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PME 3380 - Modelagem de Sistemas Dinâmicos

Exercícios aula 22/10/2020

1) $\underbrace{2\ddot{x} + 7\dot{x} + 3x = 0}$; $x(0) = x_0$; $\dot{x}(0) = 0$

$\downarrow \mathcal{L}$

$$2s^2X - 2x_0 s + 7sX - 7x_0 + 3X = 0$$

$$(2s^2 + 7s + 3)X = (2s + 7)x_0$$

$$X(s) = \frac{2s+7}{2s^2+7s+3}x_0$$

Se $u(t) = \delta(t) \Rightarrow U(s) = 1$ e $X(s)$ é a própria F.T.

Polos: $2s^2 + 7s + 3 = 0$

$$s = \frac{-7 \pm \sqrt{49-24}}{4} = \frac{-7 \pm 5}{4} \quad s_1 = -3 \quad s_2 = -\frac{1}{2}$$

$$X(s) = \frac{2s+7}{2(s+3)(s+\frac{1}{2})}x_0$$

$$X(s) = \left(\frac{a}{s+3} + \frac{b}{s+\frac{1}{2}}\right)\frac{x_0}{2}$$

$$a = \left.\frac{2s+7}{s+1/2}\right|_{s=-3} = \frac{-6+7}{-3+1/2} \Rightarrow \boxed{a = -\frac{2}{5}}$$

$$b = \frac{2s+7}{s+3}\bigg|_{s=-1/2} = \frac{-1+7}{-1/2+3} \Rightarrow \boxed{b = \frac{12}{5}}$$

$$X(s) = \left[-\frac{2}{5(s+3)} + \frac{12}{5(s+1/2)}\right]\frac{x_0}{2}$$

$$\boxed{X(s) = -\frac{x_0}{5(s+3)} + \frac{6x_0}{5(s+1/2)}}$$

$\mathcal{L}^{-1}$

$$\boxed{x(t) = -\frac{x_0}{5}e^{-3t} + \frac{6x_0}{5}e^{-t/2}}$$

② $\dddot{x} + 2\ddot{x} + 7\dot{x} = \ddot{u} + 7\dot{u} + 5u$, com $\begin{cases} \ddot{x}(0) = 2 \\ \dot{x}(0) = 1 \\ x(0) = 9 \\ u(0) = 1 \\ \dot{u}(0) = 0 \end{cases}$

Degrau unitário: $u(t) = 1$.

Aplicando $\mathcal{L}$ na EDO:

$$s^3X - s^2x(0) - s\dot{x}(0) - \ddot{x}(0) + 2\left(s^2X - s\,x(0) - \dot{x}(0)\right) +$$
$$+ 7\left(sX - x(0)\right) = s^2U - su(0) - \dot{u}(0) + 7\left(sU - u(0)\right) + 5U$$

$$(s^3 + 2s^2 + 7s)X - s^2x(0) - s\left(\dot{x}(0) + 2x(0)\right) - \left(\ddot{x}(0) + 2\dot{x}(0) + 7x(0)\right) =$$
$$= (s^2 + 7s + 5)U - su(0) - \left(\dot{u}(0) + 7u(0)\right)$$

$$(s^3 + 2s^2 + 7s)X = (s^2 + 7s + 5)U + s^2x(0) + s\left(\dot{x}(0) + 2x(0) - u(0)\right) +$$
$$+ \left(\ddot{x}(0) + 2\dot{x}(0) + 7x(0) - \dot{u}(0) - 7u(0)\right)$$

$$(s^3+2s^2+7s)X = (s^2+7s+5)U + 9s^2+18s+60$$

Para $u(t) = 1 \xrightarrow{\mathcal{L}} U(s) = \frac{1}{s}$, e então

$$X(s) = \underbrace{\frac{s^2+7s+5}{s^2(s^2+2s+7)}}_{\text{Referente à solução particular (reg. permanente)}} + \underbrace{\frac{9s^2+18s+60}{s(s^2+2s+7)}}_{\text{Referente à solução homogênea (transitória)}}$$

Função de transferência (cond. iniciais nulas):

$$G(s) = \frac{X(s)}{U(s)}$$

$$\boxed{G(s) = \frac{s^2+7s+5}{s(s^2+2s+7)}}$$

Sistema estável!

Polos do sistema: $s=0$; $s \approx -1+j2{,}449$; $s \approx -1-j2{,}449$

Reescrevemos $X(s)$ na forma de frações parciais, mas antes rearranjamos a expressão racional para simplificá-la:

$$X(s) = \frac{9s^3+19s^2+67s+5}{s^2(s^2+2s+7)}$$

$$X(s) = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d}{s^2+2s+7}$$

$$b = s^2 X(s)\Big|_{s=0} = \frac{9s^3 + 19s^2 + 67s + 5}{s^2 + 2s + 7}\Big|_{s=0} \Rightarrow \boxed{b = \frac{5}{7}}$$

$$a = \frac{d}{ds}\left[s^2 X(s)\right]_{s=0}$$

$$a = \frac{(27s^2 + 38s + 67)(s^2 + 2s + 7) - (9s^3 + 19s^2 + 67s + 5)(2s + 2)}{(s^2 + 2s + 7)^2}\Bigg|_{s=0}$$

$$a = \frac{67 \cdot 7 - 5 \cdot 2}{49} \Rightarrow \boxed{a = \frac{459}{49}}$$

$$X(s) = \frac{459}{49s} + \frac{5}{7s^2} + \frac{cs + d}{s^2 + 2s + 7}$$

$$X(s) = \frac{459s(s^2 + 2s + 7) + 35(s^2 + 2s + 7) + 49s^2(cs + d)}{49s^2(s^2 + 2s + 7)}$$

Igualamos os numeradores de $X(s)$ conhecido e o obtido acima, para determinar c e d:

$$(459 + 49c)s^3 + (918 + 35 + 49d)s^2 + (3213 + 70)s + 245 =$$
$$= 49 \cdot (9s^3 + 19s^2 + 67s + 5)$$

Donde obtemos

$$\begin{cases} 459 + 49c = 49 \cdot 9 \\ 918 + 35 + 49d = 49 \cdot 19 \end{cases} \Rightarrow \boxed{\begin{cases} c = -18/49 \\ d = -22/49 \end{cases}}$$

Assim:

$$X(s) = \frac{459}{49s} + \frac{5}{7s^2} - \frac{18s/49 + 22/49}{s^2+2s+7}$$

$$X(s) = \frac{1}{49}\left[\frac{459}{s} + \frac{245}{7s^2} - \frac{18s+22}{s^2+2s+7}\right]$$

$$X(s) = \frac{1}{49}\left[\frac{459}{s} + \frac{245}{7s^2} - 18\frac{(s+1)}{(s+1)^2+6} - 4\cdot\frac{1}{(s+1)^2+6}\right]$$

Aplicando a transformada inversa a cada um dos termos:

$$\mathcal{L}^{-1}[X(s)] = x(t)$$

$$\mathcal{L}^{-1}\left[\frac{459}{49s}\right] = \frac{459}{49} \qquad \mathcal{L}^{-1}\left[\frac{5}{7s^2}\right] = \frac{5}{7}t$$

$$\mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+6}\right] = e^{-t}\cos(\sqrt{6}t)$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+6}\right] = \mathcal{L}^{-1}\left[\frac{1}{\sqrt{6}}\frac{\sqrt{6}}{(s+1)^2+6}\right] = \frac{1}{\sqrt{6}}\cdot e^{-t}\operatorname{sen}(\sqrt{6}t)$$

Finalmente:

$$\boxed{x(t) = \frac{459}{9} + \frac{5}{7}t - \frac{18}{49}e^{-t}\cos(\sqrt{6}t) - \frac{4}{49\sqrt{6}}e^{-t}\operatorname{sen}(\sqrt{6}t)}$$