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PME 3380 - Modelagem de Sistemas Dinâmicos

Exercícios aula

① $\mathcal{L}[A f(t)] = \int_0^\infty A f(t)\, e^{-st}\, dt = A \underbrace{\int_0^\infty f(t)\, e^{-st}\, dt}_{F(s)}$

$$\boxed{\mathcal{L}[A f(t)] = A F(s)} \blacksquare$$

② $\mathcal{L}[f_1(t) \pm f_2(t)] = \int_0^\infty (f_1(t) \pm f_2(t))\, e^{-st}\, dt$

$$= \int_0^\infty \left[f_1(t) e^{-st} \pm f_2(t) e^{-st}\right] dt$$

$$= \underbrace{\int_0^\infty f_1(t)\, e^{-st}\, dt}_{F_1(s)} \pm \underbrace{\int_0^\infty f_2(t)\, e^{-st}\, dt}_{F_2(s)}$$

$$\Rightarrow \boxed{\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)} \blacksquare$$

③ Seja $g(t) = \int f(t)\, dt \iff f(t) = \frac{dg(t)}{dt}$

$$\mathcal{L}\left[\int f(t)\, dt\right] = \mathcal{L}[g(t)] = \int_0^\infty g(t)\, e^{-st}\, dt$$

Integração por partes: $\begin{cases} u = g(t) \\ dv = e^{-st} \end{cases} \Rightarrow \int_0^\infty u\, dv = (uv)\Big|_0^\infty + \int_0^\infty v\, du$

$$\Rightarrow du = \frac{dg}{dt} \qquad v = -\frac{e^{-st}}{s}$$

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$$\mathcal{L}[g(t)] = \int_0^\infty g(t)\, e^{-st}\, dt = -\frac{g(t)e^{-st}}{s}\Bigg|_0^\infty - \int_0^\infty \frac{dg(t)}{dt}\cdot\left(-\frac{e^{-st}}{s}\right)dt$$

Mas: $\begin{cases} \dfrac{dg(t)}{dt} = f(t) \\ \lim\limits_{t\to\infty} e^{-st} = 0 \Rightarrow -\dfrac{g(t)e^{-st}}{s}\Bigg|_0^\infty = -\cancel{\dfrac{g(\infty)e^{-s\infty}}{s}}^{0} + \dfrac{g(0)}{s} \end{cases}$

$$\Rightarrow \mathcal{L}[g(t)] = \frac{g(0)}{s} + \frac{1}{s}\underbrace{\int_0^\infty f(t)\, e^{-st}\, dt}_{F(s)}$$

Além disso, $g(0) = \left[\int f(t)\, dt\right]_{t=0}$

$$\Rightarrow \boxed{\mathcal{L}\left[\int f(t)\, dt\right] = \frac{F(s)}{s} + \frac{1}{s}\left[\int f(t)\, dt\right]_{t=0}} \quad \blacksquare$$

4 $\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-at} f(t)\, e^{-st}\, dt$

$$= \int_0^\infty f(t)\, e^{-(s+a)t}\, dt$$

Fazendo $u = s + a$:

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty f(t)e^{-ut}dt = F(u) = F(s+a)$$

$$\Rightarrow \boxed{\mathcal{L}[e^{-at}f(t)] = F(s+a)} \quad \blacksquare$$

⑤ Mudança de variável: $u = \frac{t}{a} \iff du = \frac{dt}{a} \iff dt = a\,du$

$$\mathcal{L}[f(\tfrac{t}{a})] = \int_0^\infty f(\tfrac{t}{a}) e^{-st} dt$$

$$\Rightarrow \int_0^\infty f(u) e^{-sau} a\, du$$

Fazendo $b = sa$:

$$\Rightarrow \mathcal{L}[f(\tfrac{t}{a})] = a \int_0^\infty f(u) e^{-bu} du = a\,F(b)$$

$$\Rightarrow \boxed{\mathcal{L}[f(\tfrac{t}{a})] = a\,F(as)}$$

⑥ Teorema do Valor Final: apresentado e demonstrado em aula. $\Rightarrow \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ ✓

Teorema do Valor Inicial:

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

Admitindo que $\begin{cases} \exists\, \mathcal{L}[f(t)], \mathcal{L}\left[\frac{df(t)}{dt}\right] & (i) \\ \exists \lim_{t \to 0^+} f(t) & (ii) \end{cases}$

Das condições (i) e (ii):

$$\mathcal{L}_+\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+) \rightarrow \left(\begin{array}{c}\text{Demonstrado} \\ \text{em aula}\end{array}\right)$$

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Escrevendo explicitamente $\mathcal{L}\left[\frac{d}{dt}f(t)\right]$ e tomando o limite para $s \to \infty$:

$$\underbrace{\lim_{s\to\infty} \int_{0^+}^{\infty} \frac{df(t)}{dt} e^{-st} dt}_{\mathrm{I}} = \underbrace{\lim_{s\to\infty} \left[sF(s) - f(0^+)\right]}_{\mathrm{II}}$$

Calculando I:

$$\lim_{s\to\infty} \int_{0^+}^{\infty} \frac{df(t)}{dt} e^{-st} dt = \int_{0^+}^{\infty} \frac{df(t)}{dt} \left(\underbrace{\lim_{s\to\infty} e^{-st}}_{0}\right) dt = 0$$

Desta forma:

$$\lim_{s\to\infty} \left[sF(s) - f(0^+)\right] = 0$$

$$\lim_{s\to\infty} sF(s) - \lim_{s\to\infty} f(0^+) = 0 \qquad (*)$$

Mas $\exists\, f(0^+) = \lim_{t\to 0^+} f(t) \Leftrightarrow f(0^+) =$ constante real.

Assim: $\lim_{s\to\infty} f(0^+) = f(0^+) = \lim_{t\to 0^+} f(t)$. $\quad (**)$

Substituindo $(**)$ em $(*)$:

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to\infty} sF(s)}$$

■

⑦ $\mathcal{L}[e^{-at}] = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt$

$$= -\frac{e^{-(s+a)t}}{s+a}\Bigg|_0^\infty = \underbrace{\lim_{t\to 0}\left(-\frac{e^{-(s+a)t}}{s+a}\right)}_{\to 0} + \frac{\cancelto{1}{e^{0}}}{s+a}$$

$$\Rightarrow \boxed{\mathcal{L}[e^{-at}] = \frac{1}{s+a}} \quad ■$$

⑧ $\mathcal{L}[te^{-at}] = \int_0^\infty t\, e^{-(s+a)t} dt$

Por partes:

$$\mathcal{L}[te^{-at}] = -\frac{te^{-(s+a)t}}{s+a}\Bigg|_0^\infty - \int_0^\infty -\frac{e^{-(s+a)t}}{s+a} dt$$

$$\mathcal{L}[te^{-at}] = \underbrace{\lim_{t\to\infty}\left(\frac{-te^{-(s+a)t}}{s+a}\right)}_{\to 0} + \cancelto{0}{\frac{0\cdot e^{0}}{s+a}} + \frac{1}{s+a}\underbrace{\int_0^\infty e^{-(s+a)t} dt}_{\mathcal{L}[e^{-at}]}$$

Mas do exercício ⑦, sabemos que $\mathcal{L}[e^{-at}] = \frac{1}{s+a}$.

Assim: $\mathcal{L}[te^{-at}] = \frac{1}{s+a}\cdot\mathcal{L}[e^{-at}]$

$$\Rightarrow \boxed{\mathcal{L}[te^{-at}] = \frac{1}{(s+a)^2}} \quad ■$$

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9) $\mathcal{L}[\operatorname{sen} \omega t] = \int_0^\infty e^{-st} \operatorname{sen} \omega t \, dt$

Por partes:

$$\int_0^\infty e^{-st} \operatorname{sen}\omega t \, dt = -\frac{e^{-st}}{s} \operatorname{sen} \omega t \Big|_0^\infty - \int_0^\infty -\frac{e^{-st}}{s} \omega \cos\omega t \, dt$$

$$= \underbrace{\lim_{t\to\infty} \left(-\frac{e^{-st}}{s}\right)}_{\to 0} + \frac{\cancel{e^{0}}^{1}}{s} \cancel{\operatorname{sen} 0}^{0} + \underbrace{\int_0^\infty \frac{e^{-st}}{s} \omega \cos\omega t \, dt}_{(*)}$$

Calculando a integral $(*)$ por partes:

$$\int_0^\infty \frac{e^{-st}}{s} \omega \cos\omega t \, dt = -\frac{e^{-st}}{s^2} \omega \cos \omega t \Big|_0^\infty - \int_0^\infty + \frac{e^{-st}}{s^2} \omega^2 \operatorname{sen}\omega t \, dt$$

$$= \underbrace{\lim_{t\to\infty} \left(\underbrace{-\frac{e^{-st}}{s^2}}_{\to 0} \underbrace{\omega \cos\omega t}_{\text{limitada}} \right)}_{\to 0} + \frac{\cancel{e}^{1}}{s^2} \omega \cancel{\cos 0}^{1} - \frac{\omega^2}{s^2} \int_0^\infty e^{-st} \operatorname{sen}\omega t \, dt$$

$$\int_0^\infty \frac{e^{-st}}{s} \omega \cos\omega t \, dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2} \int_0^\infty e^{-st} \operatorname{sen} \omega t \, dt \qquad (**)$$

Substituindo $(**)$ em $(*)$:

$$\int_0^\infty e^{-st} \operatorname{sen} \omega t \, dt = \frac{\omega}{s^2} - \frac{\omega^2}{s^2} \int_0^\infty e^{-st} \operatorname{sen} \omega t \, dt$$

$$\left(1 + \frac{\omega^2}{s^2}\right) \int_0^\infty e^{-st} \operatorname{sen}\omega t \, dt = \frac{\omega}{s^2}$$

$$\int_0^\infty e^{-st} \operatorname{sen} \omega t \, dt = \frac{\omega}{\cancel{s^2}} \cdot \frac{\cancel{s^2}}{s^2+\omega^2}$$

$$\Rightarrow \boxed{\int_0^\infty e^{-st} \operatorname{sen} \omega t \, dt = \frac{\omega}{s^2+\omega^2}}$$

$\blacksquare$

10) $$\mathcal{L}[f*g] = \int_0^\infty (f*g)\, e^{-st}\, dt$$

$$= \int_0^\infty e^{-st} \left(\int_0^\infty f(t-\tau)\, g(\tau)\, d\tau \right) dt$$

$$= \int_0^\infty \int_0^\infty f(t-\tau)\, g(\tau)\, e^{-st}\, dt\, d\tau$$

Mudança de variáveis, $\alpha = t - \tau \Leftrightarrow d\alpha = dt$

$$\mathcal{L}[f*g] = \int_0^\infty \int_0^\infty f(\alpha)\, g(\tau)\, e^{-s(\alpha+\tau)}\, d\alpha\, d\tau$$

$$= \underbrace{\left(\int_0^\infty f(\alpha)\, e^{-s\alpha}\, d\alpha \right)}_{F(s)} \cdot \underbrace{\left(\int_0^\infty g(\tau)\, e^{-s\tau}\, d\tau \right)}_{G(s)}$$

$$\Rightarrow \boxed{\mathcal{L}[f*g] = F(s)\, G(s)}$$

$\blacksquare$

(11) $y(t) = \frac{7}{3}e^{-t} + \frac{3}{2}e^{-2t} - \frac{1}{6}e^{-4t}$

$$\Big\downarrow \mathcal{L}$$

$$Y(s) = \frac{7}{3}\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s+2} - \frac{1}{6}\cdot\frac{1}{s+4}$$

Impulso: $u(t) = \delta(t) \xrightarrow{\mathcal{L}} U(s) = 1$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Y(s)}{1} \Rightarrow G(s) = Y(s)$$

$$\boxed{G(s) = \frac{7}{3(s+1)} + \frac{3}{2(s+2)} - \frac{1}{6(s+4)}}$$

(12) $y(t) = 1 - \frac{7}{3}e^{-t} + \frac{3}{2}e^{-2t} - \frac{1}{6}e^{-4t}$

$$\Big\downarrow \mathcal{L}$$

$$Y(s) = \frac{1}{s} - \frac{7}{3}\cdot\frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s+2} - \frac{1}{6}\cdot\frac{1}{s+4}$$

Degrau unitário de entrada: $u(t) = 1 \Rightarrow U(s) = \frac{1}{s}$

$$G(s) = \frac{Y(s)}{U(s)} \Rightarrow \boxed{G(s) = 1 - \frac{7s}{3(s+1)} + \frac{3s}{2(s+2)} - \frac{s}{6(s+4)}}$$

(13) Para o motor C.C. temos:

$$\begin{cases} L_a D i_a + R_a i_a = e_a - K_b \omega \\ J\ddot{\theta} + B\dot{\theta} = K i_a \end{cases}$$

$$\Big\Downarrow \mathcal{L}$$

$$\begin{cases} (L_a s + R_a) I_a = E_a - K_b s \Theta \\ (J s^2 + B s)\Theta = K I_a \end{cases}$$

Tomando $Y(s) = \Theta(s)$ e $U(s) = E_a(s)$, resolvemos o sistema de equações algébricas (dedução nos slides de aula) para escrever a função de transferência

$$G_1(s) = \frac{\Theta(s)}{E_a(s)}$$

$$\boxed{G_1(s) = \frac{K}{J L_a s^3 + (R_a J + B L_a)s^2 + (R_a B + K K_b)s}}$$

Polos do sistema $\Rightarrow$ Raízes de $E_a(s) = 0$:

$$s\left[J L_a s^2 + (R_a J + B L_a)s + R_a B + K K_b\right] = 0$$

Um dos polos é $s = 0$. Os outros dois vêm de:

$$J L_a s^2 + (R_a J + B L_a)s + R_a B + K K_b = 0$$

Por Bhaskara:

$$S = \frac{-(R_aJ + BL_a) \pm \sqrt{(R_aJ + BL_a)^2 - 4JL_a(R_aB + KK_b)}}{2JL_a}$$

$$S = \frac{1}{2}\left[-\frac{R_a}{L_a} - \frac{B}{J} \pm \sqrt{\left(\frac{R_a}{L_a} + \frac{B}{J}\right)^2 - 4\,\frac{(R_aB + KK_b)}{JL_a}}\right]$$

Os três polos do sistema serão, portanto:

$$\boxed{\begin{aligned} S_1 &= \frac{1}{2}\left[-\frac{R_a}{L_a} - \frac{B}{J} - \sqrt{\left(\frac{R_a}{L_a} + \frac{B}{J}\right)^2 - 4\,\frac{(R_aB + KK_b)}{JL_a}}\right] \\ S_2 &= \frac{1}{2}\left[-\frac{R_a}{L_a} - \frac{B}{J} + \sqrt{\left(\frac{R_a}{L_a} + \frac{B}{J}\right)^2 - 4\,\frac{(R_aB + KK_b)}{JL_a}}\right] \\ S_3 &= 0 \end{aligned}}$$