PME 3380 - Aula 15.10

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1. $\mathcal{L}[A f(t)] = \int_0^\infty A f(t) e^{-st} dt = A \underbrace{\int_0^\infty f(t) e^{-st} dt}_{F(s)} \Rightarrow$

$\Rightarrow \boxed{\mathcal{L}[A f(t)] = A F(s)}$ ∎

2. $\mathcal{L}[f_1(t) \pm f_2(t)] = \int_0^\infty (f_1(t) \pm f_2(t)) e^{-st} dt = \underbrace{\int_0^\infty f_1(t) e^{-st} dt}_{F_1(s)} \pm \underbrace{\int_0^\infty f_2(t) e^{-st} dt}_{F_2(s)}$

$\Rightarrow \boxed{\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)}$ ∎

3. Seja $f(t) = \frac{dg(t)}{dt}$

por partes

$\mathcal{L}[\int f(t) dt] = \mathcal{L}[g(t)] = \int_0^\infty g(t) e^{-st} dt = -\frac{g(t) e^{-st}}{s}\Big|_0^\infty - \int_0^\infty \underbrace{\frac{dg(t)}{dt}}_{f(t)} \cdot \left(-\frac{e^{-st}}{s}\right) dt$

Como $\lim_{t \to \infty} e^{-st} = 0 \Rightarrow -g(t) \frac{e^{-st}}{s}\Big|_0^\infty = \cancel{\frac{-g(\infty) e^{-s\infty}}{s}}^{0} + \frac{g(0)}{s}$

$\mathcal{L}[g(t)] = \frac{g(0)}{s} + \frac{1}{s} \underbrace{\int_0^\infty f(t) e^{-st} dt}_{F(s)} \Rightarrow \boxed{\mathcal{L}[\int f(t) dt] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t) dt\right]_{t=0}}$ ∎

4. $\mathcal{L}[e^{-at} f(t)] = \int_0^\infty e^{-at} f(t) e^{-st} dt = \int_0^\infty e^{-(s+a)t} f(t) dt$

Com $v = s+a$

$\mathcal{L}[e^{-at} f(t)] = \int_0^\infty f(t) e^{-vt} dt = F(v) = F(s+a) \Rightarrow \boxed{\mathcal{L}[e^{-at} f(t)] = F(s+a)}$ ∎

5. Sendo $u = \frac{t}{a} \Rightarrow du = \frac{dt}{a} \Rightarrow dt = a\,du$

$$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = \int_0^\infty f\left(\frac{t}{a}\right) e^{-st}\,dt = \int_0^\infty f(u)\, e^{-sau}\, a\,du$$

Sendo $b = as$:

$$\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = a\int_0^\infty f(u)\, e^{-bu}\, du = a\,F(b) \Rightarrow$$

$$\Rightarrow \boxed{\mathcal{L}\left[f\left(\frac{t}{a}\right)\right] = a\,F(as)}$$ ■

6. Sabemos por demonstrações em aula: (TVF) e (TVI)

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \quad \text{e} \quad \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

Admitindo-se $\exists\ \mathcal{L}[f(t)]$, $\mathcal{L}\left[\frac{df(t)}{dt}\right]$ e $\exists \lim_{t\to 0^+} f(t)$

Temos também: $\mathcal{L}_t\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$
(demonstrado)

Fazendo $s \to \infty$:

$$\lim_{s\to\infty} \int_{0^+}^\infty \frac{df(t)}{dt} e^{-st}\,dt = \lim_{s\to\infty}\left[sF(s) - f(0^+)\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \left(\cancelto{0}{\lim_{s\to\infty} e^{-st}}\right) \frac{df(t)}{dt}\,dt = \lim_{s\to\infty} sF(s) - \lim_{s\to\infty} f(0^+) \Rightarrow \lim_{s\to\infty} sF(s) - \lim_{s\to\infty} f(0^+) = 0 \Rightarrow$$

$$\Rightarrow \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} f(0^+) = f(0^+) = \lim_{t\to 0^+} f(t) \Rightarrow$$ uma vez que $\exists f(0^+) = \lim_{t\to 0^+} f(t)$

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to\infty} sF(s)}$$ ■

7. $\mathcal{L}[e^{-at}] = \int_0^\infty e^{-at} e^{-st}\,dt = \int_0^\infty e^{-(s+a)t}\,dt = -\frac{e^{-(s+a)t}}{s+a}\Big|_0^\infty =$

$= \lim_{t\to\infty} \left(-\cancel{\frac{e^{-(s+a)t}}{s+a}}^{\to 0}\right) + \frac{e^0}{s+a} \Rightarrow \boxed{\mathcal{L}[e^{-at}] = \frac{1}{s+a}}$ ■

8.

$\mathcal{L}[t e^{-at}] = \int_0^\infty t\, e^{-(s+a)t}\,dt \overset{\text{partes}}{=} \frac{-t e^{-(s+a)t}}{s+a}\Big|_0^\infty - \int_0^\infty -\frac{e^{-(s+a)t}}{s+a}\,dt =$

$= \lim_{t\to\infty}\left(\cancel{\frac{-t e^{-(s+a)t}}{s+a}}^{\to 0}\right) + \cancel{\frac{0\, e^0}{s+a}}^{0} + \frac{1}{s+a}\int_0^\infty e^{-(s+a)t}\,dt = \frac{1}{s+a}\mathcal{L}[e^{-at}]$

Lembrando que no ex 7 $\mathcal{L}[e^{-at}] = \frac{1}{s+a}$ logo:

$\mathcal{L}[t e^{-at}] = \frac{1}{s+a}\cdot \mathcal{L}[e^{-at}] \Rightarrow \boxed{\mathcal{L}[t e^{-at}] = \frac{1}{(s+a)^2}}$ ■

9.

$\mathcal{L}[\sin \omega t] = \int_0^\infty e^{-st} \sin \omega t\,dt \overset{\text{partes}}{=} -\frac{e^{-st}}{s}\sin\omega t\Big|_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\omega\cos\omega t\,dt =$

$= \lim_{t\to\infty} \cancel{\frac{-e^{-st}}{s}}^{\to 0} + \frac{e^0}{s}\cancel{\sin 0}^{\to 0} + \underbrace{\int_0^\infty \frac{e^{-st}}{s}\omega\cos\omega t\,dt}_{(*)}$

(*) por partes

$\int_0^\infty \frac{e^{-st}}{s}\omega\cos\omega t\,dt = -\frac{e^{-st}}{s^2}\omega\cos\omega t\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{s^2}\omega^2 \sin\omega t\,dt =$

$= \lim_{t\to\infty}\Big(\underbrace{-\frac{e^{-st}}{s^2}}_{\to 0}\underbrace{\omega\cos\omega t}_{\text{limitado}}\Big) + \frac{e^0}{s^2}{}^{\to 1}\,\omega\cancel{\cos 0}^{1} - \frac{\omega^2}{s^2}\int_0^\infty e^{-st}\sin\omega t\,dt$

Logo voltando:

$\int_0^\infty e^{-st}\sin\omega t = \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\int_0^\infty e^{-st}\sin\omega t\,dt \Rightarrow \int_0^\infty e^{-st}\sin\omega t\,dt = \frac{\frac{\omega}{s^2}}{1+\frac{\omega^2}{s^2}} \Rightarrow$

$\Rightarrow \boxed{\mathcal{L}[\sin\omega t] = \int_0^\infty e^{-st}\sin\omega t\,dt = \frac{\omega}{s^2+\omega^2}}$ ■

10. $\mathcal{L}[f \cdot g] = \int_0^{\infty} (f \cdot g)\, e^{-st}\, dt = \int_0^{\infty} e^{-st} \left(\int_0^{\infty} f(t-\tau) g(\tau)\, d\tau \right) dt =$

$= \int_0^{\infty}\int_0^{\infty} f(t-\tau)\, g(\tau)\, e^{-st}\, dt\, d\tau$ Fazendo $\alpha = t - \tau$

$$= \int_0^{\infty}\int_0^{\infty} f(\alpha)\, g(\tau)\, e^{-s(\alpha+\tau)}\, d\alpha\, d\tau = \underbrace{\int_0^{\infty} f(\alpha) e^{-s\alpha} d\alpha}_{F(s)} \cdot \underbrace{\int_0^{\infty} g(\tau) e^{-s\tau} d\tau}_{G(s)}$$

$\Rightarrow \boxed{\mathcal{L}[f \cdot g] = F(s) \cdot G(s)}$ ∎

11. $y(t) = \frac{7}{3} e^{-t} + \frac{3}{2} e^{-2t} - \frac{1}{6} e^{-4t} \xrightarrow{\mathcal{L}} Y(s) = \frac{7}{3}\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s+2} - \frac{1}{6}\cdot\frac{1}{s+4}$

P/ o impulso: $u(t) = \delta(t) \xrightarrow{\mathcal{L}} U(s) = 1$

$G(s) = \frac{Y(s)}{U(s)} \Rightarrow G(s) = Y(s) \Rightarrow \boxed{G(s) = \frac{7}{3(s+1)} + \frac{3}{2(s+2)} - \frac{1}{6(s+4)}}$

12. $y(t) = 1 - \frac{7}{3} e^{-t} + \frac{3}{2} e^{-2t} - \frac{1}{6} e^{-4t} \xrightarrow{\mathcal{L}} Y(s) = \frac{1}{s} - \frac{7}{3}\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s+2} - \frac{1}{6}\frac{1}{s+4}$

P/ o impulso: $u(t) = 1 \Rightarrow U(s) = \frac{1}{s}$

$G(s) = \frac{Y(s)}{U(s)} \Rightarrow \boxed{G(s) = 1 - \frac{7s}{3(s+1)} + \frac{3s}{2(s+2)} - \frac{s}{6(s+4)}}$

13. Para o motor C.C:

$$\begin{cases} L_a\, Di_a\, R_a\, i_a = e_a - K_b\,\omega \\ J\ddot{\theta} + B\dot{\theta} = K i_a \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} (L_a s + R_a) I_a = E_a - K_b s\, \Theta \\ (J s^2 + B s)\, \Theta = K I_a \end{cases}$$

Fazendo $Y(s) = \Theta(s)$ e $U(s) = E_a(s)$, achamos a solução em aula e escrevemos:

$G_1(s) = \frac{\Theta(s)}{E_a(s)} \Rightarrow G_1(s) = \frac{K}{J L_a s^3 + (R_a J + B L_a) s^2 + (R_a B + K K_b) s}$

P/ acharmos os polos, fazemos as raízes de s em $E_a(s)=0$:

$$s\left[JL_a s^2 + (R_a J + BL_a)s + R_a B + KK_b\right] = 0$$

Primeiro polo $s=0$

$$JL_a s^2 + (R_a J + BL_a)s + R_a B + KK_b = 0$$

Resolvendo eq 2º grau, chegamos nos 3 polos:

$$S_1 = \frac{1}{2}\left[-\frac{R_a}{L_a} - \frac{B}{J} - \sqrt{\left(\frac{R_a}{L_a} + \frac{B}{J}\right)^2 - 4\,\frac{(R_a B + KK_b)}{JL_a}}\right]$$

$$S_2 = \frac{1}{2}\left[-\frac{R_a}{L_a} - \frac{B}{J} + \sqrt{\left(\frac{R_a}{L_a} + \frac{B}{J}\right)^2 - 4\,\frac{(R_a B + KK_b)}{JL_a}}\right]$$

$$S_3 = 0$$