r) - \omega_l(-k_0)\chi^+_{k_0,l}(r)\right) = \equiv 0$.

3.4 Levinson's theorem

In order to proove Levinson's theorem, we first need to proove 4 facts:

__Lema 3.2__: The zeros of ω_ℓ at bound states are always simple:

If $\omega_\ell(\varkappa_0) = 0$ at the positive imaginary axis, $\phi_{\varkappa_0,\ell} = c\,\chi^+_{\varkappa_0,\ell}$, since $\omega_\ell(\varkappa) = \frac{1}{\varkappa} W(\chi^+_{\varkappa_0,\ell}, \phi_{\varkappa_0,\ell})$.

Let $\frac{d}{d\varkappa} = \dot{}$, $\dot{\omega} = \frac{1}{\varkappa}\left(W(\chi^+, \dot{\phi}) + W(\dot{\chi}^+, \phi)\right)$. Since both ϕ and χ^+ solve (2.6), we have:

$$W'(\chi^+_{\varkappa_0}, \phi_\varkappa) = \chi^+_{\varkappa_0}\phi''_\varkappa - \chi^{+''}_{\varkappa_0}\phi_\varkappa =$$

$$= \left(\frac{\ell(\ell+1)}{r^2} + U(r) - \varkappa^2\right)\chi^+_{\varkappa_0}\phi_\varkappa - \left(\frac{\ell(\ell+1)}{r^2} + U(r) - \varkappa_0^2\right)\chi^+_{\varkappa_0}\phi_\varkappa$$

$$= \left(\varkappa_0^2 - \varkappa^2\right)\chi^+_{\varkappa_0}\phi_\varkappa,$$ so $W(\chi^+_{\varkappa_0}, \phi_\varkappa) = (\varkappa_0^2 - \varkappa^2)\int_0^r dr\,\chi^+_{\varkappa_0}\phi_\varkappa$,

and $$W(\chi^+_{\varkappa_0}, \dot{\phi}_\varkappa) = -2\varkappa\int_0^r dr'\,\chi^+_{\varkappa_0}\phi_\varkappa;$$

$$W(\chi^+_\varkappa, \dot{\phi}_\varkappa) = -2\varkappa\int_0^r dr'\,\chi^+_\varkappa\phi_\varkappa.$$ Similarly, we have $$W(\dot{\chi}^+_\varkappa, \phi_\varkappa) = -2\varkappa\int_r^\infty dr'\,\chi^+_\varkappa\phi_\varkappa,$$ so

$$\dot{\omega}(\varkappa_0) = \frac{1}{\varkappa_0}\left(-2\varkappa_0\int_0^r dr'\,\chi^+_{\varkappa_0}\phi_{\varkappa_0} - 2\varkappa_0\int_r^\infty dr'\,\chi^+_{\varkappa_0}\phi_{\varkappa_0}\right)$$

$$= -2c\int_0^r dr'\,(\chi^+_{\varkappa_0})^2.$$ $\chi_{\varkappa_0} = i^\ell$ (real function)

because $\varkappa_0$ is pure imaginary, $\dot{\omega}(\varkappa_0) \neq 0$, the zero $\omega(\kappa_0) = 0$ is of order 1.

<u>Lemma 3.3</u>: If $V(r) = O(r^{-2+\eta})$ as $r \to 0$ (for $0 < \eta < 1$), then $|\omega_e(\kappa) - 1| = O(|\kappa|^{-\eta})$ as $|\kappa| \to \infty$ on the upper half plane:

From (2.10), $|\omega_e(\kappa) - 1| \leqslant \frac{\text{const.}}{|\kappa|} \int_0^\infty dr \frac{|\kappa| r}{1 + |\kappa| r} |U(r)| e^{(|\mathrm{Im}\,\kappa| - \mathrm{Im}\,\kappa) r} =$

$$\leqslant \text{const} \left(\int_0^1 dr \frac{r}{1+|\kappa| r} |U(r)| + c' \int_1^\infty dr \frac{r}{1+|\kappa| r} \frac{1}{r^{2-\eta}} \right)$$

$$\longrightarrow O(|\kappa|^{-1}) + O(|\kappa|^{-\eta}) = O(|\kappa|^{-\eta}).$$

<u>Lemma 3.4</u>: If ω_e vanishes at the origin, it does so linearly or quadratically:

For a reasonable potential, f_e is analytic at 0, so it assumes a power series expansion $\omega_e(\kappa) = \sum_{n \geq 0} c_n \kappa^n$. We have

$$\omega_e = 1 + \frac{1}{\kappa} \int \hat{h}^+ U \phi = 1 + \frac{1}{p} \int \hat{\eta} U \phi + \frac{i}{p} \int \hat{\gamma} U \phi .$$

Recall that $\hat{\gamma}$ and $\hat{\phi}$ are expanded in powers of κ^2 times $\kappa^{\ell+1}$, and $\hat{\eta}$ in powers of κ^2 times $\kappa^{-\ell}$. So

$$\omega_e(\kappa) = 1 + (\alpha_e + \beta_e \kappa^2 + O(\kappa^4)) + i(\gamma_e \kappa^{2\ell+1} + O(\kappa^{2\ell+3}))$$

with real coefficients.

Now for $l=0$, we have

$$\omega_0(\kappa) = 1+\alpha_0 + i\gamma_0 \kappa + O(\kappa^2), \qquad (3.2)$$

and if $\omega_0(\kappa=0)=0$, $1=-\alpha_0$, and it vanishes linearly. For $l>1$, we have

$$\omega_l(\kappa) = 1+\alpha_l + \beta_l \kappa^2 + O(\kappa^3 \text{ or } \kappa^4), \qquad (3.3)$$

again, if it vanishes it does so quadratically.

<u>Lema 3.5</u>: If f is analytic with a simple zero at z_0, then f'/f has a simple pole of residue 1 at z_0:

Since f has a simple zero at z_0, $\exists g$ analytic such that $f(z)=(z-z_0)g(z)$, with $g(z_0)\neq 0$. Now

$$\frac{f'(z)}{f(z)} = \frac{g(z)+(z-z_0)g'(z)}{(z-z_0)g(z)} = \frac{1}{z-z_0} + \frac{g'(z)}{g(z)},$$

since $g'(z)/g(z)$ is analytic at z_0 because $g(z_0)\neq 0$, f'/f has a simple pole at z_0 with residue 1.

Lemma 3.6 : There are no zero energy bound states for $l=0$, and for $l>0$ there is a zero energy bound state of angular momentum l if and only if we have $f_l(0)=0$.

For zero energy the radial equation (2.6) becomes $\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-U(r)\right]y(r)=0$, where $\frac{l(l+1)}{r^2}$ dominates for small and large r.

Without the potential, the solutions are r^{l+1} and r^{-l} (for small and large r).

In the case $l=0$, r, 1 are not normalizable, so a bound state of $l=0$ and zero energy is impossible. For $l>0$,

let $\tilde{\phi}_l := \frac{\phi_l}{\kappa^{l+1}} \xrightarrow{r\to 0} \frac{\hat{j}_l}{\kappa^{l+1}} \xrightarrow{r\to 0} \frac{r^{l+1}}{(2l+1)!!}$, normalizable at the origin

$\tilde{\chi}^{\pm}_l := \kappa^l \chi^{\pm}_l \xrightarrow{r\to\infty} \kappa^l \hat{h}^{\pm}_l \xrightarrow{r\to\infty} r^{-l}$, normalizable at infinity

appropriate zero energy basis. Since

$$\omega_l(\kappa) = \frac{1}{\kappa} W(\chi^{\pm},\phi) = \frac{1}{\kappa}\kappa^{l+1}\frac{1}{\kappa^l} W(\tilde{\chi}^{\pm},\tilde{\phi}) = W(\tilde{\chi}^{\pm},\tilde{\phi}),$$

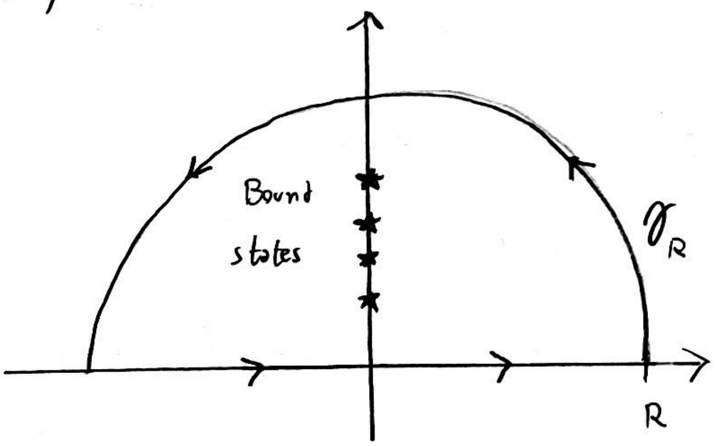
we have a normalizable solution if and only if $\omega_l(0)=0$.

Now we are ready to proove

Theorem 3.6 (Levinson's theorem): For any spherical potential subject to our usual assumptions, the phase shifts satisfy $\delta_l(0) - \delta_l(\infty) = n_l \pi$, where n_l is the number of bound states with angular momentum l, if $\omega_0(0) \neq 0$. In the case that $\omega_0(0) = 0$, we have $\delta_0(0) - \delta_0(\infty) = (n_0 + \frac{1}{2})\pi$.

Proof: According to the Lema 3.3, for large enough $|k|$ the Jost function is zeroless, and according to Lema 3.4 the zeros don't accumulate at 0. So we conclude that, since zeros of analytic functions are isolated, there is a finite amount n_l of zeros of $\omega_l(k)$ in the upper half plane, all of than in the imaginary axis. Let's suppose that $\omega_l(0) \neq 0$. Then we consider the closed contour integral

$I = \oint_{\gamma_R} d\kappa \, \dfrac{\dot{\omega}_\ell(\kappa)}{\omega_\ell(\kappa)}$, with a semicircle contour large enough to contain all the zeros of ω_ℓ. Using Cauchy's theorem and Lema 3.5, we see that $I = 2\pi i M_\ell$, and the radius of the semicircle can be



taken to infinity without changing this result. Since Lema 3.3 ensures that the contribution of the semicircle goes to zero, we have

$$2\pi i M_\ell = \oint_{\gamma_R} d\kappa \frac{\dot{\omega}_\ell(\kappa)}{\omega_\ell(\kappa)} = \oint_{\gamma_R} d\ln \omega_\ell(\kappa) = \lim_{R\to\infty} \oint_{\gamma_R} d\ln \omega_\ell(\kappa)$$

$$= \int_{-\infty}^{+\infty} d\ln \omega_\ell(\kappa)\, d\kappa = I .$$ Finally, since we have from our discussion on section 2.4 that $\omega_\ell(\kappa) = |\omega_\ell(\kappa)|\, e^{-i\delta_\ell(\kappa)}$, and since for real κ $\omega_\ell(-\kappa) = \omega_\ell(\kappa^*)^* = \omega_\ell(\kappa)^*$,

$\ln \omega_\ell(\kappa) = \ln|\omega_\ell(\kappa)| - i\delta_\ell(\kappa)$ for $\kappa \geq 0$

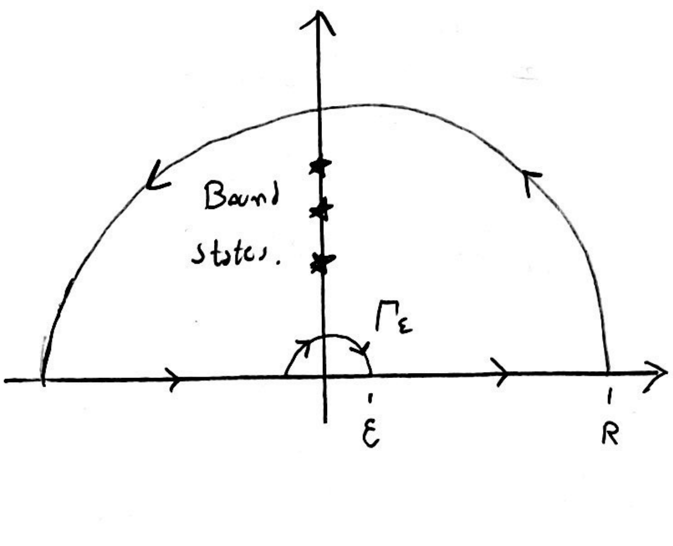
$\ln \omega_\ell(\kappa) = \ln|\omega_\ell(\kappa)| + i\delta_\ell(\kappa)$ for $\kappa < 0$.

Since $I = 2\pi i n_\ell$ is purely imaginary, the real part contribution vanishes, so

$$I = -2i \int_0^{+\infty} d\delta_\ell(k) = 2i\left(\delta_\ell(0) - \delta_\ell(\infty)\right) = 2\pi i n_\ell ,$$

so we finally obtain $\delta_\ell(0) - \delta_\ell(\infty) = n_\ell \pi$.

Now let's suppose that $\omega_\ell(0) = 0$. The contour has to be modified to exclude the singularity of $\dot{\omega}/\omega$: We can exclude it by adding a smaller semicircle Γ_ε, small enough to only contain the $\omega_\ell(0) = 0$ zero of ω_ℓ, as Lemma 3.4 allows us to do:



Now $I = \oint_{\gamma_{R,\varepsilon}} dk \dfrac{\dot{\omega}_\ell(k)}{\omega_\ell(k)}$,

$I = 2\pi i\,(\text{zeros in } \mathrm{Im}\, k > 0) =$

$= 2i\left(\delta_\ell(0) - \delta_\ell(\infty)\right) +$

$+ \lim\limits_{\varepsilon \to 0} \displaystyle\int_{\Gamma_\varepsilon} dk \frac{\dot{\omega}_\ell(k)}{\omega_\ell(k)}$. The expansions (3.2) and (3.3) guarantee that the zero of ω_ℓ at 0 is simple if $\ell = 0$ or double if $\ell > 0$ (only for $\ell = 0$ this can be a bound state zero!).

We know that $\frac{\dot{\omega}_l(\kappa)}{\omega_l(\kappa)}$ has a simple pole of residue $m = 1, 2$ depending on l, because as it's clear from the expansion (3.3) the zero of ω_l is either simple or double, so we have $\frac{\dot{\omega}_l(\kappa)}{\omega_l(\kappa)} = \frac{m}{\kappa} + O(|\kappa|)$ near the origin but $\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \frac{\dot{\omega}_l(\kappa)}{\omega_l(\kappa)} d\kappa =$

$$= \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} \frac{m\,dz}{z} = m \lim_{\varepsilon \to 0} \int_{\pi}^{0} \frac{\varepsilon i e^{i\theta}\, d\theta}{\varepsilon e^{i\theta}} = m i \int_{\pi}^{0} d\theta = -m i \pi,$$

so for $l > 0$ $I = 2\pi i (n_l - 1) = 2i(\delta_l(0) - \delta_l(\infty)) - 2i\pi$, the result is the same (there's an missing bound state!). However for $l = 0$, we obtain $2\pi i n_0 = 2i(\delta_0(0) - \delta_0(\infty)) - i\pi$, and we have $\delta_0(0) - \delta_0(\infty) = (n_0 + \frac{1}{2})\pi$.

3.5 Scattering length

At small energies, the scattering amplitude is dominated by the scattering length. Recall that the partial-wave amplitude is $f_\ell(\kappa) = \dfrac{S_\ell(\kappa) - 1}{2i\kappa} = \dfrac{\omega_\ell(-\kappa) - \omega_\ell(\kappa)}{2i\kappa\, \omega_\ell(\kappa)}$.

Using the integral form (2.9) of $\omega_\ell(\kappa)$,

$$\omega(\kappa) = 1 + \frac{1}{\kappa}\int dr\, \hat{h}^+(\kappa r)\, U(r)\, \phi_\kappa(r),$$

and since $\hat{h}^+(-\kappa r)\, \phi_{-\kappa}(r) = (-1)^\ell\, \hat{h}^-(\kappa r)\, (-1)^{\ell+1}\phi_\kappa(r) = -\hat{h}^-(\kappa r)\, \phi_\kappa(r)$, we obtain

$$f_\ell(\kappa) = \frac{1}{\omega_\ell(\kappa)\kappa^2}\int_0^\infty dr \left(\frac{\hat{h}^+(\kappa r) - \hat{h}^-(\kappa r)}{2i}\right) U(r)\, \phi_{\kappa,\ell}(r)$$

$$= \frac{1}{\omega_\ell(\kappa)\kappa^2}\int_0^\infty dr\, \hat{\jmath}_\ell(\kappa r)\, U(r)\, \phi_{\kappa,\ell}(r).$$

For physical κ, both $|\phi_\ell|, |\hat{\jmath}_\ell| \leqslant \left(\frac{|\kappa| r}{1+|\kappa| r}\right)^{\ell+1}$, so for a neighborhood of the origin, provided that $f_\ell(0) \neq 0$,

$$|f_\ell(\kappa)| \leq \frac{\text{const.}}{|f_\ell(0)|}\,\frac{1}{\kappa^2}\int_0^\infty dr\,|U(r)|\left(\frac{|\kappa| r}{1+|\kappa| r}\right)^{2\ell+2} \leq \frac{\text{const.}}{|f_\ell(0)|}\,\kappa^{2\ell}\int_0^\infty dr\, r^{2\ell+2}|U(r)|$$

so provided that the integral converges (V vanish eventually, V exponentially bounded, etc.), the expansion of f_ℓ around $\kappa = 0$ will have the form

$f_\ell(\kappa) = -a_\ell \kappa^{2\ell} + O(\kappa^{2\ell+1})$, where a_ℓ is the definition of scattering lenght.

4. Resonances

4.1 Resonances vs. poles of the S matrix elements

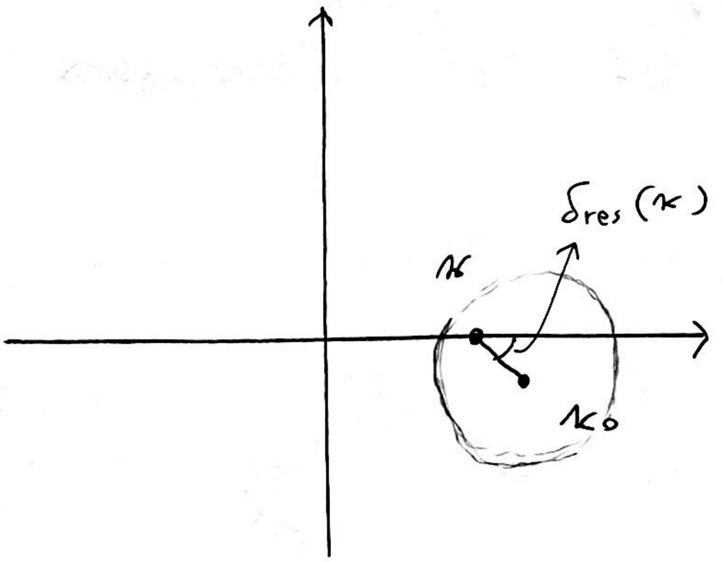
Let's assume that the potential is reasonable, and that $w_\ell(\kappa_0) = 0$ for some κ_0 with $\operatorname{Im}\kappa_0 < 0$, $\kappa = \kappa_R - i\kappa_I$. By exactly the same arguments used to proove theorem 2.1, we can see that $\phi_{\kappa_0,\ell}$ is not normalizable, so it can't represent a bound state. Further, suppose that κ_0 is a simple zero

So since ω_ℓ is analytic at k_0, we can approximate $\omega_\ell(k) \cong \dot{\omega}_\ell(k_0)(k-k_0)$ near k_0. We've already seen that $\arg(\omega_\ell(k)) = -\delta_\ell(k)$, so we can approximate the phase shift to

$$\delta_\ell(k) \cong -\arg \dot{\omega}_\ell(k_0) - \arg(k-k_0) \qquad (4.1)$$

$$\equiv \delta_{bg} + \delta_{res}(k),$$

where $\delta_{res}(k) = -\arg(k-k_0)$. If k_0 is close to the real line this approximation is good for physical k near k_R:



As $k < k_R$ approaches k_R, $\delta(k)$ increases fast from δ_{bg} to $\delta_{bg} + \pi$. This rapid increase defines a resonance of angular momentum ℓ.

Notice that

$$\sin \delta_{res}(k) = \frac{k_I}{\sqrt{(k-k_R)^2 + k_I^2}} \qquad (4.2)$$

so for a small δ_{bg} (pure Breit-Wigner)

we have $\sigma_\ell(k) \propto \sin^2 \delta_\ell(k) = \dfrac{k_I^2}{(k-k_R)+k_I^2}$,

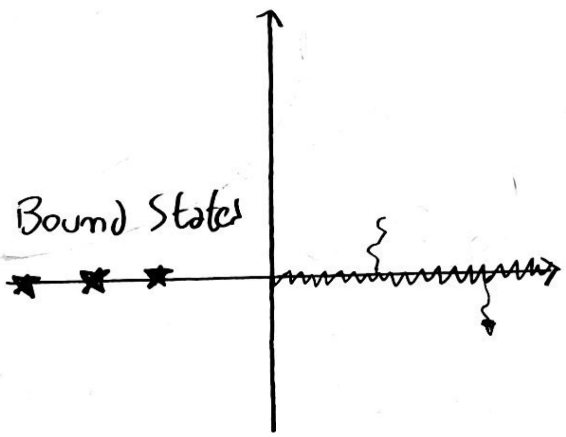
with a peak at $k = k_R$.

Notice however that not always there is a correspondence between zeros of $\omega_\ell(k)$ on $\mathrm{Im}\, k < 0$ (or poles of S_ℓ) and resonances. If the zero is far away from the real axis it will have no physical effect whatsoever.

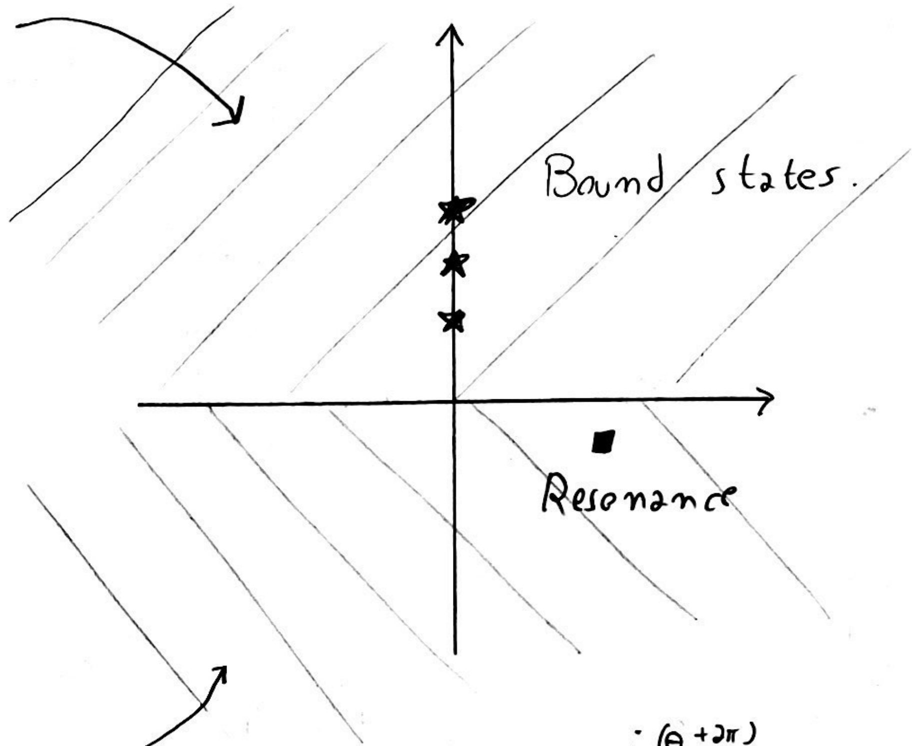
4.2 Expressing σ_ℓ in terms of E.

Since $k = \sqrt{2mE}$ and the square root is in fact a double valued function, it is defined on a Riemann surface $\mathcal{R}$, that can be covered by two sheets:

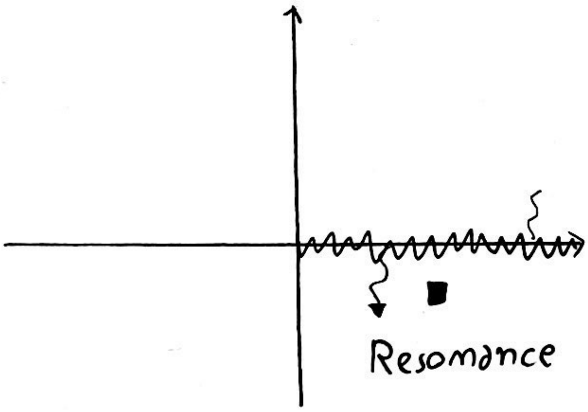
Physical sheet



$re^{i\theta} \rightarrow \sqrt{r}\, e^{i\theta/2}$ k Plane



Second sheet



$re^{i\theta} \rightarrow -\sqrt{r}\, e^{i\theta/2} = \sqrt{r}\, e^{i\frac{(\theta+2\pi)}{2}}$

So in terms of E, $\omega_\ell(E)$ is defined on a Riemann surface $\omega_\ell : R \rightarrow \mathbb{C}$.

On the second sheet, we define a resonance via $E_0 = \frac{k_0^2}{2m} = E_R - i\frac{\Gamma}{2}$, a pure Breit-Wigner has the form (4.2)

$$\sigma_\ell(E) \propto \frac{(\Gamma/2)^2}{(E-E_R) + (\Gamma/2)^2}$$

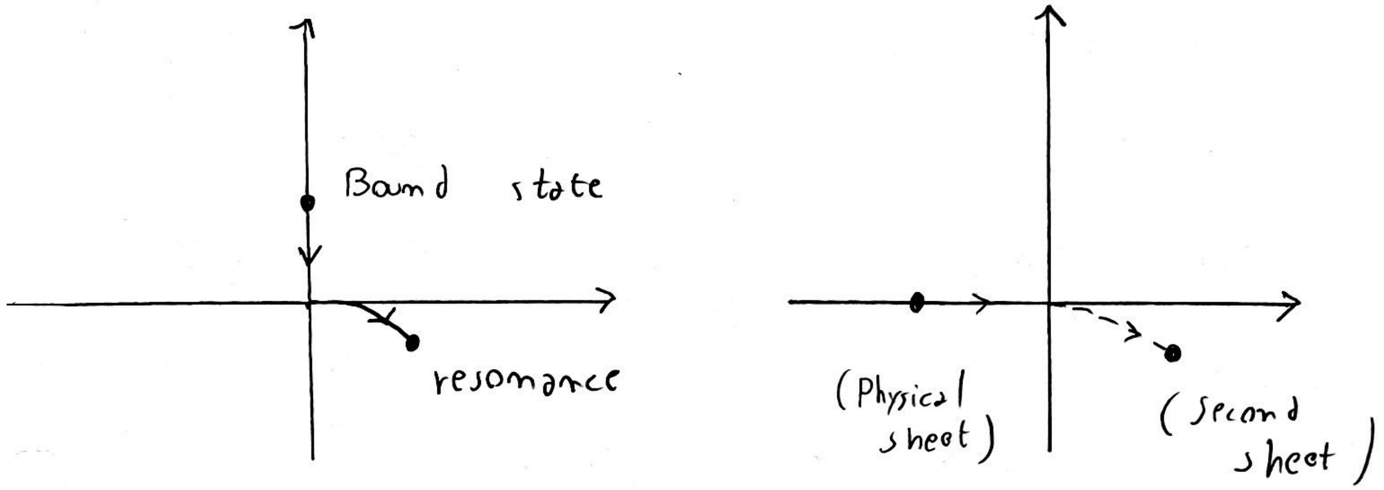
4.3 Bound states and resonances:

We will now use de dependence of $\omega_\ell(\kappa, \lambda)$ on the strengh parameter λ to show that resonances can be regarded as "would-be bound states". For a reasonable potential, $\omega_\ell(\kappa,\lambda)$ is analytic at $\kappa=0$ and entire on λ. Let's more λ around physical (real) values near λ_0, and expand $\omega_\ell(\kappa,\lambda)$ as

$$\omega_\ell(\kappa,\lambda) = \sum_{m,n\geq 0} \alpha_{mn} \kappa^m (\lambda-\lambda_0)^n$$

Now suppose that $\ell>0$, and for $\lambda=\lambda_0$ $\omega_\ell(0,\lambda_0)=0$. Using the expansion (3.3), $\omega_0(P,\lambda_0) = \alpha\kappa^2 + \beta(\lambda-\lambda_0) + \cdots$, so for λ close to λ_0 the zeros are located at $\kappa_0 \simeq \pm i\sqrt{\frac{\beta}{\alpha}(\lambda-\lambda_0)}$. On the κ plane, as $\lambda>\lambda_0$ approaches a bound state (zero on $\mathrm{Im}\,\kappa>0$) approaches the origin, and splits into two zeros, one of than possibly a resonance.

On the Riemann surface, this correspond to a migration of the zero E_0 from the physical sheet from a resonance position on the second sheet:



$\varkappa$ Plane R surface.

For those resonances close to the origin "emerging" from bound states, δ_{bg} is always small. Since as long $\varkappa_0$ represent a bound state we have $\varkappa_0$ pure imaginary, we see that $\omega(\varkappa_0) = \omega(-\varkappa_0^*)^* = \omega(\varkappa_0)^*$, $\omega(\varkappa_0)$ is real, so $\dot{\omega}(\varkappa_0)$ also is.

As λ varies slowly and $\varkappa_0$ becomes a resonance, $\dot{\omega}(\varkappa_0)$ remains predominantly real, so $\delta_{bg} = -\arg \dot{\omega}(\varkappa_0)$ is small.

4.4 Resonances viewed as "unstable bound states"

Let's study the scattering of a wave package $|\psi_{in}\rangle = \int d^3k\, \phi(\vec{k})\, |\vec{k}\rangle$, were $\phi(\vec{k})$ is peaked about $\vec{k} = \vec{k}_0$. The scattering wave function will be (recall our discussion on section 1.1)

$$\psi(\vec{r},t) = \langle \vec{r}\,|U(t)|\psi\rangle = \langle \vec{r}\,|U(t)\Omega_+|\psi_{in}\rangle =$$

$$= \langle \vec{r}\,|U(t)\Omega_+ \int d^3k\, \phi(\vec{k})\, |\vec{k}\rangle$$

$$= \langle \vec{r}\,|U(t) \int d^3k\, \phi(\vec{k})\, |\vec{k}+\rangle = \int d^3k\, e^{-iE_k t}\, \phi(\vec{k}) \langle \vec{r}\,|\vec{k}+\rangle$$

$$\xrightarrow{r\to\infty} \frac{1}{(2\pi)^{3/2}} \int d^3k\, \phi(\vec{k})\, e^{-iE_k t} \left(e^{i\vec{k}\cdot\vec{r}} + f(k\hat{r}\,|\vec{k})\, \frac{e^{ikr}}{r} \right)$$

$$= \psi_{in}(\vec{r},t) + \psi_{sc}(\vec{r},t)$$

, a free propagating wave package and an outgoing spherical wave package. We now suppose that the only appreciable contribution to the scattering amplitude comes from a pure Breit-Wigner resonance

$E_0 = E_R - i\frac{\Gamma}{2}$, with width much smaller then the spread ΔE of $\phi(\vec{\kappa})$, $\Gamma << \Delta E$. In that case,

$$f(\kappa\hat{r}|\vec{\kappa}) = \sum_{l\geq 0}(2l+1)\, f_l(E)\, P_l(\cos\theta)$$

$$= (2l+1)\, f_l(E)\, P_l(\cos\theta)$$

$$= -\frac{1}{2\kappa}(2l+1)\frac{\Gamma}{E-E_R+i\Gamma/2}\, P_l(\cos\theta).$$

So we'll have:

$$\psi_{sc}(\vec{r},t) \xrightarrow{r\to\infty} \text{const} \times \int d^3\kappa\, \phi(\vec{\kappa})\, e^{-iE_\kappa t} f(\kappa\hat{r}|\vec{\kappa})\frac{e^{i\kappa r}}{r}$$

$$= \text{const} \times \frac{\Gamma}{r}\int d^3\kappa\, \frac{\phi(\vec{\kappa})\, P_l(\cos\theta)}{\kappa(E_\kappa - E_R + i\Gamma/2)}\, e^{i(\kappa r - E_\kappa t)}$$

Aligning $\vec{\kappa}$ and $\hat{z}$, and assuming $\phi(\vec{\kappa})$ invariant under rotations about $\hat{z}$, we must have $\phi(\vec{\kappa}) = \sum_{l\geq 0}\phi_l(\kappa)\, P_l(\cos\theta) =$

$$= \frac{1}{\sqrt{m\kappa}}\sum_{l\geq 0}\phi_l(E_\kappa)\, P_l(\cos\theta).$$

An integration on the angular part of $d^3\kappa$ will them selecionate only $\phi_l(E_\kappa)$ wher l is the angular momentum of the

$E_0 = E_R - i\frac{\Gamma}{2}$, with width much smaller then the spread ΔE of $\phi(\vec{\kappa})$, $\Gamma \ll \Delta E$. In that case,

$$f(\kappa\hat{r}|\vec{\kappa}) = \sum_{l\geq 0}(2l+1) f_l(E) P_l(\cos\theta)$$

$$= (2l+1) f_l(E) P_l(\cos\theta)$$

$$= -\frac{1}{2\kappa}(2l+1)\frac{\Gamma}{E-E_R+i\Gamma/2} P_l(\cos\theta).$$

So we'll have:

$$\Psi_{sc}(\vec{r},t) \xrightarrow{r\to\infty} const \times \int d^3\kappa\, \phi(\vec{\kappa})\, e^{-iE_\kappa t} f(\kappa\hat{r}|\vec{\kappa})\frac{e^{i\kappa r}}{r}$$

$$= const \times \frac{\Gamma}{r}\int d^3\kappa \frac{\phi(\vec{\kappa}) P_l(\cos\theta)}{\kappa(E_\kappa - E_R + i\Gamma/2)} e^{i(\kappa r - E_\kappa t)}$$

Aligning $\vec{\kappa}$ and $\hat{z}$, and assuming $\phi(\vec{\kappa})$ invariant under rotations about $\hat{z}$, we must have $\phi(\vec{\kappa}) = \sum_{l\geq 0}\phi_l(\kappa) P_l(\cos\theta) = \frac{1}{\sqrt{m\kappa}}\sum_{l\geq 0}\phi_l(E_\kappa) P_l(\cos\theta)$. An integration on the angular part of $d^3\kappa$ will then selecionate only $\phi_l(E_\kappa)$ wher l is the angular momentum of the resonance. So we're left with

$$\psi_{sc}(\vec{r},t) \xrightarrow{r\to\infty} \text{const} \times \frac{\Gamma P_\ell(\cos\theta)}{r} \int_0^\infty d\kappa\, \kappa^{-1/2} \frac{\phi_\ell(E_\kappa)\, e^{i(\kappa r - E_\kappa t)}}{E_\kappa - E_R + i\Gamma/2}$$

Since the integrand is mostly appreciable near E_R, we can approximate

$$\kappa \cong \kappa_R + \left.\frac{d\kappa}{dE}\right|_{E_R} (E_\kappa - E_R) = \kappa_R - \frac{E_\kappa - E_R}{v_R} \text{, hence}$$

$$e^{i(\kappa r - E_\kappa t)} \cong e^{i(\kappa_R r - E_R t)}\, e^{-i(E_\kappa - E_R)(t - \frac{r}{v_R})}.$$

And since $\Delta E \gg \Gamma$, $\phi_\ell(E_\kappa)$ varies slowly near E_R, so we can take it out of the integral as $\phi_\ell(E_R)$ (the point where the integrand is most prominant).

$$\psi_{sc}(\vec{r},t) \xrightarrow{r\to\infty} \text{const.} \times \Gamma P_\ell(\cos\theta)\, \phi_\ell(E_R)\, \frac{e^{i(\kappa_R - E_R t)}}{\sqrt{\kappa_R}\; r} \times$$

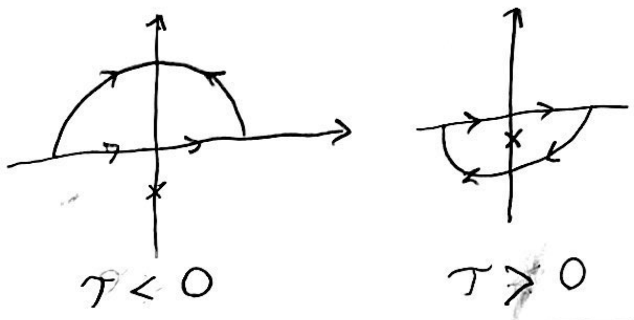
$$\times \int_0^\infty dE\, \frac{e^{-i(E-E_R)(t - r/v_R)}}{E - E_R + i\Gamma/2}$$

This integral can be extended to $-\infty$ without harm, since the integrand is only appreciable near E_R.

We now study the integral

$$\int_{-\infty}^{+\infty} dz \frac{e^{-iz\tau}}{z+i\Gamma/2}$$

It can be solved by contour integration in the complex plane of z. At $z=-i\Gamma/2$, the integrand has a simple pole, with residue $e^{-i(-i\Gamma/2)\tau} = e^{-\Gamma/2\,\tau}$. If $\tau>0$, we have to close the contour in the lower half plane, and by Cauchy's theorem $\int_{-\infty}^{+\infty} dz \frac{e^{-iz\tau}}{z+i\Gamma/2} = -2\pi i\, e^{-\Gamma/2\,\tau}$. And for $\tau<0$ we close the contour on the upper half plane were the integrand is analytic, so the integral vanishes.



$$, \int_{-\infty}^{+\infty} dz \frac{e^{-iz\tau}}{z+i\Gamma/2} = \Theta(\tau)\left(-2\pi i\, e^{-\Gamma/2\,\tau}\right)$$