PME3380 - Modelagem

Exc. 06/10

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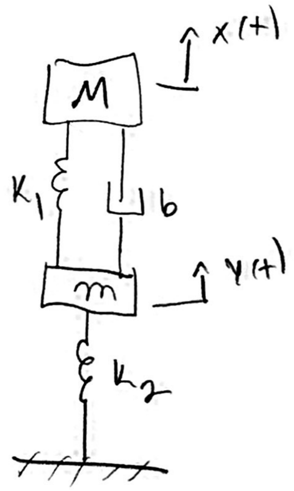

$\longrightarrow$ $v(t)$

$$\begin{cases} M\ddot{x} + b\dot{x} + K_1 x = b\dot{y} + K_1 y \\ m\ddot{y} + b\dot{y} + (K_1 + K_2)y = b\dot{x} + K_1 x + K_2 z \end{cases}$$

Usando $w(t) = \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}$ e $u(t) = z(t)$

Em E.E.:

$$\begin{cases} \dot{w}(t) = A\,w(t) + B\,u(t) \\ r(t) = C\,w(t) + D\,u(t) \end{cases}$$

$\hookrightarrow$ vetor de saída

Onde:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-K_1}{M} & \frac{K_1}{M} & \frac{-b}{M} & \frac{b}{M} \\ \frac{K_1}{m} & \frac{-(K_1+K_2)}{m} & \frac{b}{m} & \frac{-b}{m} \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_2}{m} \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} ; \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Com $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

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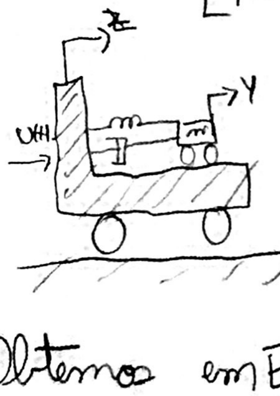


$$M\ddot{z} + b\dot{z} + kz = b\dot{y} + ky + u(t)$$

$$m\ddot{y} + b\dot{y} + ky = b\dot{z} + kz$$

Com $x(t) = \begin{bmatrix} y \\ z \\ \dot{y} \\ \dot{z} \end{bmatrix}$

Obtemos em EE:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k}{m} & \frac{k}{m} & \frac{-b}{m} & \frac{b}{m} \\ \frac{k}{M} & \frac{-k}{M} & \frac{b}{M} & \frac{-b}{M} \end{bmatrix}}_{A} \begin{bmatrix} y \\ z \\ \dot{y} \\ \dot{z} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \end{bmatrix}}_{B} u(t)$$

Com o vetor de saída $r(t) = \begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix}$

$$r = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} y \\ z \\ \dot{y} \\ \dot{z} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D} u(t)$$

Lembrando: $m\ddot{y} + M\ddot{z} = u(t) \xrightarrow{\text{com } m \gg M} m\ddot{y} = u(t)$

vetor entrada $x(t) = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$; vet saída $r(t) = \dot{y}$

EE:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t) \qquad \dot{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + 0\,u(t)$$

4) a) P/ esse sistema suas eqs são

$$\begin{cases} m_1\ddot{x}_1 + b_1\dot{x}_1 + (k_{p_1} + k_1)x_1 = k_{p_1} z(t) + k_1(x_G + l_1 \sin\theta) + b_1(\dot{x}_G + l_1\dot{\theta}\cos\theta) \\ m_2\ddot{x}_2 + b_2\dot{x}_2 + (k_{p_2} + k_2)x_2 = k_{p_2} z(t-\alpha) + k_2(x_G - l_2\sin\theta) + b_2(\dot{x}_G - l_2\dot{\theta}\cos\theta) \\ M\ddot{x}_G + (b_1+b_2)\dot{x}_G + (k_1+k_2)x_G = k_1(x_1 - l_1\sin\theta) + k_2(x_2 + l_2\sin\theta) + b_1(\dot{x}_1 - l_1\dot{\theta}\cos\theta) + b_2(\dot{x}_2 + l_2\dot{\theta}\cos\theta) \\ J_G\ddot{\theta} + (b_1 l_1^2 + b_2 l_2^2)\dot{\theta}\cos^2\theta + (k_1 l_1^2 + k_2 l_2^2)\sin\theta\cos\theta = \cos\theta\left[k_1 l_1(x_1 - x_G) + k_2 l_2(x_G - x_2) + b_1 l_1(\dot{x}_1 - \dot{x}_G) + b_2 l_2(\dot{x}_G - \dot{x}_2)\right] \end{cases}$$

Com $w(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_G \\ \theta \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_G \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{bmatrix}$

Logo;

$$\begin{cases} \dot{w}_1 = w_5 \\ \dot{w}_2 = w_6 \\ \dot{w}_3 = w_7 \\ \dot{w}_4 = w_8 \\ \dot{w}_5 = \frac{1}{m_1}\left[k_{p_1} z(t) + k_1(w_3 + l_1\sin w_4) + b_1(w_7 + l_1 w_8\cos w_4) - b_1 w_5 - (k_{p_1} + k_1)w_1\right] \\ \dot{w}_6 = \frac{1}{m_2}\left[k_{p_2} z(t-\alpha) + k_2(w_3 - l_2\sin w_4) + b_2(w_7 - l_2 w_8\cos w_4) - b_2 w_6 - (k_{p_2} + k_2)w_2\right] \end{cases}$$

$$
\begin{cases}
\dot{\omega}_7 = \dfrac{1}{M}\Big[K_1(\omega_1 - l_1 \sin\omega_4) + K_2(\omega_2 + l_2 \sin\omega_4) + b_1(\omega_1 - l_1\omega_8\cos\omega_4) + b_2(\omega_6 + l_2\omega_8\cos\omega_4 \\
\qquad - (b_1+b_2)\,\omega_7 - (K_1+K_2)\,\omega_3\Big] \\
\dot{\omega}_8 = \dfrac{1}{J_G}\Big\{\Big[K_1 l_1(\omega_1-\omega_3) + K_2 l_2(\omega_3-\omega_2) + b_1 l_1(\omega_5-\omega_7) + b_2 l_2(\omega_7-\omega_6)\Big]\cos\omega_4 - \\
\qquad (b_1 l_1^2 + b_2 l_2^2)\,\omega_8\cos^2\omega_4 - (K_1 l_1^2 + K_2 l_2^2)\sin\omega_4\cos\omega_4\Big\}
\end{cases}
$$

b) p/ pequenos movs.

$$
\begin{cases}
m_1\ddot{x}_1 + b_1\dot{x}_1 + (K_{p_1}+k_1)x_1 = K_{p_1} z(t) + K_1(x_G + l_1\theta) + b_1(\dot{x}_G + l_1\dot{\theta}) \\
m_2\ddot{x}_2 + b_2\dot{x}_2 + (K_{p_2}+K_2)x_2 = K_{p_2} z(t-\alpha) + K_2(x_G - l_2\theta) + b_2(\dot{x}_G - l_2\dot{\theta}) \\
M\ddot{x}_G + (b_1+b_2)\dot{x}_G + (K_1+K_2)x_G = K_1(x_1 - l_1\theta) + K_2(x_2 + l_2\theta) + b_1(\dot{x}_1 - l_1\dot{\theta}) + b_2(\dot{x}_2 + l_2\dot{\theta}) \\
J_G\ddot{\theta} + (b_1 l_1^2 + b_2 l_2^2)\dot{\theta} + (K_1 l_1^2 + K_2 l_2^2)\theta = K_1 l_1(x_1 - x_G) + K_2 l_2(x_G - x_2) + b_1 l_1(\dot{x}_1 - \dot{x}_G) + b_2 l_2(\dot{x}_G - \dot{x}_2
\end{cases}
$$

com $x(t) = [x_1\ x_2\ x_G\ \theta\ \dot{x}_1\ \dot{x}_2\ \dot{x}_G\ \dot{\theta}]^T$ em EE temos;

$$
\begin{cases}
\dot{x} = Ax + Bu \\
y = Cx + Du
\end{cases}
$$

Onde;

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{-(k_1+k_1)}{m_1} & 0 & \frac{K_1}{m_1} & \frac{K_1 l_1}{m_1} & \frac{b_1}{m_1} & 0 & \frac{b_1}{m_1} & \frac{b_1 l_1}{m_1} \\
0 & \frac{-(K_{p_2}+K_2)}{m_2} & \frac{K_2}{m_2} & \frac{-K_2 l_2}{m_2} & 0 & \frac{-b_2}{m_2} & \frac{b_2}{m_2} & \frac{-b_2 l_2}{m_2} \\
\frac{K_1}{M} & \frac{K_2}{M} & \frac{-(K_1+K_2)}{M} & \frac{K_2 l_2 - K_1 l_1}{M} & \frac{b_1}{M} & \frac{b_2}{M} & \frac{-(b_1+b_2)}{M} & \frac{b_2 l_2 - b_1 l_1}{M} \\
\frac{K_1 l_1}{J_G} & \frac{-K_2 l_2}{J_G} & \frac{K_2 l_2 - K_1 l_1}{J_G} & \frac{-(K_1 l_1^2 + K_2 l_2^2)}{J_G} & \frac{b_1 l_1}{J_G} & \frac{-b_2 l_2}{J_G} & \frac{b_2 l_2 - b_1 l_1}{J_G} & \frac{-(b_1 l_1^2 + b_2 l_2^2)}{J_G}
\end{bmatrix}
$$

$$u(t) = \begin{bmatrix} z(t) \\ z(t-\alpha) \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{K_{p_1}}{m_1} & 0 \\ 0 & \frac{K_{p_2}}{m_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

e se $y(t) = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_G \\ \dot{\theta} \end{bmatrix}$:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5 $$\begin{cases} (M+m)\ddot{x} = u + ml\left(\dot{\theta}^2 \sin\theta + \ddot{\theta}\cos\theta\right) \\ \frac{4}{3} l\ddot{\theta} - g\sin\theta = -\ddot{x}\cos\theta \end{cases}$$

Vet Estados:

$$w(t) = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$, com entrada $u = u(t)$

Reescrevendo na forma do EE:

$$\begin{cases} \dot{w}_1 = w_3 \\ \dot{w}_2 = w_4 \\ \dot{w}_3 = \left[\left(g - \frac{m l w_4^2}{M+m}\cos w_2\right)\sin w_2 - \frac{u(t)\cos w_2}{M+m}\right]\left(\frac{4}{3} l + \frac{m l \cos^2 w_2}{M+m}\right)^{-1} \\ \dot{w}_4 = \left(u(t) + m l w_4^2 \sin w_2 + \frac{3}{4} mg \sin w_2 \cos w_2\right)\left(M + m + \frac{3}{4} m \cos^2 w_2\right)^{-1} \end{cases}$$

⑥ a) $\begin{cases} m\ddot{x} = mg - \dfrac{ki^2}{x^2} \\ L\dfrac{di}{dt} + Ri = V(t) \end{cases}$ $\quad w(t) = \begin{bmatrix} x \\ \dot{x} \\ i \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

Reescrevendo em EE:

$$\begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = g - \dfrac{k w_3^2}{m w_1^2} \\ \dot{w}_3 = \dfrac{V(t)}{L} \dfrac{R}{L} w_3 \end{cases}$$

Saída
$y(t) = x(t) = w_1(t)$

b) Com linearização

$$\begin{bmatrix} \delta\dot{x} \\ \delta\ddot{x} \\ \delta\dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{2k\,\dot{x}_{eq}^2}{m x_{eq}^3} & 0 & \dfrac{-2k i_{eq}}{m x_{eq}^2} \\ 0 & 0 & -\dfrac{R}{L} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta \dot{x} \\ \delta i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix} \delta V(t)$$

Onde $\begin{cases} \delta x = x - x_{eq} \\ \delta\dot{x} = \dot{x} - \dot{x}_{eq} \\ \delta i = i - i_{eq} \\ \delta V = V - V_{eq} \end{cases}$

Saída:
$y(t) = x(t)$

⑦ a) Teríamos que

$$L\frac{di}{dt} + R_{eq} i + K_b \dot{\theta}_1 = V_a(t)$$

E por TQM no disco e pinhão

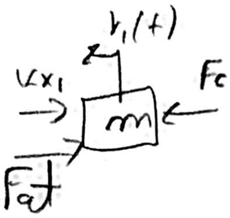
$$J_m \ddot{\theta}_1 + 2K_s |\theta_1 - \theta_2| (\theta_1 - \theta_2) + B_m \dot{\theta}_1 = K_a i$$

$$J_p \ddot{\theta}_2 = T_{NLK} - F_c R$$

$\dot{x}_1 = \theta_2 R \Rightarrow \ddot{x}_1 = \ddot{\theta}_2 R$

Supondo $\theta_2(0) = 0$

$$F_c = \frac{1}{R}\left[2K_s\left|\theta_1 - \frac{x_1}{R}\right|\left(\theta_1 - \frac{x_1}{R}\right) - J_p\frac{\ddot{x}_1}{R}\right]$$

P/ o carro 

$$F_{at} = 2\dot{x}_1^3$$

Aplicando TMB:

$$\left(m + \frac{J_p}{R^2}\right)\ddot{x}_1 + 2K\dot{x}_1^3 + Kx_1 = \frac{2K_s\left|\theta_1 - \frac{x_1}{R}\right|\left(\theta_1 - \frac{x_1}{R}\right)}{R}$$

Logo as eqs que regem:

$$L\frac{di}{dt} + R_{el}\, i + K_b\dot{\theta}_1 = V_a(t)$$

$$J_m\ddot{\theta}_1 + 2K_s\left|\theta_1 - \frac{x_1}{R}\right|\left(\theta_1 - \frac{x_1}{R}\right) + B_m\dot{\theta}_1 = K_a i$$

$$\left(m + \frac{J_p}{R^2}\right)\ddot{x}_1 + 2K\dot{x}_1^3 + kx_1 = \frac{2K_s\left|\theta_1 - \frac{x_1}{R}\right|\left(\theta_1 - \frac{x_1}{R}\right)}{R}$$

b) Reescrevendo e isolando

$$f_1 = \frac{di}{dt} = \frac{1}{L}\left(V_a(t) - K_b\dot{\theta}_1 - R_{el}\, i\right)$$

$$f_2 = \ddot{\theta}_1 = \frac{1}{J_m}\left[K_a i - 2K_s\left|\theta_1 - \frac{x_1}{R}\right|\left(\theta_1 - \frac{x_1}{R}\right) - B_m\dot{\theta}_1\right]$$

$$f_3 = \ddot{x}_1 = \frac{R^2}{mR^2 + J_p}\left[\frac{2K_s\left|\theta_1 - \frac{x_1}{R}\right|\left(\theta_1 - \frac{x_1}{R}\right)}{R} - 2\dot{x}_1^3 - Kx_1\right]$$

Assim, resulta-se no seguinte sistema linearizado

$$L\frac{di}{dt} = \delta V_a - K_b\,\delta\dot{\theta}_1 - R_{el}\,\delta i$$

Se $\theta_1 \geq \frac{x_1}{R}$: $J_m\ddot{\theta}_1 = K_a\,\delta i - 4K_s\left(\theta_{1eq} - x_{1eq}\right)\delta\theta_1 + \frac{4K_s}{R}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right)\delta x_1 - B_m\dot{\theta}_1$

Se $\theta_1 < \frac{x_1}{R}$: $J_m\ddot{\theta}_1 = K_a\,\delta i + 4K_s\left(\theta_{1eq} - x_{1eq}\right)\delta\theta_1 - \frac{4K_s}{R}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right)\delta x_1 - B_m\dot{\theta}_1$

Se $\theta_1 \geq \frac{x_1}{R}$: $\left(m + \frac{J_p}{R^2}\right)\ddot{x}_1 = \frac{4K_s}{R}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right)\delta\theta_1 - \delta x_1\left[K + \frac{4K_s}{R^2}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right)\right] - 6\,\dot{x}_{1eq}^2\,\delta\dot{x}_1$

Se $\theta_1 < \frac{x_1}{R}$: $\left(m + \frac{J_p}{R^2}\right)\ddot{x}_1 = -\frac{4K_s}{R}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right)\delta\theta_1 - \delta x_1\left[K - \frac{4K_s}{R^2}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right)\right] - 6\,\dot{x}_{1eq}^2\,\delta\dot{x}_1$

c) Definimos $\delta x(t) = \begin{bmatrix} \delta x_1 \\ \delta\dot{x}_1 \\ \delta\theta_1 \\ \delta\dot{\theta}_1 \\ \delta i \end{bmatrix}$ com $\begin{cases} \delta\dot{x}_1 = A\,\delta x + B\,\delta u \\ y = C\,\delta x + D\,\delta u \end{cases}$

Logo p/ $\theta_1 \geq x_1/R$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{-R^2}{mR^2+J_p}\left[K + \frac{4K_s}{R^2}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right)\right] & \frac{-6R^2\dot{x}_{eq}^2}{mR^2+J_p} & \frac{-4RK_s}{mR^2+J_p}\left(\theta_{1eq} - x_{eq}\right) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{4K_s}{J_mR}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right) & 0 & -\frac{4K_s}{J_m}\left(\theta_{1eq} - \frac{x_{1eq}}{R}\right) & -\frac{B_m}{J_m} & \frac{K_a}{J_m} \\ 0 & 0 & 0 & \frac{-K_b}{L} & \frac{-R_{el}}{L} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{U} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

P/ $\theta_1 < \frac{x_1}{R}$!

Apenas subs $A_{2,1}$ por $\frac{R^2}{mR^2 + J_p}\left[k - \frac{4ks}{R^2}\left(\theta_{1eq} - \frac{x_{eq}}{R}\right)\right]$

e multiplica (-1) $A_{2,3}$, $A_{4,1}$ e $A_{4,3}$

8 Por TMQM temos

$$J_x \ddot{\theta}_x = -2B\dot{\theta}_x - 2kd\theta_x - Jw\theta_z \Rightarrow$$

$$\Rightarrow \boxed{J_x \ddot{\theta}_x + 2B\dot{\theta}_x + 2kd\theta_x = -Jw\theta_z}$$

Vet estados

$$x(t) = \begin{bmatrix} \theta_x \\ \dot{\theta}_x \end{bmatrix}$$

entrada

$$u(t) = \dot{\theta}_z(t)$$

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ \frac{-2kd}{J_x} & \frac{-2B}{J_x} \end{bmatrix} \begin{bmatrix} \theta_x \\ \dot{\theta}_x \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{Jw}{J_x} \end{bmatrix} \int \dot{\theta}_z dt$$

Vetor de saídas

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_x \\ \dot{\theta}_x \end{bmatrix}$$