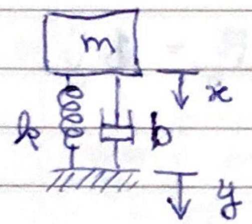
- Exercícios de Modelagem -

Carolina Carvalho Silva - 10705933

1- Sismógrafo



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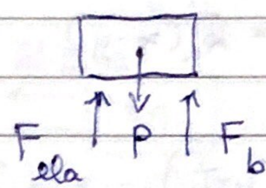
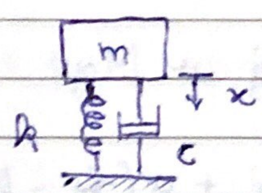


$$F_{ela} = k(x-y)$$

$$F_b = b(\dot{x}-\dot{y})$$

$$m(\ddot{x}+\ddot{y}) = -k(x-y) - b(\dot{x}-\dot{y}) + m \cdot g$$

1- b) Acelerômetro



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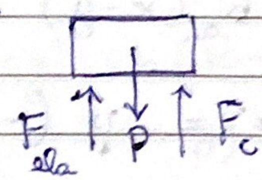
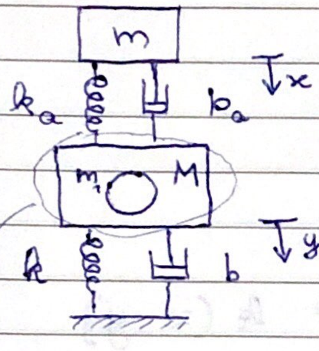


$$F_{ela} = k \cdot x$$

$$F_c = c \cdot \dot{x}$$

$$m(\ddot{x}+\ddot{y}) = -k \cdot x - c \cdot \dot{x} + m \cdot g$$

2- Máquina rotativa com absorvedor de vibração



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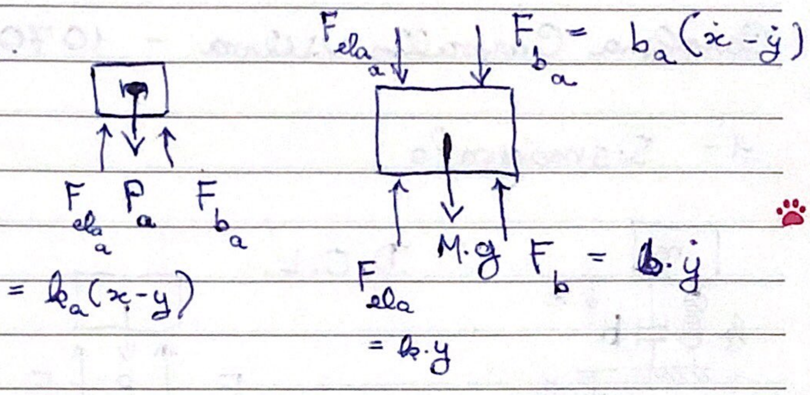


$$M \cdot y_{CM} = M \cdot y + m_1 r \operatorname{sen}(\omega t)$$

$$\Rightarrow M \cdot \ddot{y}_{CM} = M \cdot \ddot{y} - m_1 r \operatorname{sen}(\omega t) \cdot \omega^2$$

$$(1) \quad m(\ddot{x} + \ddot{y}) = -k_a(x-y) - b_a(\dot{x}-\dot{y}) + mg$$

$$(2) \quad M \cdot \ddot{y}_{CM} = k_a(x-y) + b_a(\dot{x}-\dot{y}) - k \cdot y - b \cdot \dot{y} + Mg$$

$$\Rightarrow M \cdot \ddot{y} - m_1 r \operatorname{sen}(\omega t)\, \omega^2 = \dot{y}(-b_a - b) + b_a \dot{x} + y(-k_a - k) + k_a x + Mg$$

$$\Rightarrow M\ddot{y} + (b_a + b)\dot{y} - b_a \dot{x} + (k_a + k)y - k_a x = m_1 r \operatorname{sen}(\omega t)\omega^2 + Mg$$

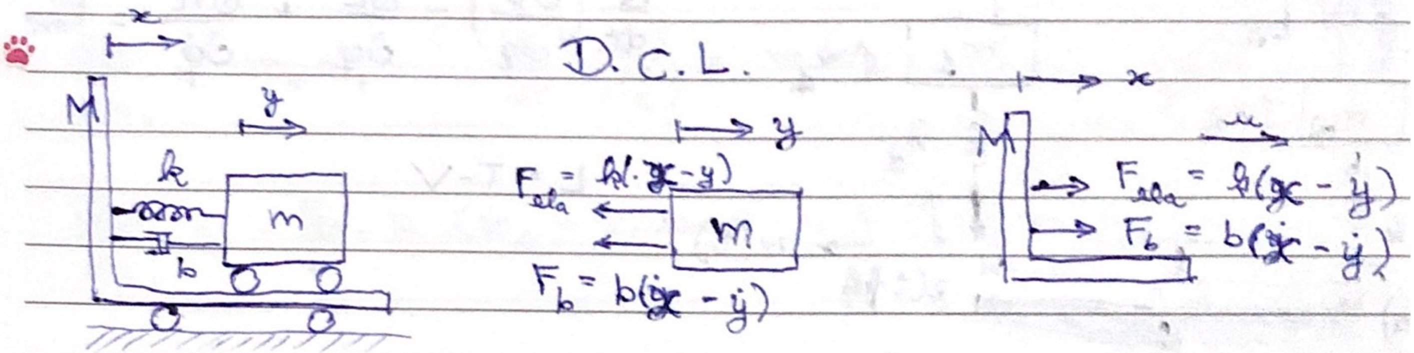
Na forma matricial:

$$\begin{bmatrix} m & m \\ 0 & M \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} b_a & -b_a \\ -b_a & b_a + b \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} k_a & -k_a \\ -k_a & k_a + k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} mg \\ m_1 r \operatorname{sen}(\omega t)\omega^2 + Mg \end{bmatrix}$$

3- Carrinho de transporte

3.1 - Massa da carreta desprezível

3.2 - " " " não desprezível



(1) $m(\ddot{x}+\ddot{y}) = -k(x-y) - b(\dot{x}-\dot{y})$

(2) $M\cdot\ddot{x} = k(x-y) + b(\dot{x}-\dot{y}) + u$

3.1 - $m >> M$:

$$k(x-y) + b(\dot{x}-\dot{y}) + u = 0$$

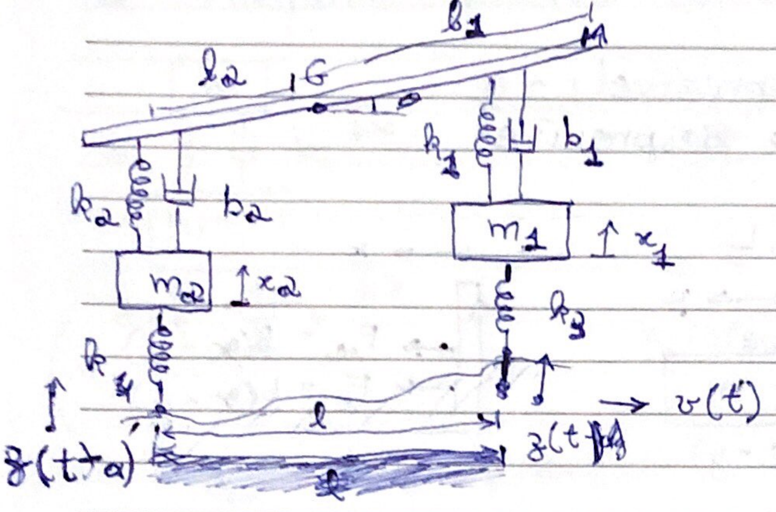
$$\Rightarrow u = -k(x-y) - b(\dot{x}-\dot{y})$$

$$\Rightarrow u = m(\ddot{x}+\ddot{y})$$

3.2 - M não desprezível :

$$\begin{bmatrix} m & m \\ M & 0 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}$$

4- Carro



Equação de Lagrange:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial R}{\partial \dot{q}} = F_{ext}$$

$$L = T - V$$

$$T = T_1 + T_2 + T_{barra} = \left(\frac{m_1 \cdot \dot{x}_1^2}{2}\right) + \left(\frac{m_2 \cdot \dot{x}_2^2}{2}\right) + \left(\frac{M \cdot \dot{x}_G^2}{2} + \right.$$

$$\left. + M \cdot \dot{\vec{x}}_G \left(\vec{\omega} \wedge \cancelto{0}{(G-G)}\right) + \frac{1}{2} \left[\cancel{\omega_x}\ \cancel{\omega_y}\ \omega_z\right] [J]_{Gxyz} \begin{bmatrix} \cancel{\omega_x} \\ \cancel{\omega_y} \\ \omega_z \end{bmatrix}\right)$$

$$\Rightarrow T = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} + \frac{M \dot{x}_G^2}{2} + \frac{J_z \cdot \dot{\theta}^2}{2}$$

$$V = \frac{k_1 (x_G + l_1 \operatorname{sen}\theta - x_1)^2}{2} + \frac{k_2 (x_G - l_2 \operatorname{sen}\theta - x_2)^2}{2} +$$

$$+ \frac{k_3 (x_1 - z(t))^2}{2} + \frac{k_4 (x_2 - z(t-a))^2}{2}$$

$$L = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} + \frac{M \dot{x}_G^2}{2} + \frac{J_z \dot{\theta}^2}{2} - \frac{k_1 (x_G + l_1 \operatorname{sen}\theta - x_1)^2}{2}$$

$$- \frac{k_2 (x_G - l_2 \operatorname{sen}\theta - x_2)^2}{2} - \frac{k_3 (x_1 - z(t))^2}{2}$$

$$- \frac{k_4 (x_2 - z(t-a))^2}{2}$$

$$R = \frac{b_1(\dot{x}_G + l_1 \cos\theta\,\dot{\theta} - \dot{x}_1)^2}{2} + \frac{b_2(\dot{x}_G - l_2 \cos\theta\,\dot{\theta} - \dot{x}_2)^2}{2}$$

Coordenadas generalizadas: x_1, x_2, x_G, θ

a)

$$\frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1 \quad \Big| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$-\frac{k_3}{2}\cdot 2(x_1 - z(t))\cdot 1$$

$$\frac{\partial L}{\partial x_1} = -\frac{k_1}{2}\cdot 2(x_G + l_1 \operatorname{sen}\theta - x_1)\cdot(-1) = k_1(x_G + l_1 \operatorname{sen}\theta - x_1) - k_3(x_1 - z(t))$$

$$\frac{\partial R}{\partial \dot{x}_1} = -b_1(\dot{x}_G + l_1 \cos\theta\cdot\dot{\theta} - \dot{x}_1)$$

(1) $$m_1\ddot{x}_1 - k_1(x_G + l_1 \operatorname{sen}\theta - x_1) + k_3(x_1 - z(t)) - b_1(\dot{x}_G + l_1\cos\theta\cdot\dot{\theta} - \dot{x}_1) = 0$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2 \quad \Big| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial L}{\partial x_2} = k_2(x_G - l_2 \operatorname{sen}\theta - x_2) - k_4(x_2 - z(t-a))$$

$$\frac{\partial R}{\partial \dot{x}_2} = -b_2(\dot{x}_G - l_2\cos\theta\cdot\dot{\theta} - \dot{x}_2)$$

(2) $$m_2\ddot{x}_2 - k_2(x_G - l_2 \operatorname{sen}\theta - x_2) + k_4(x_2 - z(t-a)) - b_2(\dot{x}_G - l_2\cos\theta\,\dot{\theta} - \dot{x}_2) = 0$$

$$\frac{\partial L}{\partial \dot{x}_G} = M \cdot \dot{x}_G \quad \Big| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_G}\right) = M \ddot{x}_G$$

$$\frac{\partial L}{\partial x_G} = -k_1(x_G + l_1 \operatorname{sen}\theta - x_1) - k_2(x_G + l_2 \operatorname{sen}\theta - x_2)$$

$$\frac{\partial R}{\partial \dot{x}_G} = b_1(\dot{x}_G + l_1 \cos\theta \cdot \dot{\theta} - \dot{x}_1) + b_2(\dot{x}_G - l_2 \cos\theta \cdot \dot{\theta} - \dot{x}_2)$$

$$M\ddot{x}_G + k_1(x_G + l_1 \operatorname{sen}\theta - x_1) + k_2(x_G + l_2 \operatorname{sen}\theta - x_2) +$$

$$+ b_1(\dot{x}_G + l_1 \cos\theta \cdot \dot{\theta} - \dot{x}_1) + b_2(\dot{x}_G - l_2 \cos\theta \cdot \dot{\theta} - \dot{x}_2) = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = J_g \dot{\theta} \quad \Big| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J_g \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -k_1(x_G + l_1 \operatorname{sen}\theta - x_1)\, l_2 \cos\theta - k_2(x_G - l_2 \operatorname{sen}\theta - x_2) \cdot (-l_2 \cos\theta)$$

$$= -k_1(x_G + l_1 \operatorname{sen}\theta - x_1)\, l_1 \cos\theta +$$

$$+ k_2(x_G - l_2 \operatorname{sen}\theta - x_2)\, l_2 \cos\theta$$

$$\frac{\partial R}{\partial \dot{\theta}} = b_1(\dot{x}_G + l_1 \cos\theta \cdot \dot{\theta} - \dot{x}_1)\, l_1 \cos\theta$$

$$- b_2(\dot{x}_G - l_2 \cos\theta \cdot \dot{\theta} - \dot{x}_2)\, l_2 \cos\theta$$

$$J_g \ddot{\theta} + k_1(x_G + l_1 \operatorname{sen}\theta - x_1)\, l_1 \cos\theta - k_2(x_G - l_2 \operatorname{sen}\theta - x_2)\, l_2 \cos\theta +$$

$$+ b_1(\dot{x}_G + l_1 \cos\theta \cdot \dot{\theta} - \dot{x}_1)\, l_1 \cos\theta - b_2(\dot{x}_G - l_2 \cos\theta \cdot \dot{\theta} - \dot{x}_2)\, l_2 \cos\theta = 0$$

b) Ângulo de inclinação pequeno:

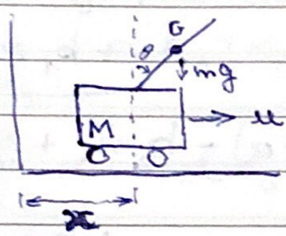
$$m_1\ddot{x}_1 - k_1 x_G - k_1 \cdot l_1 \theta + k_1 x_1 + k_3 x_1 - k_3 \cdot z(t) - b_1 \dot{x}_G - b_1 l_1 \dot{\theta} +$$
$$+ b_1 \dot{x}_1 = 0$$

$$m_2\ddot{x}_2 - k_2 x_G + k_2 l_2 \theta + k_2 x_2 + k_4 x_2 - k_4 \cdot z(t-\alpha) - b_2 \dot{x}_G + b_2 l_2 \dot{\theta}$$
$$+ b_2 \dot{x}_2 = 0$$

$$M\ddot{x}_G + k_1 x_G + k_1 l_1 \theta - k_1 x_1 + k_2 x_G + k_2 l_2 \theta - k_2 x_2 + b_1 \dot{x}_G +$$
$$+ b_1 l_1 \dot{\theta} - b_1 \dot{x}_1 + b_2 \dot{x}_G - b_2 l_2 \dot{\theta} - b_2 \dot{x}_2 = 0$$

$$J_3 \ddot{\theta} + k_1 (x_G + l_1 \theta - x_1) l_1 - k_2 (x_G - l_2 \theta - x_2) l_2 +$$
$$+ b_1 (\dot{x}_G + l_1 \dot{\theta} - \dot{x}_1) l_1 - b_2 (\dot{x}_G - l_2 \dot{\theta} - \dot{x}_2) l_2 = 0$$

5- Pêndulo invertido



a) Leis de Newton

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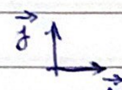
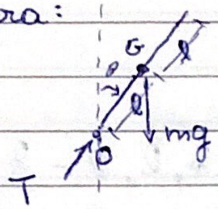


barra:



$$\frac{d\vec{H}_O}{dt} = \vec{M}_O$$

$$\vec{H}_O = (G-O) \wedge m\vec{v}_O + [\vec{i}\ \vec{j}\ \vec{k}]\,[J]_{Oxyz} \begin{bmatrix} \cancel{\omega_x} \\ \cancel{\omega_y} \\ \omega_z \end{bmatrix}$$

$$\Rightarrow \vec{H}_O = (l \cancel{sen} \operatorname{sen}\theta\,\vec{i} + l\cos\theta\,\vec{j}) \wedge m \cdot \dot{x}\,\vec{i} + J_z \cdot \dot{\theta}\,\vec{k}$$

$$\Rightarrow \vec{H}_O = -ml\cos\theta\,\dot{x}\,\vec{k} + J_z\,\dot{\theta}\,\vec{k}$$

$$\Rightarrow \frac{d\vec{H}_O}{dt} = \left(-ml\cos\theta\,\ddot{x} + ml\operatorname{sen}\theta \cdot \dot{\theta}\,\dot{x} + \frac{m(2l)^2}{3} \cdot \frac{\ddot{\theta}}{2}\right)\vec{k}$$

$$\vec{M}_O = -mg \cdot l\operatorname{sen}\theta\,\vec{k}$$

$$\therefore\ -mgl\operatorname{sen}\theta = -ml\cos\theta \cdot \ddot{x} + ml\operatorname{sen}\theta \cdot \dot{\theta}\,\dot{x} + \frac{4ml^2}{2\cdot 3}\ddot{\theta}$$

$$\Rightarrow ml\cos\theta \cdot \ddot{x} - \frac{2ml^2}{3} \cdot \ddot{\theta} - ml\operatorname{sen}\theta \cdot \dot{\theta} \cdot \dot{x} - mgl\operatorname{sen}\theta = 0$$

$$\vec{R} = m \cdot \vec{a}_G$$

$$\vec{a}_G = \vec{a}_O + \dot{\vec{\omega}} \wedge (G-O) + \vec{\omega} \wedge (\vec{\omega} \wedge (G-O))$$

$$= \ddot{x}\,\vec{i} - \ddot{\theta}\,\vec{k} \wedge (l \operatorname{sen}\theta\,\vec{i} + l\cos\theta\,\vec{j}) +$$

$$- \dot{\theta}\,\vec{k} \wedge (-\dot{\theta}\,\vec{k} \wedge (l \operatorname{sen}\theta\,\vec{i} + l\cos\theta\,\vec{j}))$$

$$\Rightarrow \vec{a}_G = \ddot{x}\,\vec{i} - l \operatorname{sen}\theta \cdot \ddot{\theta}\,\vec{j} + l\cos\theta \cdot \ddot{\theta}\,\vec{i} +$$

$$- \dot{\theta}\,\vec{k} \wedge (-l \operatorname{sen}\theta \cdot \dot{\theta}\,\vec{j} + l\cos\theta \cdot \dot{\theta}\,\vec{i})$$

$$\Rightarrow \vec{a}_G = \ddot{x}\,\vec{i} + l\cos\theta \cdot \ddot{\theta}\,\vec{i} - l \operatorname{sen}\theta \cdot \ddot{\theta}\,\vec{j} - l \operatorname{sen}\theta \cdot \dot{\theta}^2\,\vec{i} - l\cos\theta \cdot \dot{\theta}^2\,\vec{j}$$

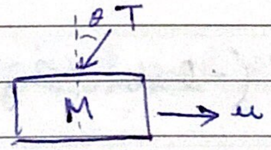
$$\Rightarrow \vec{a}_G = (\ddot{x} + l\cos\theta \cdot \ddot{\theta} - l \operatorname{sen}\theta \cdot \dot{\theta}^2)\,\vec{i} - (l \operatorname{sen}\theta \cdot \ddot{\theta} + l\cos\theta \cdot \dot{\theta}^2)\,\vec{j}$$

$$\vec{R} = -mg\,\vec{j} + T \operatorname{sen}\theta\,\vec{i} + T\cos\theta\,\vec{j}$$

$$\begin{cases} m(\ddot{x} + l\cos\theta \cdot \ddot{\theta} - l \operatorname{sen}\theta \cdot \dot{\theta}^2) = T \operatorname{sen}\theta & (1) \\ -m(l \operatorname{sen}\theta \cdot \ddot{\theta} + l\cos\theta \cdot \dot{\theta}^2) = -mg + T\cos\theta & (2) \end{cases}$$

D.C.L.

carrinho



$$\vec{R} = m \cdot \vec{a}_G$$

$$M \cdot \ddot{x}\,\vec{i} = u\,\vec{i} - T \operatorname{sen}\theta\,\vec{i}$$

$$\Rightarrow T \operatorname{sen}\theta = u - M\ddot{x} \Rightarrow T = \frac{u - M\ddot{x}}{\operatorname{sen}\theta}$$

Substituindo T em (1):

$$m(\ddot{x} + l\cos\theta \cdot \ddot{\theta} - l\,\mathrm{sen}\,\theta \cdot \dot{\theta}^2) = \frac{u - M\ddot{x}}{\cancel{\mathrm{sen}\theta}} \cdot \cancel{\mathrm{sen}\,\theta}$$

$$\boxed{\Rightarrow (M+m)\ddot{x} + ml\cos\theta \cdot \ddot{\theta} - ml\,\mathrm{sen}\,\theta \cdot \dot{\theta}^2 = u}$$

b) Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \cancel{\frac{\partial R}{\partial \dot{q}}} = F_{ext}$$

$L = T - V$

$T = T_{carrinho} + T_{barra}$ | $T_{carrinho} = \frac{M \cdot \dot{x}^2}{2}$

$$T_{barra} = T_0 = \frac{m \cdot \dot{x}^2}{2} + m \cdot \dot{x}\vec{i}\left(-\dot{\theta}\vec{k} \wedge (l\,\mathrm{sen}\,\theta\,\vec{i} + l\cos\theta\,\vec{j})\right) +$$

$$+ \frac{1}{2}\left[\cancel{\omega_x}\ \cancel{\omega_y}\ -\dot{\theta}\right][J]_{O_{xyz}}\begin{bmatrix}\cancel{\omega_x} \\ \cancel{\omega_y} \\ -\dot{\theta}\end{bmatrix}$$

$$\Rightarrow T_{barra} = \frac{m\dot{x}^2}{2} + m\dot{x}\vec{i}(-l\,\mathrm{sen}\,\theta \cdot \dot{\theta}\,\vec{j} + l\cos\theta \cdot \dot{\theta}\,\vec{i}) + \frac{J_z \dot{\theta}^2}{2}$$

$$= \frac{m\dot{x}^2}{2} + ml\cos\theta \cdot \dot{\theta} \cdot \dot{x} + \frac{m(2l)^2}{3} \cdot \frac{\dot{\theta}^2}{2}$$

$$\therefore T = \frac{M\dot{x}^2}{2} + \frac{m\dot{x}^2}{2} + ml\cos\theta \cdot \dot{\theta}\dot{x} + \frac{2ml^2}{3} \cdot \dot{\theta}^2$$

$V = mgl\cos\theta$

$$L = \frac{M\dot{x}^2}{2} + \frac{m\dot{x}^2}{2} + \frac{\cancel{2}\,ml^2}{3}\cdot\dot{\theta}^2 + ml\cos\theta\cdot\dot{\theta}\cdot\dot{x} - mgl\cos\theta$$

Coodernadas generalizadas: x, θ

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + ml\cos\theta\cdot\dot{\theta} \quad \Bigg| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m\ddot{x} + ml\cos\theta\cdot\ddot{\theta} + - ml\,\mathrm{sen}\,\theta\cdot\dot{\theta}^2$$

$$\frac{\partial L}{\partial x} = 0 \quad \Bigg| \quad F_{ext} = u$$

$$\therefore \boxed{M\ddot{x} + m\ddot{x} + ml\cos\theta\cdot\ddot{\theta} - ml\,\mathrm{sen}\,\theta\cdot\dot{\theta} = u}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{4ml^2}{3}\cdot\dot{\theta} + ml\cos\theta\cdot\dot{x} \quad \Bigg| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{4ml^2}{3}\cdot\ddot{\theta} + ml\cos\theta\cdot\ddot{x} + - ml\,\mathrm{sen}\,\theta\cdot\dot{\theta}\cdot\dot{x}$$

$$\frac{\partial L}{\partial \theta} = -ml\,\mathrm{sen}\,\theta\cdot\dot{\theta}\cdot\dot{x} + mgl\,\mathrm{sen}\,\theta \quad \Bigg| \quad F_{ext} = 0$$

$$\therefore \frac{4ml^2}{3}\cdot\ddot{\theta} + ml\cos\theta\cdot\ddot{x} - \cancel{ml\,\mathrm{sen}\,\theta\cdot\dot{\theta}\cdot\dot{x}} + \cancel{ml\,\mathrm{sen}\,\theta\cdot\dot{\theta}\cdot\dot{x}} - mgl\,\mathrm{sen}\,\theta = 0$$

$$\Rightarrow \boxed{\frac{4ml^2}{3}\cdot\ddot{\theta} + ml\cos\theta\cdot\ddot{x} - mgl\,\mathrm{sen}\,\theta = 0}$$