are the associated Legendre polynomials.

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m}[P_l(x)]$$

$$Q_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m}[Q_l(x)]$$

where $P_l(x)$ and $Q_l(x)$ are Legendre functions of the first and second kind.

(for l/m complex we have Legendre function in terms of hypergeometric functions.)

Let's first consider the case $\epsilon^2 > 0$, corresponding to discrete levels

↳ boundary conditions

$$\tilde{x}_\lambda(\xi=-1)=0 \Rightarrow \begin{array}{l} Q_2^\epsilon(-1) \to \infty \\ P_2^\epsilon(-1) \to 0 \end{array} \Rightarrow C_2 = 0$$

$$\tilde{x}_\lambda(\xi=1)=0 \Rightarrow P_2^0(1) = 1$$

$$P_2^{1,2}(1) = 0 \Rightarrow$$

to $\tilde{x}_\lambda(1) = C_1 P_2^\epsilon(1) = 0$ be nontrivially satisfied $\epsilon = 1, 2$

Therefore

↳ $\epsilon = 1 \Rightarrow \epsilon^2 = 1 = 4 - \lambda/\alpha^2 \Rightarrow \lambda = 3\alpha^2$

↳ $\epsilon = 2 \Rightarrow \epsilon^2 = 4 = 4 - \lambda/\alpha^2 \Rightarrow \lambda = 0$

Thus we have 2 discrete levels, with eigenvalues

$$\lambda_0 = 0 \tag{97}$$

$$\lambda_1 = 3\alpha^2 \tag{98}$$

The lowest level is the zero mode. The corresponding eingenfunction $\tilde{x}_0(\xi) = C_1 P_2^2(\xi) = C_1\ 3(1-\xi^2) = C_1 \dfrac{3}{\cosh^2(\alpha\tau)}$ is proportional to (60).

The contribution from the continuum states with $\epsilon^2 < 0$ is more involved. Since is difficult to deal with associated Legendre function for non-integer ϵ, we rewrite

$$\tilde{x}_\lambda(\xi) = (1-\xi^2)^{\epsilon/2} w(\xi)$$

(92)

$$\Rightarrow \left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}(1-\xi^2)^{\epsilon/2} - \epsilon^2(1-\xi^2)^{\epsilon/2-1} + 6(1-\xi^2)^{\epsilon/2}\right] w(\xi) = 0 \quad (99)$$

Now

$$\Rightarrow \frac{d}{d\xi}(1-\xi^2)^{\epsilon/2} w = \frac{\epsilon}{2}(1-\xi^2)^{\epsilon/2-1}(-2\xi) w + (1-\xi^2)^{\epsilon/2}\frac{dw}{d\xi}$$

$$\Rightarrow \frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}(1-\xi^2)^{\epsilon/2} w = \frac{d}{d\xi}\left[-\epsilon\xi(1-\xi^2)^{\epsilon/2} w + (1-\xi^2)^{\epsilon/2+1}\frac{dw}{d\xi}\right]$$

$$= -\epsilon\left((1-\xi^2)^{\epsilon/2} + \xi\frac{\epsilon}{2}(1-\xi^2)^{\epsilon/2-1}(-2\xi)\right) w - \epsilon\xi(1-\xi^2)^{\epsilon/2}\frac{dw}{d\xi}$$

$$+ \left(\frac{\epsilon}{2}+1\right)(1-\xi^2)^{\epsilon/2}(-2\xi)\frac{dw}{d\xi} + (1-\xi^2)^{\epsilon/2+1}\frac{d^2}{d\xi^2} w$$

$$= \left\{-\epsilon(1-\xi^2)^{\epsilon/2}\left[1 - \xi^2\epsilon(1-\xi^2)^{-1}\right] - (1-\xi^2)^{\epsilon/2}\left[+\epsilon\xi + \left(\frac{\epsilon}{2}+1\right)2\xi\right]\frac{d}{d\xi} + (1-\xi^2)^{\epsilon/2+1}\frac{d^2}{d\xi^2}\right\} w(\xi)$$

Replacing in (99) and factoring out $(1-\xi^2)^{\epsilon/2}$, we obtain

$$\left\{-\epsilon + \xi^2\epsilon^2(1-\xi^2)^{-1} - (2\epsilon\xi + 2\xi)\frac{d}{d\xi} + (1-\xi^2)\frac{d^2}{d\xi^2} - \epsilon^2(1-\xi^2)^{-1} + 6\right\} w = 0$$

$$\Rightarrow \left\{ (1-\xi^2)\frac{d^2}{d\xi^2} - 2(\epsilon+1)\xi\frac{d}{d\xi} - \epsilon^2 \frac{\cancel{(1-\xi^2)}}{\cancel{(1-\xi^2)}} - \epsilon + 6 \right\} w(\xi) = 0$$

$$\Rightarrow \left\{ (1-\xi^2)\frac{d^2}{d\xi^2} - 2(\epsilon+1)\xi\frac{d}{d\xi} - (\epsilon-2)(\epsilon+3) \right\} w(\xi) = 0 \qquad (100)$$

def. the variable u

$$u = \frac{1-\xi}{2}, \quad \xi = 1-2u, \quad 1-\xi^2 = 4u(1-u) \qquad (101)$$

$$\frac{dw}{du} = -2\frac{dw}{d\xi} \quad ; \quad \frac{d^2w}{du^2} = 4\frac{d^2w}{d\xi^2}$$

$$\Rightarrow \left\{ u(1-u)\frac{d^2}{du^2} + (\epsilon+1)(1-2u)\frac{d}{du} - (\epsilon-2)(\epsilon+s+1) \right] w(u) = 0 \qquad (102)$$

with $s=2$. This is the hypergeometric differential eq.

The general solution for $\epsilon^2<0$ reads

$$w(u) = c_3 \, {}_2F_1\left[\frac{iK}{\alpha}-s, s+\frac{iK}{\alpha}+1, 1+\frac{iK}{\alpha}, u\right] + c_4 \, {}_2F_1\left[-s, s+1, 1-\frac{iK}{\alpha}, u\right] \qquad (103)$$

The opposite sign $K\to-K$ interchanges incoming and outgoing solutions. For our case, $s=2$

$$\overset{(103)}{\Rightarrow} w(u) = C_3(1-u)^{-iK/\alpha}\left[K^2 - 3iK\alpha(1-2u) - 2\alpha^2(1-6(1-u))\right] + C_4 u^{-iK/\alpha}\left[1 + \frac{6u\alpha(-iK+2\alpha(1-u))}{(K+i\alpha)(K+2i\alpha)}\right] \qquad (104)$$

or, transforming back to $\xi = 1-2u$

$$w(\xi) = C_3 \left(\frac{1+\xi}{2}\right)^{-iK/\alpha} \left[K^2 - 3iK\alpha\xi + 2\alpha^2(2+3\xi)\right]$$

$$+ C_4 \left(\frac{1-\xi}{2}\right)^{-iK/\alpha} \left[1 + \frac{3\alpha(1-\xi)(-iK+\alpha(1+\xi))}{(K+i\alpha)(K+2i\alpha)}\right] \quad (106)$$

and $\tilde{x}_\lambda(\xi) = (1-\xi^2)^{iK/2\alpha} \; w(\xi)$

$$\Rightarrow \tilde{x}_\lambda(\xi) = C_3 \left(\frac{\sqrt{1+\xi}}{2\sqrt{1-\xi}}\right)^{-iK/\alpha} \left[K^2 - 3iK\alpha\xi + 2\alpha^2(2+3\xi)\right]$$

$$+ C_4 \left(\frac{\sqrt{1-\xi}}{2\sqrt{1+\xi}}\right)^{-iK/\alpha} \left[1 + \frac{3\alpha(1-\xi)(-iK+\alpha(1+\xi))}{(K+i\alpha)(K+2i\alpha)}\right] \quad (110)$$

now, using $\xi = \tanh(\alpha\tau)$ and

$$\left(\frac{\sqrt{1\pm\xi}}{2\sqrt{1\mp\xi}}\right)^{-iK/\alpha} = e^{-\frac{iK}{\alpha}\ln\left(\frac{\sqrt{1\pm\xi}}{2\sqrt{1\mp\xi}}\right)} \quad (111)$$

with

$$\ln\left(\frac{\sqrt{1\pm\xi}}{2\sqrt{1\mp\xi}}\right) = \ln\left(\frac{1}{2}\sqrt{\frac{\cosh(\alpha\tau)\pm\sinh(\alpha\tau)}{\cosh(\alpha\tau)\mp\sinh(\alpha\tau)}}\right) = \ln\left(\frac{e^{\pm\alpha\tau}}{2}\right)$$

$$= \pm\alpha\tau - \ln 2 \quad (112)$$

Therefore, we can write the full set of solutions for $\epsilon^2<0$ as

$$\tilde{x}_\lambda(\tau) = \tilde{C}_3\, e^{-iK\tau} \left[K^2 - 3iK\alpha\tanh(\alpha\tau) + 2\alpha^2(2+3\tanh(\alpha\tau))\right] \quad (113)$$

$$+ \tilde{C}_4\, e^{iK\tau} \left[1 + \frac{3\alpha(1-\tanh(\alpha\tau))(-iK+\alpha(1+\tanh(\alpha\tau)))}{(K+i\alpha)(K+2i\alpha)}\right]$$

To proceed we will use the fact that (89) is a real eq. Thus we can obtain the eigenvalues in the asymptotic region $|\tau|\to\infty$ where the potential induced by the instanton field vanishes (similarly to what we did in elastic scattering). Then, (89) simplifies to

$$\left[\frac{d^2}{d\tau^2} + \lambda - 4\alpha^2\right]\tilde{x}_\lambda(\tau) = 0$$

$$\Rightarrow \quad \left[\frac{d^2}{d\tau^2} + k^2\right]\tilde{x}_\lambda(\tau) = 0 \qquad (114)$$

where the momentum k now labels continuum modes

$$k^2 \equiv \lambda - 4\alpha^2 \qquad (115)$$

The solution of (114) are 'plane waves'. Since their normalization is fixed (elastic scattering) the only effect of the instanton-induced pot. can be a k-dependent phase shift.

Also, the asymptotic solution reveals that no reflection occurs in this potential. Thus

$$\tilde{x}_\lambda(\tau) \propto e^{ik\tau + i\delta_k} \quad \text{for } \tau \to -\infty \qquad (116)$$

$$\tilde{x}_\lambda(\tau) \propto e^{ik\tau} \quad \text{for } \tau \to +\infty \qquad (117)$$

So all the required info about the eigenvalues is contained in the phase shifts of (113) in the lim $\tau \to -\infty$ where $\tanh(\alpha\tau) \to -1$ and thus

$$\tilde{x}_\lambda(\tau) \to \tilde{C}_3\, e^{-ik\tau}\left[k^2 + 3ik\alpha - 2\alpha^2\right] + \tilde{C}_4\, e^{ik\tau}\underbrace{\left[1 + \frac{6\alpha(-ik)}{(i\alpha)^2\left(1+\frac{k}{i\alpha}\right)\left(2+\frac{k}{i\alpha}\right)}\right]}_{= 1 + \frac{6ki}{\alpha\left(1+\frac{k}{i\alpha}\right)\left(2+k/i\alpha\right)}}$$

$$\therefore\ \tilde{x}_\lambda(\tau) \to \tilde{C}_3\, e^{-iK\tau}\left[K^2 + 3iK\alpha - 2\alpha^2\right] + \tilde{C}_4\, e^{iK\tau}\left[\frac{(1+iK/\alpha)(2+iK/\alpha)}{(1-iK/\alpha)(2-iK/\alpha)}\right] \qquad (118)$$

comparing with (116) we obtain

$$\delta_K = -i\ln\left[\left(\frac{1+iK/\alpha}{1-iK/\alpha}\right)\left(\frac{2+iK/\alpha}{2-iK/\alpha}\right)\right] \qquad (119)$$

To establish the relation between δ_K and the eigenvalues we will use the band cond. $\tilde{x}(\pm T/2)=0$

Starting from the general solution

$$\tilde{x}_{gen,\lambda}(\tau) = A\,\tilde{x}_\lambda(\tau) + B\,\tilde{x}_\lambda(-\tau) \qquad (120)$$

$$\Rightarrow \tilde{x}(T/2)=0 \Rightarrow A\,\tilde{x}_\lambda\left(\tfrac{T}{2}\right) + B\,\tilde{x}_\lambda\left(-\tfrac{T}{2}\right) = 0$$

$$\Rightarrow \tilde{x}(-T/2)=0 \Rightarrow A\,\tilde{x}_\lambda\left(-\tfrac{T}{2}\right) + B\,\tilde{x}_\lambda\left(\tfrac{T}{2}\right) = 0 \qquad (121)$$

$$\therefore\ A + B\,\frac{\tilde{x}_\lambda(-T/2)}{\tilde{x}_\lambda(T/2)} = A + B\,\frac{\tilde{x}_\lambda(T/2)}{\tilde{x}_\lambda(-T/2)} = 0 \Rightarrow \left(\frac{\tilde{x}_\lambda(-T/2)}{\tilde{x}_\lambda(+T/2)}\right)^2 = 1$$

$$\therefore\ \frac{\tilde{x}_\lambda(-T/2)}{\tilde{x}_\lambda(+T/2)} = \pm 1 = \frac{e^{-iKT/2}}{e^{+iKT/2 + i\delta_K}} = e^{-iKT - i\delta_K} \qquad (122)$$

because
$$t = -\frac{T}{2} \to \tau = \frac{T}{2} \Rightarrow \tau \to +\infty$$
$$t = +\frac{T}{2} \to \tau = -\frac{T}{\alpha} \Rightarrow \tau \to \infty \qquad (t = -i\tau)$$

from (122) $\Rightarrow \cos(KT+\delta_K) - i\,\mathrm{sen}(KT+\delta_K) = \pm 1$

$$\therefore \quad KT + \delta_K = n\pi \quad \Rightarrow \quad K_n = \frac{n\pi - \delta_K}{T} \qquad (123)$$

) Due to boundary conditions the K_n are discrete for finite T and become continuous in the limit $T \to \infty$ to be taken at the end.

It remains to calculate

$$\frac{\det \hat{F}[x_I]'}{\omega^{-2} \det \hat{F}[x_{ho}]} = \frac{\lambda_1}{\lambda_{ho,2}} \frac{\prod_{n=1} (K_n^2 + 4\alpha^2)}{\prod_{n=3} (K_{ho,n}^2 + \omega^2)} = \frac{3}{4} \frac{\prod_{n=1} (K_n^2 + 4\alpha^2)}{\prod_{n=3} (K_{ho,n}^2 + 4\alpha^2)} \qquad (124)$$

$$\lambda_{ho,1} = \omega^2 \; (T \to \infty)$$

)

where we used

$$\frac{\lambda_1}{\lambda_{ho,2}} = \frac{3\alpha^2}{\omega^2} = \frac{3}{4} \qquad (T \to \infty) \qquad (125)$$

and $\quad K_{ho,n} = \dfrac{n\pi}{T}$; we choose $\omega = 2\alpha$.

) Since the relevant range of K-values in the eigenvalue products of (124) will turn out not to contain small K the contribution of the 2 lowest H.O. eigenvalues can be multiplied with negligible effect

$$\Rightarrow \frac{\prod_{n=1} (K_n^2 + 4\alpha^2)}{\prod_{n=3} (K_{ho,n}^2 + 4\alpha^2)} = \exp \sum_{n=1}^{\infty} \ln \left[\frac{K_n^2 + 4\alpha^2}{K_{ho,n}^2 + 4\alpha^2} \right] \qquad (126)$$

) with

$$K_n^2 = \left(K_{ho,n} - \frac{\delta_K}{T} \right)^2 \simeq K_{ho,n}^2 - \frac{2\delta_K K_{ho,n}}{T} \qquad (127)$$

(since $\delta_K / T \ll 1$ in the n region which contribute for $T \to \infty$)

$$\Rightarrow \ln\left[\frac{K_n^2+4\alpha^2}{K_{no,n}^2+4\alpha^2}\right] \simeq \ln\left[1-\frac{1}{T}\frac{2\delta_K K_{no,n}}{K^2_{no,n}+4\alpha^2}\right] \simeq -\frac{1}{T}\frac{2\delta_K K_{no,n}}{K^2_{no,n}+4\alpha^2} \tag{128}$$

$\ln(1+x) \simeq x$

Now, in the continuum limit

$$\sum_{n=1}^{\infty} f(K_n) = \frac{1}{\Delta K}\sum_{n=1}^{\infty}\Delta K\, f(K_n) \rightarrow \frac{T}{\pi}\int_0^{\infty} dK\, f(K) \tag{129}$$

where $\Delta K = K_{no,n+1} - K_{no,n} = \dfrac{\pi}{T}$

We then obtain

$$\frac{\prod_{n=1}(K_n^2+4\alpha^2)}{\prod_{n=1}(K^2_{no,n}+4\alpha^2)} = \exp\left(-\frac{1}{\pi}\int_0^{\infty} dK\,\frac{2\delta_K K}{K^2+4\alpha^2}\right) \tag{130}$$

To evaluate the integral we use

$$\frac{d}{dK}\ln\left[1+\frac{K^2}{4\alpha^2}\right] = \frac{2K}{K^2+4\alpha^2} \tag{131}$$

$$\Rightarrow \int_0^{\infty} dK\,\frac{2K\delta_K}{K^2+4\alpha^2} = -\int_0^{\infty} dK\,\frac{d\delta_K}{dK}\ln\left[1+\frac{K^2}{4\alpha^2}\right] = -\int_0^{\infty} d\varkappa\,\frac{d\delta_\varkappa}{d\varkappa}\ln[1+\varkappa^2] \tag{132}$$

integration by parts (the surface term vanishes since $\delta_{K=0} = \delta_{K=\infty} = 0$ [eq. 119])

where $\varkappa = \dfrac{K}{2\alpha}$ is a dimensionless variable and

$$\frac{d\delta_\varkappa}{d\varkappa} = -i\frac{d}{d\varkappa}\ln\left[\frac{1+2i\varkappa}{1-2i\varkappa}\,\frac{1+i\varkappa}{1-i\varkappa}\right] = \frac{2}{1+\varkappa^2} + \frac{4}{1+4\varkappa^2} \tag{133}$$

so that the remaining integral can be done analytically:

$$\int_0^\infty dK \frac{2K\,\delta_K}{K^2+4\alpha^2} = -\int_0^\infty dx \left(\frac{2}{1+x^2}+\frac{4}{1+4x^2}\right)\ln[1+x^2] = \pi \ln 9 \qquad (134)$$

$$\Rightarrow \frac{\det \hat{F}[x_I]'}{\omega^{-2}\det \hat{F}[x_{ho}]} = \frac{3}{4}\, e^{-\frac{1}{\pi}\pi \ln 9} = \frac{3}{4}\,\frac{1}{9} = \frac{1}{12} \qquad (135)$$

Finally, we obtain our final result for $Z_I(-x_0, x_0)$ at large T:

$$Z_I(-x_0,x_0) = \underbrace{\sqrt{\frac{m\hbar\omega}{\pi}}\, e^{-\omega T/2}}_{Z_{ho}(0,0)}\; \omega T \sqrt{\frac{S_E}{2\pi\hbar m}}\; e^{-S_E/\hbar} \left[\frac{1}{12}\right]^{-1/2}$$

$$\boxed{Z_I(-x_0,x_0) = \sqrt{\frac{m\hbar\omega}{\pi}}\, e^{-\omega T/2}\, \omega T \sqrt{\frac{6S_E[x_I]}{\pi\hbar m}}\; e^{-S_E[x_I]/\hbar}} \qquad (136)$$

This is the quantum mechanical propagator of the double well tunneling problem, to $O(\hbar)$ in the SC approx. around a single instanton.

2.4- Dilute Instanton Gas

Up to now we have concentrated on the saddle points which correspond to single-instanton solutions. However, there are additional (approx.) saddle pnts which also contribute to the semiclassical (SC) tunneling amplitude for large T. $\Rightarrow$ 6 multi-instanton solutions?

DOUBLE-WELL POTENTIAL

Since the instanton deviates only in a small time interval $\Delta\tau = 1/2\alpha$ appreciably from x_0 or $-x_0$, and since the overlap between neighboring instantons and anti-instantons is exp. small, multi-(anti)-instanton solutions of (26) can be approx. written as a chain (ordered superposition) of N alternating (anti-) instantons, sufficiently far separated in time by the (averaged) interval

$$\bar{\Delta}_\tau = \frac{T}{N} \gg \frac{1}{2\alpha} \qquad (137)$$

Those chains correspond to N tunneling processes, back and forth between both minima of the pot.

The approx. N-instanton solutions, composed of single, alternating ins/anti-instantons at times $\tau_{0,K}$ is

$$x_N(\tau) = \sum_{K=1}^{N} x_{I,\bar{I}}(\tau - \tau_{0,K}) \qquad (139)$$

$$= \sum_{K=1}^{(N+1)/2} x_I(\tau - \tau_{0,2K-1}) + \sum_{K=1}^{(N-1)/2} x_{\bar{I}}(\tau - \tau_{0,2K}) \qquad (138)$$

where N must be odd in order to satisfy the bound. conditions (46) and (47).

(138) → more explicit expression ⇒ the first and last transitions in the chain must be instantons, and that those inbetween consist of 1 anti-inst + $(N-3)/2$ pairs inst/anti-inst

(139) become exact solutions for infinite separations $|\tau_{0,K+1} - \tau_{0,K}| \to \infty$.

The (anti)-instanton centers are ordered in Euclidean time as

$$-\frac{T}{2} \ll \tau_{0,1} \ll \tau_{0,2} \ll \cdots \ll \tau_{0,N} \ll \frac{T}{2} \qquad (140)$$

All multi-instanton solutions have to be included as additional saddle-points in the SC approx..

The corresponding expression for $Z(-x_0, x_0)$ by using the approx. solution (139) is called 'dilute instanton gas approximation' (DIGA). The contribution to the path integral is

$$Z_N \simeq \mathcal{N} \int D[\eta]\, e^{-S_E[x_N+\eta]/\hbar} \qquad (141)$$

We now write the fluctuations $\eta(\tau)$ as a sum of independent, localized fluctuations $\eta_K(\tau)$ around the single instantons and $\eta_0(\tau)$ around the approx. constant pieces $x(\tau) = \pm x_0$, between them:

$$\eta(\tau) = \eta_0(\tau) + \sum_{K=1}^{N} \eta_K(\tau) \qquad (142)$$

This implies that $\eta_0(\tau)$ can be finite over $[-T/2, T/2]$ (except at the boundaries) while the $\eta_K(\tau)$ are time localized around the K-th (anti) instanton.

The action then decomposes into the sum of actions for single (well-separated) (anti-) inst + 'vacuum' piece

$$S_E[x_N+\eta] \simeq S_E[x_0+\eta_0] + \sum_{K=1}^{N} S_E[x_I+\eta_k] \qquad (143)$$

where

$S[x_{cl} = \pm x_0] = 0$

$S[x_0+\eta] = S[-x_0+\eta]$ due to the sym. of the pot.

This formula expresses the (approx.) fact that the (anti-) inst. have too little overlap to interact. The measure factorizes into integrals over the localized fluctuations around the instantons and in between.

$$Z_N(-x_0,x_0) \simeq \mathcal{N}\int D[\eta_0]\, e^{-S_E[x_0+\eta_0]/\hbar} \times \prod_{K=1} \mathcal{N}\int D[\eta_K]\, e^{-\frac{S_E[x_I+\eta_K]}{\hbar}}$$

$$= Z_0(x_0,x_0)\,[Z_I(-x_0,x_0)]^N \qquad (145)$$

(145) is not formally correct since the N factors $Z_I(-x_0,x_0)$ have different bound. conditions since the endpoints $\pm x_0$ are (almost) reached at $\neq$ times. However, due to time translation invariance this makes no difference for the value of the $Z_I(-x_0,x_0)$ except through the zero-mode contribution.

↳ we need to integrate over τ_0

$$Z_I = Z_I' \sqrt{\frac{S_I}{2\pi\hbar m}} \int d\tau_0 \equiv \tilde{Z}_I \int d\tau_0 \qquad (146)$$

where the range of the center $\tau_{0,K}$ is restricted by the condition that it occurs after the $(K-1)$-th, i.e.: $\tau_{0,K-1} < \tau_{0,K} < T/2$. Therefore

$$\int_{-T/2}^{T/2} d\tau_{0,1} \int_{\tau_{0,1}}^{T/2} d\tau_{0,2} \cdots \int_{\tau_{0,N-1}}^{T/2} d\tau_{0,N} = \frac{T^N}{N!} \qquad (147)$$

$$\Rightarrow \quad Z_N \simeq Z_0 \frac{(\tilde{Z}_I T)^N}{N!} \qquad (148)$$

This expression determines the N-instanton contribution explicitly since we had already calculated Z_0 in (84), with $\omega = 2\alpha$, and $\tilde{Z}_I$ can be obtained from (136) and (58):

$$Z_0(\pm x_0, \pm x_0) = \mathcal{N} \left(\det[-\partial_\tau^2 + \omega^2]\right)^{-1/2} \rightarrow \left(\frac{m\hbar\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \quad (149)$$

$$\tilde{Z}_I = 2\alpha \sqrt{\frac{6 S_E[x_I]}{\pi\hbar m}}\, e^{-\frac{S_E[x_I]}{\hbar}} = 4\sqrt{\frac{2\alpha^3 x_0^2}{\pi\hbar}}\, e^{-\frac{4}{3}\alpha m x_0^2/\hbar} \quad (150)$$

In order to collect the multi-instanton contributions to $Z(x_0, -x_0)$ we have to sum over all odd N

$$Z_{DIGA}(x_0, -x_0) = Z_0 \sum_{N\,\mathrm{odd}} \frac{(\tilde{Z}_I T)^N}{N!} = \frac{Z_0}{2}\left\{e^{\tilde{Z}_I T} - e^{-\tilde{Z}_I T}\right\}$$

$$= Z_0 \sinh\left(\tilde{Z}_I T\right) \quad (151)$$

An analogous expression $Z_{DIGA}(-x_0, -x_0)$ is obtained when only contributions from even number of inst. are summed

$$Z_{DIGA}(\pm x_0, -x_0) = \frac{1}{2}\left(\frac{m\hbar\omega}{\pi}\right)^{1/2} e^{-\omega T/2}\left\{e^{\tilde{Z}_I T} \mp e^{-\tilde{Z}_I T}\right\}$$

$$= \frac{1}{2}\left(\frac{m\hbar\omega}{\pi}\right)^{1/2}\left\{e^{-\left(\frac{\omega}{2} - \tilde{Z}_I\right)T} \mp e^{-\left(\omega/2 + \tilde{Z}_I\right)T}\right\} \quad (153)$$

from which the 2 lowest energy levels (GS and 1st excited) can be found

$$E_0 = -\hbar \lim_{T\to\infty} \frac{1}{T} \ln Z_{DIGA}(\pm x_0, -x_0) = -\hbar \lim_{T\to\infty} \frac{1}{T}\left[-\left(\frac{\omega}{2} - \tilde{Z}_I\right)T\right]$$

$\hookrightarrow$

$$E_0 = -\hbar \lim_{T\to\infty} \frac{1}{T} \ln\left\{\frac{1}{2}\left(\frac{m\hbar\omega}{\pi}\right)^{1/2} e^{-\frac{\omega T}{2}} \,\mathrm{sinh}/\mathrm{cosh}\left(\tilde{Z}_I T\right)\right\}$$

using $\ln(ab) = \ln(a) + \ln(b)$

$$E_0 = -\hbar \lim_{T\to\infty} \frac{1}{T}\left[\ln\left(\left(\frac{m\cancel{\hbar}\omega}{\pi}\right)^{1/2}\frac{1}{2}\right) + \ln e^{-\frac{\omega T}{2}} \begin{matrix}\sinh\\ \cosh\end{matrix} / (\tilde{z}_I T)\right]$$

$$= -\hbar \lim_{T\to\infty} \frac{1}{T} \ln\left[e^{-\frac{\omega T}{2}} \begin{matrix}\sinh\\ \cosh\end{matrix} / (\tilde{z}_I T)\right]$$

↙ for both (Mathematica)

$$= -\hbar\left[\tilde{z}_I - \frac{\omega}{2}\right] = \frac{\hbar\omega}{2} - \hbar\tilde{z}_I \qquad (154)$$

$$E_I = -\hbar \lim_{T\to\infty}\left[-\left(\frac{\omega}{2} + \tilde{z}_I\right)T\right] = \frac{\hbar\omega}{2} + \hbar\tilde{z}_I \qquad (155)$$

The effect of tunneling is to split the degenerate GS energies $\frac{\hbar\omega}{2}$ of the wave functions $|\pm x_0\rangle$ centered in each of the 2 minima of the potential

The corresponding energy eigenstates are obtained from (153) using (16):

$$|0\rangle = \frac{1}{\sqrt{2}}\{|x_0\rangle + |-x_0\rangle\} \quad \text{symmetric} \qquad (158)$$

$$|1\rangle = \frac{1}{\sqrt{2}}\{|x_0\rangle - |-x_0\rangle\} \quad \text{antisymmetric} \qquad (159)$$

These are the standard WKB results for tunneling amp. with the typical splitting

$$\Delta E \sim e^{-\frac{S_E[x_I]}{\hbar}} \qquad (160)$$

of the energy levels of the states connected by tunneling.

(158) → parity invariant ⇒ the artificially broken parity in the absence of tunneling is restored

Note that when summing over the dilute instanton gas in eq. (151), for an increasing N of (anti-) instantons in the constant interval T, the diluteness condition (137) is less and less satisfied, implying that for some large N the corresponding terms in the sum will violate the fundamental DIGA requirement (137). However, Stirling's formula $n! \simeq \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ for large n shows that (151) is dominated by terms:

$$\frac{(\tilde{Z}_I T)^N}{N!} \sim \left(\frac{\tilde{Z}_I T}{N}\right)^N \frac{e^N}{\sqrt{2\pi N}} \quad \Rightarrow \quad \frac{\tilde{Z}_I T}{N} \sim \mathcal{O}(1) \qquad (161)$$

and that contributions from larger N are rapidly suppressed. As a consequence, only N with

$$\frac{N}{T} \lesssim C e^{-\frac{S_E[x_I]}{\hbar}} \qquad (162)$$

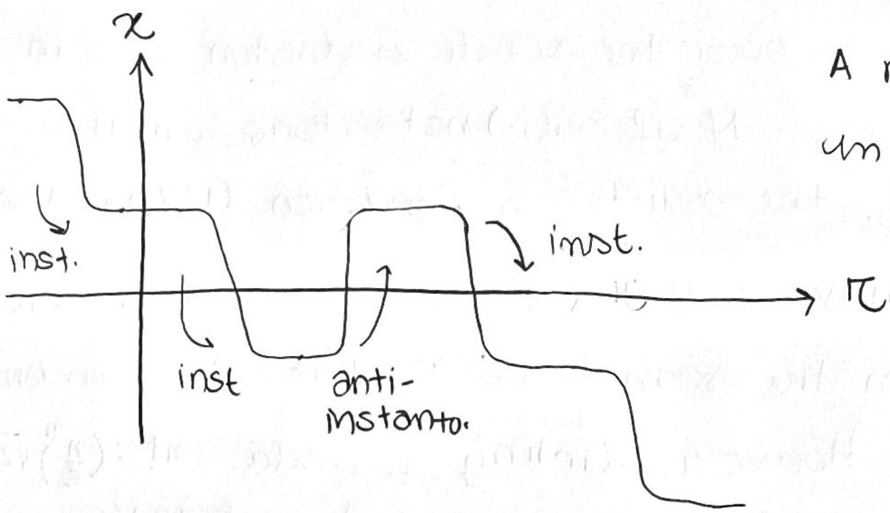
i.e, with an exponentially small instanton density in the SC limit (which can be improved at will by increasing α since $S_E[x_I] = \frac{4}{3}\alpha m x_0^2$) contribute significantly to the sum.

This ensures that Z_{DIGA} is dominated by terms in which the DIGA diluteness requirement (137) is satisfied.

PERIODIC POTENTIAL

Let's consider the periodic extension of the double well pot. (resembles the situation encountered in QCD vacuum)

In a periodic pot with degenerate minima, instantons and anti-inst. can arbitrarily follow each other, starting out at the minima where the previous ended, i.e, connecting adjacent minima $x_{0,n}$ and $x_{0,n\pm1}$.



A multi-instanton solution in the periodic pot.

The integer values of x correspond to the minima $x_{0,n}$ of the periodic pot.

We will refer to an instanton (anti-inst.) as the segment of the solution which interpolates between neighboring minima to the left (righ), ie. which decreases (increases) n.)

Getting from the minimum with index n_i to the one with n_f, correspond to the bound. cond.

$$x\left(-\frac{T}{2}\right) = x_{0,n_i} \qquad x\left(\frac{T}{2}\right) = x_{0,n_f} \tag{163}$$

requires that $N_{\bar{I}} - N_I = n_f - n_i$

$$\Rightarrow Z_{per}(x_{n_f}, x_{n_i}) \simeq Z_0 \sum_{N_I=0}^{\infty} \sum_{N_{\bar{I}}=0}^{\infty} \frac{(\tilde{Z}_I T)^{N+N_{\bar{I}}}}{N!\, N_{\bar{I}}!}\, \delta_{N_{\bar{I}} - N_I - (n_f - n_i)} \tag{164}$$

, with the representation:

$$\delta_{ab} = \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{i\theta(a-b)} \tag{165}$$

we write:

$$Z_{per}(x_{n_f}, x_{n_i}) = \left(\frac{m\hbar\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-i\theta(n_f - n_i)}$$

$$\times \sum_{N_I=0}^{\infty} \frac{(\tilde{Z}_I T e^{-i\theta})^{N_I}}{N_I!} \sum_{N_{\bar{I}}=0}^{\infty} \frac{(\tilde{Z}_I T e^{i\theta})^{N_{\bar{I}}}}{N_{\bar{I}}!} \tag{166}$$

$$\Rightarrow Z_{per}(x_{n_f}, x_{n_i}) \simeq \left(\frac{m\hbar\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-i\theta(n_f - n_i)}\, e^{\tilde{Z}_I T e^{-i\theta} + \tilde{Z}_I T e^{i\theta}}$$

$$= \left(\frac{m\hbar\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \int_0^{2\pi} \frac{d\theta}{2\pi}\, e^{-i\theta(n_f - n_i)}\, e^{-\left(\frac{\omega}{2} - 2\tilde{Z}_I \cos\theta\right)T} \qquad (169)$$

Now we can obtain the low-lying energy levels from $\lim_{T\to\infty}$. (165) show that we get a continuous 'band' energies parametrized by θ. Since (169) is the sum of contributions from the lowest energy levels in (14)

$$E_0(\theta) = -\hbar \lim_{T\to\infty} \frac{1}{T}\left[-\left(\frac{\omega}{2} - 2\tilde{Z}_I \cos\theta\right)T\right] = \frac{\hbar\omega}{2} - 2\hbar \tilde{Z}_I \cos\theta \qquad (171)$$

Our result is the analog of. e.g. the lowest-lying band of electrons states in the periodic pot of a metal. The corresponding eigenstates are the 'Floquet-Bloch waves'

$$|\theta\rangle = \frac{1}{\sqrt{2\pi}} \left(\frac{\hbar\omega}{\pi}\right)^{1/4} \sum_n e^{in\theta} |n\rangle \qquad (172)$$

$|n\rangle$ = state localized at the n-th minimum of the pot.

QM II - Seminar - Ana Foguel

Techniques and applications of path integral

- L. S. Schulman

Chapter 29 - Critical Droplets, Alias Instantons and Metastability

local minimum in an energy functional and the system tends to remain near the minimum for some time until it goes to distant but lower energy states.

Exp. one particle QM → unstable states
lifetime can be obtained by WKB barrier penetration
↳ the state is a resonance or pole in the S matrix

Exp. metastable states in statistical physics
Condensation to the stable GS via nucleation of critical droplets

Consider system of spins in 1 dim with periodic boundary conditions. Partition function

$$Z(\alpha) = \int \underbrace{d\varphi(x)}_{\text{measure}} \exp\left\{ - \int_0^L \left[\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \varepsilon\varphi^2 + \alpha\varphi^4 \right] dx \right\} \quad (1)$$

L = circumference of the ring
φ = spin field ; $\varepsilon > 0$

Thermodynamic limit $L \to \infty$ and the infinite volume free energy per unit volume is

$$\Psi(\alpha) \equiv - \lim_{L\to\infty} L^{-1} \log Z(\alpha) \quad (2)$$

For $\alpha > 0$

$$V(\varphi) = \varepsilon \varphi^2 + \alpha \varphi^4 \qquad (3)$$

is positive for all real φ

We examine what happens when the sign of α is changed → V continues to have a minimum at $\varphi = 0$ but for large values of φ, V becomes negative and (1) doesn't exist

So we need an analytic continuation of $\Psi(\alpha)$ from $Re\,\alpha > 0$ to $Re\,\alpha < 0$

Ψ is the GS energy of $\frac{1}{2}p^2 + V$ → $E_0(\alpha)$ $\{$ for $\alpha > 0$ is well defined

For $\alpha < 0$ there is no stable GS, but for $|\alpha|$ small there is an approx. stationary state whose lifetime can be estimated by barrier penetration.

↳ decay rate is related to the imaginary part of the energy

$$S(\varphi) = \int_0^L dx \left[\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \varepsilon\varphi^2 + \alpha\varphi^4 \right]$$

variation $\delta S = 0$:

$$-\frac{\delta}{\delta\varphi(x)} S(\varphi) = \frac{d^2\varphi}{dx^2} - V'(\varphi) = 0 \Rightarrow \frac{d^2\varphi}{dx^2} = 2\varepsilon\varphi^2 + 4\alpha\varphi^3 \qquad (4)$$

$$\equiv -\frac{\partial U(\varphi)}{\partial\varphi}$$

where $U(\varphi) \equiv -V(\varphi)$

First, consider constant solutions $\Rightarrow \frac{d\varphi}{dx} = 0$

$\varphi = 0$ is always a solution

Solutions of (4) → boundary condition $\varphi(0) = \varphi(L)$

we can consider paths that start and end near $\varphi = 0$

For $\alpha > 0$ there are no paths that start and end at $\varphi = 0$ except $\varphi(x) = 0$.

For $\alpha < 0$, the constant solution $\varphi(x) = 0$ is still valid but there are additional constant solutions

$$\overset{(4)}{\Rightarrow} \quad 2\varepsilon\varphi - 4|\alpha|\varphi^3 = \varphi(2\varepsilon - 4|\alpha|\varphi^2) = 0$$

$$\Rightarrow \quad 2\varepsilon = 4|\alpha|\varphi^2 \rightsquigarrow \varphi^2 = \frac{2\varepsilon}{4|\alpha|}$$

$$\boxed{\varphi = \pm\sqrt{-\frac{\varepsilon}{2\alpha}}}$$

Note that with $\varphi(0) = \varphi(L)$ condition, these paths do not appear but they "nearly" do. → there is a path that "hurries" to $\varphi = \pm\sqrt{\frac{-\varepsilon}{2\alpha}}$, remains there until $t = L$ and then hurries back to zero.

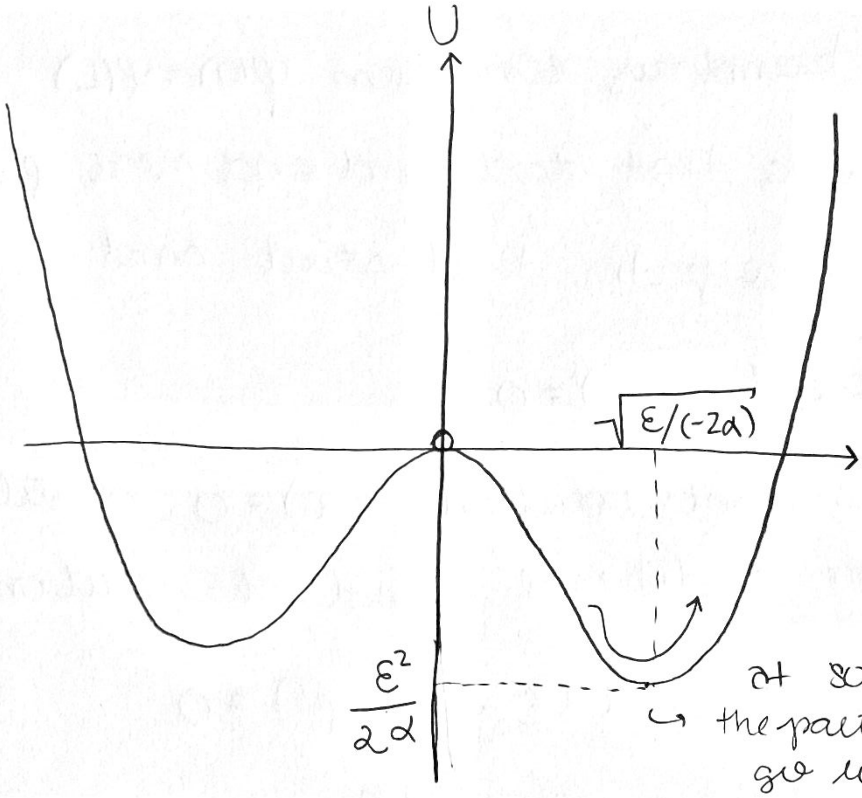
The action of this constant solutions differs by terms of $O(L)$ from that of $\varphi = 0$ → so they drop out

$$\Psi \sim \frac{1}{L}\log Z = \frac{1}{L}\log\left(Z_{\varphi=0} + Z_{\varphi=\sqrt{-\varepsilon/2\alpha}}\right)$$

$$= \frac{1}{L}\log\left\{Z_{\varphi=0}\left[1 + O(e^{-\text{const}\,L})\right]\right\}$$

$$= \frac{1}{L}\log Z_{\varphi=0} + \frac{1}{L}\exp(-\text{const}\,L)\}$$

Now, however there are solutions satisfying $\varphi(0) = \varphi(L) \sim 0$ which are NOT constant.



$U(\varphi) = -\varepsilon\varphi^2 - \alpha\varphi^4$)

for $\alpha<0$

at some intermediate time
↳ the particle pass through and go up the other side

particle starts very close to $\varphi=0$ and takes a long time to move away from $\varphi=0$ → it starts with little energy and spends almost all the time L getting away from $\varphi=0$ and falling back up the hill as $x\to L$.

Solution

$$\varphi_z(x) = \pm\left(\frac{\varepsilon}{-\alpha}\right)^{1/2} \operatorname{sech}\left[\sqrt{2\varepsilon}\,(x-z)\right] \qquad (5)$$

z arbitrary → freedom of choice anywhere in $[0, L]$

The action is

$$S(\varphi_z) = \frac{(2\varepsilon)^{3/2}}{-3\alpha} \qquad (6)$$

The general structure of the partition function is ($\alpha<0$)

$$Z = Z_0 + Z_1 + Z' \qquad (7)$$

Z_0 is the $\varphi=0$ contribution

Z_1 is the contribution from φ_z

Z' ⇒ everything else

Now we consider small oscillation integral for φ_z

$\eta \equiv \varphi - \varphi_z$:

$$S(\varphi) = S(\varphi_z) + \int_0^L dx \left[\frac{1}{2}\left(\frac{d\eta}{dx}\right)^2 + \varepsilon \eta^2 + 6\alpha \varphi_z^2 \eta^2 \right] \quad (8)$$

the variation will be diagonalized with operator

$$-\frac{1}{2}\frac{d^2\eta_j}{dx^2} - 6(-\alpha)\varphi_z^2(x)\eta_j + \varepsilon \eta_j = \lambda_j \eta_j \quad (9)$$

$$\eta(x) = \sum_j a_j \eta_j(x) \quad (10)$$

$$S(\varphi) = S(\varphi_z) + \sum_j a_j^2 \eta_j(x) \quad (11)$$

we change $d\varphi \to d\eta$ => integration vol $\prod_j \frac{da_j}{\sqrt{\pi}}$

(5) in (9)

=> $$-\frac{1}{2}\frac{d^2\eta_j}{dx^2} - 6\varepsilon\, \mathrm{sech}^2[\sqrt{2\varepsilon}(x-z)]\eta_j + \varepsilon \eta_j = \lambda_j \eta_j$$

↳ bound states with eigenfunctions / values

$$\eta_0(x) = \sqrt{\frac{3}{4}}(2\varepsilon)^{1/4} \mathrm{sech}^2[(x-z)\sqrt{2\varepsilon}] \; ; \; \lambda_0 = -3\varepsilon$$

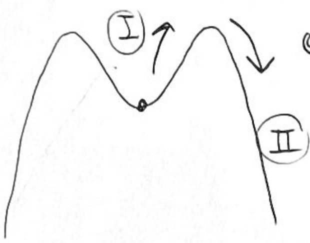
$$\eta_1(x) = \sqrt{\frac{3}{2}}(2\varepsilon)^{1/2} \frac{\sinh[(x-z)\sqrt{2\varepsilon}]}{\cosh^2[(x-z)\sqrt{2\varepsilon}]} \; ; \; \lambda_1 = 0$$

the remaining states have $\lambda_j > 0$

The integral over a_0 diverges $\frac{1}{\sqrt{\pi}}\int da_0\, e^{3\varepsilon a_0^2}$

also for a_1: $\frac{1}{\sqrt{\pi}}\int da_1$

interpretation: φ_z is a saddle point of S, $\varphi = 0$ is a local but not absolute minimum. Because $\alpha < 0$ paths with large φ will have smaller value of S.



on passing from local minimum to the depths of paths with much smaller S one must rise than descend

Among paths that rise and descend there is one (or more) that rises as little as possible → will be too an extremum of S => φ_z

properties of col → mostly the peaks rise around it ($\lambda_i > 0$)

There is at least one direction from Ⓘ → Ⓘⓘ for which the col is a maximum → mode, a destabilizing fluctuation about φ_z.

We dont have a description of the stable state at large φ (II) in theory of metastability the mode η_0 represent increase and decrease in the size of the droplet → both are destabilizing, one sends the system to the large φ global min. of S, the other back to $\varphi = 0$. The state φ_z is the main barrier the system must pass through in its decay from $\varphi = 0$ → we can look at int a_0 to find lifetime

φ_z = instanton solution

interpretation for η_1 => S is translationally invariant we have solution $\varphi_z(x)$ for each z in $[0, L]$

one-dim family of solution

zero-eigenvalue is manifestation of this invariance

$\eta_1 \propto d\varphi_z/dz$ → correspond to the fact that the critical droplet can appear anywhere throughout the volume

So we have been double counting → int. over z ~ factor L

relation between da_1 and dz => consider path

$\eta = \varphi_{z+dz} - \varphi_z$ → entirely in the 'direction' η_1 and

$$\varphi_{z+dz} - \varphi_z = \frac{d\varphi_z}{dz} dz = \eta_1 da_1$$

Jacobian $a_1 \to z$ = ratio of the norms of the functions:

$$\frac{1}{\sqrt{\pi}} \int da_1 = \frac{1}{\sqrt{\pi}} \int dz \left\| \frac{d\varphi_z}{dz} \right\| = \frac{1}{\sqrt{\pi}} \left[\int \left(\frac{d\varphi_z}{dz} \right)^2 dx \right]^{1/2} \int dz$$

$$= \frac{(2\varepsilon)^{3/4}}{(-3\alpha)^{1/2}} \frac{L}{\sqrt{\pi}} \qquad (15)$$

or

$$\frac{1}{\sqrt{\pi}} \int da_1 = \frac{1}{\sqrt{\pi}} \sqrt{S_I} \int dz = \frac{(2\varepsilon)^{3/4}}{(-3\alpha)^{1/2}} \frac{L}{\sqrt{\pi}}$$

We need to multiply (15) by 2 to allow for instantons heading either left or right

The integration over da_j $(j>1)$ => $\left({\prod_j}' \lambda_j \right)^{-1/2}$ no $j=0,1$

To calculate the int. for da_0

$$J = \frac{1}{\sqrt{\pi}} \int da_0 \, e^{3\varepsilon a_0^2} \qquad (16)$$

we are interested in the analytic continuation. Let

$$F(\mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu z^2} dz \qquad (17)$$

For $\mathrm{Re}\, \mu > 0$, $F(z) = G(\mu) \equiv \frac{1}{\sqrt{\mu}} \qquad (18)$

To continuate to $Re\,\mu \leq 0$ we let $F = G$ everywhere → branch point at $\mu = 0$

In (17) consider the change of contour from $Re\, z$ to $z = e^{-i\theta} x$, $-\infty < x < \infty$. Originally (17) was defined for $-\frac{\pi}{2} < \arg \mu < \frac{\pi}{2}$ => now $-\frac{\pi}{2} < \arg \mu - 2\theta < \frac{\pi}{2}$

Hence for $\theta = \pi/2$, F has been continued to the negative half plane

so that

cos > 0

$\Rightarrow z = x e^{-i\pi/2} = x(-i) \Rightarrow z^2 = -x^2$

$$F(\mu) = \frac{1}{\sqrt{\pi}} \int_{-i\infty}^{i\infty} e^{-\mu z^2} dz = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\mu x^2} i\, dx = i \sqrt{\frac{1}{-\mu}}$$

therefore

$$J = \frac{1}{\sqrt{\pi}} \int_0^{i\infty} da_0\, e^{3ea_0^2}$$

φ_z is a critical droplet and the mode a_0 is the unstable direction in the function space of microscopic configurations. Increase of a_0 (from 0) => grow droplet and decrease in free energy. Decrease of a_0 (to negative values) => return to metastable minimum $\varphi = 0$

∴ $(-\infty, 0]$ ⇝ contribution of metastable GS

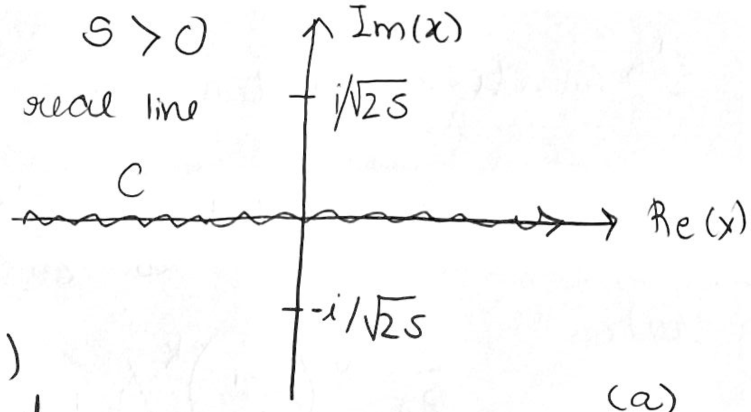
so the infinite from int $-\infty < a_0 < 0$ is fake

↳ included in int. around $\varphi = 0$ => contribute only for real part of z

So is the integral $[0, +\infty]$ that needs shifting

$$f(s) = \int_C dx\, e^{-x^2 - s x^4}$$

$s > 0$ real line



change variable

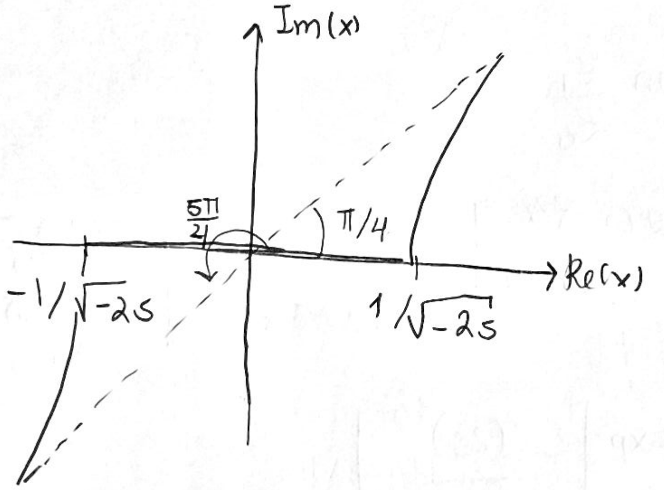
$y = \sqrt{s}\, x$

$$f(x) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} e^{-\frac{1}{s}(y^2 + y^4)}\, dy$$

(a) $s>0$

extrema at $y = \pm i/\sqrt{2}$ play no role (small s)

(b) for $s<0$: f can be analytically continued by rotating the contour so that Re $sx^4 > 0$



For $|s| \to 0$ its best to evaluate the int. passing through $0,\ x = \pm\sqrt{\frac{1}{-2s}}$

$$\therefore\ J = \pm \frac{i}{2}\sqrt{\frac{1}{3\varepsilon}}$$

=> free energy acquire imaginary part

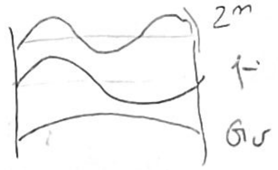
instanton start and end at zero → derivative vanish somewhere

∴ zero-energy mode vanish somewhere

↳ it is a solution with a NODE → cannot be the GS

* Node theorem [Messiah Ch3 §12]

ψ_n has $(n-1)$ nodes



So $\quad Z = Z_0 + Z_1 + Z'$

normalization constant

$$Z_0 = \int_{(0,0)}^{(0,L)} d\eta\, \exp\left\{-\int_0^L \left[\frac{1}{2}\left(\frac{d\eta}{dx}\right)^2 + \varepsilon\eta^2\right] dx\right\} = \mathcal{N} \prod_j \lambda_j^{(0)\, -1/2}$$

→ modes for $\varphi \equiv 0$

$$Z_1 = \mathcal{N}\left(\pm\frac{i}{2}\sqrt{\frac{1}{3\varepsilon}}\right)\left[\frac{2(2\varepsilon)^{3/4} L}{(-3a)^{1/2}\sqrt{\pi}}\right] \exp\left(-\frac{(2\varepsilon)^{3/2}}{(-3a)}\right) \prod_j{}' \lambda_j^{-1/2}$$

$Z' \to$ multi-instanton terms

$$Z = Z_0\left(1 + \frac{Z_1}{Z_0} + \frac{Z'}{Z_0}\right)$$

where $\frac{Z'}{Z_0} \sim \left(\frac{Z_1}{Z_0}\right)^K / K!$ instantons are identical part $K = 2, 3, \ldots$

$$\Rightarrow \quad Z = Z_0 \exp\left(\frac{Z_1}{Z_0}\right) \qquad (Z_0 \in \mathbb{R})$$

we are interested in the imaginary part of ψ

$$\mathrm{Im}\, \psi(\alpha) = -\frac{1}{L} \lim_{L\to\infty} \mathrm{Im}\, \frac{Z_1}{Z_0}$$

$$= \pm \frac{1}{\alpha} \sqrt{\frac{1}{3\varepsilon\pi}} \left[\frac{2(2\varepsilon)^{3/4}}{(-3\alpha)^{1/2}}\right] \exp\left[-\frac{(2\varepsilon)^{3/2}}{(-3\alpha)}\right] \frac{\Pi' \lambda_j^{-1/2}}{\Pi_j^{(0)-1/2}}$$

$$= \pm \frac{2^{7/4} \varepsilon^{5/4}}{(-\pi \alpha)^{1/2}} \exp\left[-\frac{(2\varepsilon)^{3/2}}{-3\alpha}\right]$$